Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 3, No. 1, (January 2012), pp. 1- 8
ISSN 2093–9310
http://www.afmi.or.kr

Fuzzy Cs-closed spaces

A. M. Zahran, A. Ghareeb

Received 1 May 2011; Accepted 5 June 2011

Abstract. In this paper the concept of Cs-closed Spaces is introduced in fuzzy topological spaces. Several characterizations and some interesting properties of these spaces are discussed. The image of fuzzy Cs-closed under some functions are also discussed.

2010 AMS Classification: 54A40.

Keywords: Fuzzy Cs-closed, Fuzzy semicontinuity, Fuzzy irresolute function, Fuzzy filter

Corresponding Author: A. Ghareeb (nasserfuzt@aim.com)

1. Preliminaries

The concept of fuzzy topology was first defined in 1968 by Chang [3] based on the concept of a fuzzy set introduced by Zadeh in [8]. Since then, various important notions in the classical topology such as compactness have been extended to fuzzy topological spaces.

The purpose of this paper is to introduce and study the concept of fuzzy Cs-closeness in fuzzy setting. Section 1 deals with preliminaries, Section 2 deals with the concept of fuzzy Cs-closeness and some of the characterizations in fuzzy topological spaces and properties are discussed. Finally the image of fuzzy Cs-closeness under some functions are also investigated.

Let (X,τ) be a fuzzy topological space (fts, for short) and let μ be any fuzzy set in X. We define the closure of μ to be $Cl(\mu) = \bigwedge\{\lambda \mid \mu \leq \lambda,\ \lambda \text{ is fuzzy closed}\}$ and the interior of μ to be $Int(\mu) = \bigvee\{\lambda \mid \lambda \leq \mu,\ \lambda \text{ is fuzzy open}\}$.

A fuzzy point [6] x_r is a fuzzy set with support x and value $r \in (0,1]$. For a fuzzy set μ in X, we write $x_r \in \mu$ iff $r \leq \mu(x)$. Evidently, every fuzzy set μ can be expressed as the union of all fuzzy points which belong to μ. A fuzzy point x_r is said to be quasi-coincident [5] with μ denoted by $x_r q\mu$ if and only if $r + \mu(x) > 1$. A fuzzy set μ is said to be quasi-coincident with λ, denoted $\mu q\lambda$ if and only if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. If μ is not quasi-coincident with λ, then we write $\lambda\bar{q}\mu$.

Definition 1.1 ([5])**.** A fuzzy set A is said to be q-neighborhood if a fuzzy point x_r if there exist a fuzzy open set U with $x_r q U \leq A$.

Definition 1.2. A fuzzy set μ in a fts X is said to be:

(i) Fuzzy semiopen [2] (fso, for short) if $\mu \leq Cl(Int(\mu))$,

(ii) Fuzzy semiclosed [2] (fsc, for short) if $Int(Cl(\mu)) \leq \mu$,

(iii) Fuzzy regular semiopen [1] ($frso$, for short) if there exists a fuzzy regular open set λ such that $\lambda \leq \mu \leq Cl(\lambda)$,

(iv) Fuzzy semiopen neighborhood [2] of a fuzzy point x_r if there exists a fuzzy semiopen set λ such that $x_r \in \lambda \leq \mu$.

Definition 1.3 ([2])**.** The intersection of all fuzzy semiclosed sets containing a fuzzy set μ is called a fuzzy semiclosure of μ and will be denoted by $Scl(\mu)$. The union of all fuzzy semiopen sets contained in a fuzzy set μ is called a fuzzy semi-interior of μ and will be denoted by $Sint(\mu)$.

Lemma 1.4. *For a fuzzy set μ in a fts X, the following are true*

(i) *Every fuzzy regular semiopen set is fuzzy semiopen and semiclosed,*

(ii) *$Scl(\mu)$ is fuzzy regular semiopen for each fuzzy semiopen set μ in X.*

Definition 1.5 ([7])**.** Let (X, τ) be a fts. A family of fuzzy sets $\xi = \{\lambda_\alpha : \alpha \in \Delta\}$ in X is said to be a cover of X if $\bigvee_{\lambda_\alpha \in \Delta} \lambda_\alpha = 1$ and a subfamily of ξ having a similar property is called a subcover of ξ. A fuzzy topological space (X, τ) is said to be a fuzzy compact (resp. semicompact) if every cover of X by fuzzy open (resp. semiopen) sets has a finite subcover.

Definition 1.6 ([2])**.** A function $f : (X, \tau) \to (Y, \delta)$ is said to be fuzzy irresolute (resp. semicontinuous) if the inverse image of a fuzzy semiopen (resp. open) set in Y is a fuzzy semiopen (resp. semiopen) set in X.

Lemma 1.7 ([2])**.** *A function $f : (X, \tau) \to (Y, \delta)$ is fuzzy irresolute if and only if $Scl(f^{-1}(\lambda)) \leq f^{-1}(Scl(\lambda))$ for each fuzzy set λ in Y*

Lemma 1.8 ([2])**.** *A function $f : (X, \tau) \to (Y, \delta)$ is fuzzy semicontinuous if and only if $Scl(f^{-1}(\lambda)) \leq f^{-1}(Cl(\lambda))$ for each fuzzy set λ in Y*

Definition 1.9 ([4])**.** A fuzzy *filter base* on X is a non-empty collection ζ of fuzzy sets on X satisfy the conditions:

(i) $0 \notin \xi$; where 0 stands for empty fuzzy set,

(ii) If $\lambda_1, \lambda_2 \in \zeta$, then $\lambda_1 \wedge \lambda_2 \in \zeta$,

(iii) If $\lambda \leq \mu$ and $\lambda \in \zeta$, then $\mu \in \zeta$.

2. Fuzzy Cs-closed space

Definition 2.1. Let (X, τ) be a fuzzy topological space. (X, τ) is said to be fuzzy Cs-closed if for any ordinary subset A of $X, A \neq X$ such that χ_A (the characteristic function of $A \subset X$) is a proper fuzzy semiclosed set and for each fuzzy semiopen cover $\beta = \{\lambda_\alpha : \alpha \in \Delta\}$ of χ_A, there exists a finite subfamily β_0 of β such that

$$\chi_A \leq \bigvee_{\lambda_\alpha \in \beta_0} Scl(\lambda_\alpha).$$

Remark 2.2. From this definition it is clear that fuzzy Cs-closed ness implies fuzzy nearly C-compactness.

Theorem 2.3. *In a fuzzy topological space (X, τ), the following statements are equivalent:*

(i) *X is fuzzy Cs-closed .*

(ii) *For each subset $A \subset X$ such that χ_A is a proper fuzzy semiclosed set and for each fuzzy regular semiopen cover $\beta = \{\lambda_\alpha \colon \alpha \in \Delta\}$ of χ_A, there exists a finite subfamily β_0 of β such that*
$$\chi_A \leq \bigvee_{\lambda_\alpha \in \beta_0} \lambda_\alpha.$$

(iii) *For each subset $A \subset X$ such that χ_A is a proper fuzzy semiclosed set and for each family of fuzzy semiclosed sets $\xi = \{\lambda_\alpha \colon \alpha \in \Delta\}$ such that*
$$\Big(\bigwedge_{\lambda_\alpha \in \xi} \lambda_\alpha\Big) \wedge \chi_A = 0,$$
there exists a finite subfamily ξ_0 of ξ such that
$$\Big(\bigwedge_{\lambda_\alpha \in \xi_0} Sint(\lambda_\alpha)\Big) \wedge \chi_A = 0.$$

(iv) *For each subset $A \subset X$ such that χ_A is a proper fuzzy semiclosed set and for each family $\xi = \{\lambda_\alpha \colon \alpha \in \Delta\}$ of fuzzy semiclosed sets, if for each finite subfamily ξ_0 of ξ we have*
$$\Big(\bigwedge_{\lambda_\alpha \in \xi_0} Sint(\lambda_\alpha)\Big) \wedge \chi_A \neq 0,$$
then
$$\Big(\bigwedge_{\alpha \in \Delta} \lambda_\alpha\Big) \wedge \chi_A \neq 0.$$

Proof. (i) ⇒ (ii) It follows from Lemma 1.4.

(ii) ⇒ (i) Let A be any subset of X such that χ_A is a proper fuzzy semiclosed set and $\beta = \{\lambda_\alpha \colon \alpha \in \Delta\}$ be a fuzzy semiopen cover of χ_A. Then $\xi = \{Scl(\lambda_\alpha) \colon \alpha \in \Delta\}$ is a fuzzy regular semiopen cover of χ_A. By (ii) there exists a finite subfamily ξ_0 of ξ such that
$$\chi_A \leq \bigvee_{\lambda_\alpha \in \xi_0} Scl(\lambda_\alpha).$$

(ii) ⇒ (iii) Let A be any subset of X such that χ_A is proper fuzzy semiclosed set and $\beta = \{\lambda_\alpha \colon \alpha \in \Delta\}$ be a fuzzy semiclosed family such that
$$\Big(\bigwedge_{\lambda_\alpha \in \beta} \lambda_\alpha\Big) \wedge \chi_A = 0.$$
Then $\xi = \{Scl(1 - \lambda_\alpha) \colon \alpha \in \Delta\}$ is a fuzzy regular semiopen cover of χ_A and hence there exists a finite subfamily ξ_0 of ξ such that
$$\chi_A \leq \bigvee_{\lambda_\alpha \in \xi_0} Scl(1 - \lambda_\alpha).$$

Thus

$$\Big(1 - \bigvee_{\lambda_\alpha \in \xi_0} Scl(1-\lambda_\alpha)\Big) \le (1-\chi_A)$$

and hence

$$\begin{aligned} \Big(\bigwedge_{\lambda_\alpha \in \xi_0} Sint(\lambda_\alpha)\Big) \wedge \chi_A &\le (1-\chi_A) \wedge \chi_A \\ &= \chi_{X-A} \wedge \chi_A \\ &= \chi_{(X-A)\cap A} = \chi_\phi = 0 \end{aligned}$$

(iii) $\Rightarrow$ (ii) Let A be any subset of X such that χ_A is a proper fuzzy semiclosed set and $\beta = \{\lambda_\alpha \colon \alpha \in \Delta\}$ be a fuzzy regular semiopen cover of χ_A. Then

$$\chi_A \le \bigvee_{\lambda_\alpha \in \beta} \lambda_\alpha$$

and hence

$$1 - \chi_A \ge (1 - \bigvee_{\lambda_\alpha \in \beta} \lambda_\alpha)$$

which implies

$$\Big(\bigwedge_{\lambda_\alpha \in \beta} (1-\lambda_\alpha)\Big) \wedge \chi_A \le \chi_{X-A} \wedge \chi_A = 0.$$

So

$$\xi = (1-\lambda_\alpha \colon \alpha \in \Delta)$$

is a family of a fuzzy semiclosed such that

$$\Big(\bigwedge_{\lambda_\alpha \in \beta} (1-\lambda_\alpha)\Big) \wedge \chi_A = 0.$$

Then by (iii) there exists a finite subfamily ξ_0 of ξ such that

$$\Big(\bigwedge_{\lambda_\alpha \in \xi_0} Sint(1-\lambda_\alpha)\Big) \wedge \chi_A = 0,$$

which implies

$$\begin{aligned} \chi_A &\le \bigvee_{\lambda_\alpha \in \xi_0} \Big(1 - Sint(1-\lambda_\alpha))\Big) \\ &\le \bigvee_{\lambda_\alpha \in \xi_0} Scl(\lambda_\alpha) \\ &= \bigvee_{\lambda_\alpha \in \xi_0} \lambda_\alpha \end{aligned}$$

(iii) $\Rightarrow$ (iv) Let A be any subset of X such that χ_A is a proper fuzzy semiclosed set and $\xi = \{\lambda_\alpha \colon \alpha \in \Delta\}$ be a family of fuzzy semiclosed sets such that for each finite family ξ_0 of ξ,

$$\Big(\bigwedge_{\lambda_\alpha \in \xi_0} Sint(\lambda_\alpha)\Big) \wedge \chi_A \ne 0.$$

Suppose that

$$(\bigwedge_{\lambda_\alpha \in \xi} \lambda_\alpha) \wedge \chi_A = 0,$$

then by (iii) there exists a finite subfamily ξ_0 of ξ such that

$$\Big(\bigwedge_{\lambda_\alpha \in \xi_0} Sint(\lambda_\alpha)\Big) \wedge \chi_A = 0,$$

which a contradiction. Hence

$$(\bigwedge_{\lambda_\alpha \in \xi} \lambda_\alpha) \wedge \chi_A \neq 0.$$

(iv) $\Rightarrow$ (iii) Let A be any subset of X such that χ_A is a proper fuzzy semiclosed set and $\xi = \{\lambda_\alpha \colon \alpha \in \Delta\}$ be a family of fuzzy semiclosed sets such that

$$(\bigwedge_{\lambda_\alpha \in \xi} \lambda_\alpha) \wedge \chi_A = 0.$$

Suppose that for every finite subfamily ξ_0 of ξ we have

$$\Big(\bigwedge_{\lambda_\alpha \in \xi_0} Sint(\lambda_\alpha)\Big) \wedge \chi_A \neq 0,$$

then by (iv) we have

$$(\bigwedge_{\lambda_\alpha \in \xi} \lambda_\alpha) \wedge \chi_A \neq 0$$

which a contradiction. Hence there exists a finite subfamily ξ_0 of ξ such that

$$\Big(\bigwedge_{\lambda_\alpha \in \xi_0} Sint(\lambda_\alpha)\Big) \wedge \chi_A = 0.$$

□

Theorem 2.4. *In a fuzzy topological space (X, τ), the following statements are equivalent:*

(i) *X is fuzzy Cs-closed.*

(ii) *If $A \subset X$ such that χ_A is a proper fuzzy semiclosed set and $\beta = \{\lambda_\alpha \colon \alpha \in \Delta\}$ is a family of fuzzy semiclosed sets of X such that*

$$\chi_A \leq 1 - \bigwedge_{\lambda_\alpha \in \beta} \lambda_\alpha,$$

then there exists a finite subfamily β_0 of β such that

$$\chi_A \leq 1 - \bigwedge_{\lambda_\alpha \in \beta_0} Sint(\lambda_\alpha).$$

Proof. (i) $\Rightarrow$ (ii) Let A be any subset of X such that χ_A is a proper fuzzy semiclosed set and $\beta = \{\lambda_\alpha \colon \alpha \in \Delta\}$ be a family of fuzzy semiclosed sets of X such that

$$\chi_A \leq 1 - \bigwedge_{\lambda_\alpha \in \beta} \lambda_\alpha = \bigvee_{\lambda_\alpha \in \beta} (1 - \lambda_\alpha).$$

Hence $\xi = \{1 - \lambda_\alpha \colon \alpha \in \Delta\}$ is a fuzzy semiopen cover of χ_A. Since X is fuzzy Cs-closed , then there exists a finite subfamily ξ_0 of ξ such that

$$\begin{aligned} \chi_A \leq \bigvee_{\lambda_\alpha \in \xi_0} Scl(1 - \lambda_\alpha) &= \bigvee_{\lambda_\alpha \in \xi_0} \Big(1 - Sint(\lambda_\alpha)\Big) \\ &= 1 - \bigwedge_{\lambda_\alpha \in \xi_0} Sint(\lambda_\alpha). \end{aligned}$$

(ii) $\Rightarrow$ (i) Let A be any subset of X such that χ_A is a proper fuzzy semiclosed set and $\beta = \{\lambda_\alpha \colon \alpha \in \Delta\}$ be a fuzzy semiopen cover of χ_A. Then $\xi = \{1 - \lambda_\alpha \colon \alpha \in \Delta\}$ is a family of fuzzy semiclosed sets such that

$$\begin{aligned} \chi_A \leq \bigvee_{\lambda_\alpha \in \beta} \lambda_\alpha &= \bigvee_{\lambda_\alpha \in \beta} \Big(1 - (1 - \lambda_\alpha)\Big) \\ &= 1 - \bigwedge_{\lambda_\alpha \in \beta} (1 - \lambda_\alpha) \end{aligned}$$

Hence by (ii) there exists a finite subfamily ξ_0 of ξ such that

$$\begin{aligned} \chi_A \leq 1 - \bigwedge_{\lambda_\alpha \in \xi_0} Sint(1 - \lambda_\alpha) &= \bigvee_{\lambda_\alpha \in \xi_0} \Big(1 - Sint(1 - \lambda_\alpha)\Big) \\ &= \bigvee_{\lambda_\alpha \in \xi_0} Scl(\lambda_\alpha). \end{aligned}$$

Thus X is fuzzy Cs-closed . □

Definition 2.5. Let (X, τ) be a fuzzy topological space. A fuzzy filter ζ in X is said to be semi adherence convergent if every fuzzy semiopen neighborhood of the adherence set of ζ contains an element of ζ, where the adherence set of ζ is defined by $\bigwedge\{Scl(\mu) \colon \mu \in \zeta\}$.

Theorem 2.6. *If (X, τ) is fuzzy Cs-closed, then every fuzzy semiopen filter is semi adherence convergent.*

Proof. Let (X, τ) be fuzzy Cs-closed, ζ be any fuzzy semiopen filter with semi adherence set λ of ζ and ρ be a fuzzy semiopen neighborhood of λ. Then $\lambda = \bigwedge\{Scl(\mu) \colon \mu \in \zeta\}$ and $\lambda \leq \rho$ and hence $1 - \rho$ is fuzzy semiclosed. Since

$$\begin{aligned} 1 - \rho \leq 1 - \lambda &= 1 - \bigwedge_{\mu \in \zeta} Scl(\mu) \\ &= \bigvee_{\mu \in \zeta} \Big(1 - Scl(\mu)\Big) \end{aligned}$$

Then $\{1 - Scl(\mu) \colon \mu \in \zeta\}$ is a fuzzy semiopen cover of a fuzzy semiclosed set $1 - \rho$ and hence there exists a finite subfamily $\{1 - Scl(\mu_\alpha) \colon \alpha = 1, 2, ..., n\}$ such that

$$\begin{aligned}
1 - \rho &\leq \bigvee_{\alpha=1}^{n} Scl\Big(1 - Scl(\mu_\alpha)\Big) \\
&= \bigvee_{\alpha=1}^{n} \Big(1 - Sint\Big(Scl(\mu_\alpha)\Big)\Big) \\
&= 1 - \bigwedge_{\alpha=1}^{n} Sint\Big(Scl(\mu_\alpha)\Big) \\
&= 1 - \bigwedge_{\alpha=1}^{n} Scl(\mu_\alpha)
\end{aligned}$$

and hence

$$\bigwedge_{\alpha=1}^{n} Scl(\mu_\alpha) \leq \rho.$$

Since $\mu_\alpha \leq Scl(\mu_\alpha)$ for each $\alpha = 1, 2, ..., n$, then

$$\bigwedge_{\alpha=1}^{n} \mu_\alpha \leq Scl(\mu_\alpha) \leq \rho$$

Put $\gamma = \bigwedge_{\alpha=1}^{n} \mu_\alpha$, then $\gamma \in \zeta$ and hence ρ contains γ. Therefore ζ is semi adherence convergent. $\square$

Theorem 2.7. *Let $f \colon (X, \tau) \to (Y, \delta)$ be a function from a fts (X, τ) to a fts (Y, δ). Then the image of a fuzzy Cs-closed space under a fuzzy irresolute function is fuzzy Cs-closed.*

Proof. Let $f \colon (X, \tau) \to (Y, \delta)$ be a fuzzy irresolute function from fuzzy Cs-closed X onto Y and let $A \subset Y$ be any subset of Y such that χ_A is fuzzy semiclosed in Y. Let $\beta = \{\lambda_\alpha \colon \alpha \in \Delta\}$ be a fuzzy semiopen cover of χ_A in Y. Since f is fuzzy irresolute, then $f^{-1}(\chi_A)$ is a fuzzy semiclosed set in X and $\{f^{-1}(\lambda_\alpha) \colon \alpha \in \Delta\}$ is a fuzzy semiopen cover of $f^{-1}(\chi_A)$ in X. Since X is fuzzy Cs-closed, then there is a finite subfamily $\{f^{-1}(\lambda_\alpha) \colon \alpha = 1, 2, ..., n\}$ such that

$$\begin{aligned}
f^{-1}(\chi_A) &\leq \bigvee_{\alpha=1}^{n} Scl\Big(f^{-1}(\lambda_\alpha)\Big) \\
&\leq \bigvee_{\alpha=1}^{n} f^{-1}\Big(Scl(\lambda_\alpha)\Big) \qquad \text{By Lemma 1.7,}
\end{aligned}$$

and hence

$$\chi_A \leq \bigvee_{\alpha=1}^{n} Scl(\lambda_\alpha)$$

Thus Y is fuzzy Cs-closed. $\square$

References

[1] M. H. Ghanim, A. S. Mashour and M. A. Fath Alla, α-sepsration axioms and α-compactness in fuzzy topological spaces, Rocky Mountain J. Math. 6 (1986) 591–600.

[2] K. K. Azad, On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl. 82 (1981) 14–32.

[3] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 141 (1968) 82–89.

[4] A. K. Katsaras, Convergence of fuzzy flters in fuzzy topological spaces, Bull. Math. Soc. Sci. Math. Roumanie 27 (1983) 131–137.

[5] Pu. Pao. Ming and Liu Ying Ming, Fuzzy topology, i. neighbourhood structure of a fuzzy points and moore-smith convergence, J. Math. Anal. Appl. 76 (1980) 571–599.

[6] M. N. Mukherjee and S. P. Sinha, On some weaker forms of fuzzy continuous and fuzzy open mappings on fuzzy topological spaces, Fuzzy Sets and Systems 32 (1989) 103–114.

[7] S. Nanda, On fuzzy topological spaces, Fuzzy Sets and Systems 19 (1986) 1–12.

[8] L. A. Zadeh, Fuzzy sets, Inform. Control 8 (1965) 338–353.

A. M. Zahran (amzahran60@aol.com) – Department of Mathematics, Faculty of Science, Taibah University, Saudi Arabia

A. Ghareeb (nasserfuzt@aim.com) – Department of Mathematics, Faculty of Science, South Valley University, Qena, Egypt