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# Characterizations of ternary semigroups by their anti fuzzy ideals

Muhammad Shabir, Noor Rehman

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Abstract. In this paper we define anti fuzzy left (right, lateral) ideals, anti fuzzy quasi-ideals, anti fuzzy bi-ideals and anti fuzzy generalized bi-ideals in ternary semigroups. We also characterize different classes of ternary semigroups by the properties of these ideals.

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Corresponding Author: Noor Rehman (noorrehman82@yahoo.com )

## 1. Introduction

The fundamental concept of fuzzy set given by L. Zadeh in his pioneering paper [13] provides a natural frame-work for generalizing several basic notions of algebra. A. Rosenfeld laid the foundations of the theory of fuzzy groups in [9]. The study of fuzzy sets in semigroups was introduced by Kuroki [5, 6]. A systematic exposition of fuzzy semigroups was given by Mordeson et al. in [8]. Biswas in [2] introduced the concept of anti fuzzy subgroups of groups. Hong and Jun modified Biswas's idea and applied it to BCK-algebras (see [3]). In [1] Akram and Dar defined anti fuzzy left $h-$ideals in hemirings. Shabir and Nawaz in [10] discussed the anti fuzzy ideals in semigroups. Jeong [4] defined anti fuzzy prime ideals in BCK-algebras. Many results on ternary semigroups can be seen in [11].

In this paper we introduce anti fuzzy left (right, lateral) ideals, anti fuzzy quasi-ideals, anti fuzzy bi-ideals and anti fuzzy generalized bi-ideals in ternary semigroups and discuss some basic properties of such ideals.

## 2. Preliminaries

A ternary semigroup is an algebraic structure $(S, f)$ such that $S$ is a non-empty set and $f : S^3 \rightarrow S$ is a ternary operation satisfying the following associative law: $f(f(a,b,c),d,e) = f(a,f(b,c,d),e) = f(a,b,f(c,d,e))$ for all $a,b,c,d,e \in S$.(cf [7]). A non-empty subset $A$ of a ternary semigroup $S$ is called a ternary subsemigroup of $S$ if $AAA \subseteq A$. By a left (right, lateral) ideal of a ternary semigroup $S$ we mean a non-empty subset $A$ of $S$ such that $SSA \subseteq A$ ($ASS \subseteq A, SAS \subseteq A$). If a non-empty subset $A$ of $S$ is a left and right ideal of $S$, then it is called a two sided ideal of $S$. If a non-empty subset $A$ of a ternary semigroup $S$ is a left, right and lateral ideal of $S$, then it is called an ideal of $S$. A non-empty subset $A$ of a ternary semigroup $S$ is called a quasi-ideal of $S$ if $ASS \cap SAS \cap SSA \subseteq A$ and $ASS \cap SSASS \cap SSA \subseteq A$. $A$ is called a bi-ideal of $S$ if it is a ternary subsemigroup of $S$ and $ASASA \subseteq A$. A non-empty subset $A$ of a ternary semigroup $S$ is called a generalized bi-ideal of $S$ if $ASASA \subseteq A$. It is clear that every left (right, lateral) ideal of $S$ is a quasi-ideal, every quasi-ideal is a bi-ideal and every bi-ideal is a generalized bi-ideal. An element $a$ of a ternary semigroup $S$ is called regular if there exists an element $x \in S$ such that $axa = a$. A ternary semigroup $S$ is called regular if every element of $S$ is regular. It is obvious that for a regular ternary semigroup the concept of quasi-ideal, bi-ideal and generalized bi-ideal coincide.

A fuzzy subset $f$ of a universe $X$ is a function from $X$ into the unit closed interval $[0,1]$, that is, $f : X \rightarrow [0,1]$. For any two fuzzy subsets $f$ and $g$, $f \leq g$ means $f(x) \leq g(x)$ for all $x \in S$. The symbols $f \wedge g$ and $f \vee g$ have the following meanings

$(f \wedge g)(x) = f(x) \wedge g(x)$ and $(f \vee g)(x) = f(x) \vee g(x)$ for all $x \in S$.

Let $f, g$ and $h$ be three fuzzy subsets of a ternary semigroup $S$. The product $f \circ g \circ h$ is a fuzzy subset of $S$ defined by:

$$(f \circ g \circ h)(a) = \begin{cases} \bigvee\limits_{a=xyz} (f(x) \wedge g(y) \wedge h(z)) & \text{if there exist } x,y,z \in S \\ & \text{such that } a = xyz, \\ 0 & \text{otherwise,} \end{cases}$$

A fuzzy subset $f$ of a ternary semigroup $S$ is called a fuzzy ternary subsemigroup of $S$ if $f(xyz) \geq \min\{f(x), f(y), f(z)\}$ and is called fuzzy left (right, lateral) ideal of $S$ if $f(xyz) \geq f(z)$ $(f(xyz) \geq f(x), f(xyz) \geq f(y))$ for all $x,y,z \in S$. A fuzzy subset $f$ of a ternary semigroup $S$ is called a fuzzy ideal of $S$ if it is a fuzzy left ideal, fuzzy right ideal and fuzzy lateral ideal of $S$. A fuzzy subset $f$ of a ternary semigroup $S$ is called a fuzzy quasi-ideal of $S$ if

$$(f \circ \mathcal{S} \circ \mathcal{S}) \wedge (\mathcal{S} \circ f \circ \mathcal{S}) \wedge (\mathcal{S} \circ \mathcal{S} \circ f) \leq f,$$
$$(f \circ \mathcal{S} \circ \mathcal{S}) \wedge (\mathcal{S} \circ \mathcal{S} \circ f \circ \mathcal{S} \circ \mathcal{S}) \wedge (\mathcal{S} \circ \mathcal{S} \circ f) \leq f$$

where $\mathcal{S}$ is the fuzzy subset of $S$ mapping every element of $S$ on 1.

A fuzzy subset $f$ of a ternary semigroup $S$ is called a fuzzy bi-ideal of $S$ if it is a fuzzy ternary subsemigroup of $S$ and $f(xuyvz) \geq \min\{f(x), f(y), f(z)\}$. A fuzzy subset $f$ of a ternary semigroup $S$ is called a fuzzy generalized bi-ideal of $S$ if $f(xuyvz) \geq \min\{f(x), f(y), f(z)\}$ for all $x,y,z,u,v \in S$.

It is obvious that every fuzzy left (right, lateral) ideal of a ternary semigroup $S$ is a fuzzy quasi-ideal and every fuzzy quasi-ideal of $S$ is a fuzzy bi-ideal and every fuzzy bi-ideal of $S$ is a fuzzy generalized bi-ideal of $S$, but the converse need not be true.

**Theorem 2.1** ([12]). *The following conditions on a ternary semigroup $S$ are equivalent.*

(1) *$S$ is regular.*
(2) *$R \cap M \cap L = RML$ for every right ideal $R$, every lateral ideal $M$ and every left ideal $L$ of $S$.*

**Theorem 2.2** ([11]). *The following conditions on a ternary semigroup $S$ are equivalent.*

(1) *$S$ is regular.*
(2) *$BSBSB = B$ for every bi-ideal $B$ of $S$.*
(3) *$QSQSQ = Q$ for every quasi-ideal $Q$ of $S$.*

## 3. Major Section

In this section we define anti fuzzy left (right, lateral) ideal, anti fuzzy quasi-ideal, anti fuzzy bi-ideal and anti fuzzy generalized bi-ideal in ternary semigroups and study some basic properties of these ideals.

**Definition 3.1.** A fuzzy subset $f$ of a ternary semigroup $S$ is called an *anti fuzzy ternary subsemigroup of* $S$ if

$$f(xyz) \leq \max\{f(x), f(y), f(z)\} \text{ for all } x, y, z \in S.$$

**Definition 3.2.** A fuzzy subset $f$ of a ternary semigroup $S$ is called an *anti fuzzy left (right, lateral) ideal of* $S$ if

$$f(xyz) \leq f(z)\,(f(xyz) \leq f(x), f(xyz) \leq f(y)) \text{ for all } x, y, z \in S.$$

A fuzzy subset $f$ of a ternary semigroup $S$ is called an anti fuzzy ideal of $S$ if it is an anti fuzzy left ideal, anti fuzzy right ideal and anti fuzzy lateral ideal of $S$.

**Definition 3.3.** An anti fuzzy ternary subsemigroup $f$ of a ternary semigroup $S$ is called an *anti fuzzy bi-ideal of* $S$ if it satisfies

$$f(xuyvz) \leq \max\{f(x), f(y), f(z)\} \text{ for all } x, y, z, u, v \in S.$$

**Definition 3.4.** A fuzzy subset $f$ of a ternary semigroup $S$ is called an *anti fuzzy generalized bi-ideal* of $S$ if

$$f(xuyvz) \leq \max\{f(x), f(y), f(z)\} \text{ for all } x, y, z, u, v \in S.$$

**Definition 3.5.** For any fuzzy subset $f$ of a ternary semigroup $S$ and $t \in [0,1]$, the set

$$L(f;t) = \{x \in S : f(x) \leq t\}$$

is called the *lower level cut* of $f$.

Next we characterize anti fuzzy ternary subsemigroups left ( right, lateral, bi) ideal of $S$ by its lower level cuts.

**Theorem 3.6.** *A fuzzy subset $f$ of a ternary semigroup $S$ is an anti fuzzy ternary subsemigroup of $S$ if and only if $\varnothing \neq L(f;t)$ is a ternary subsemigroup of $S$ for all $t \in [0,1]$.*

*Proof.* Let $f$ be an anti fuzzy ternary subsemigroup of $S$ and $x, y, z \in L(f;t)$. Then $f(x) \leq t$, $f(y) \leq t$ and $f(z) \leq t$. Since $f(xyz) \leq \max\{f(x), f(y), f(z)\}$, so $f(xyz) \leq \max\{t, t, t\} = t$. This implies that $f(xyz) \leq t$. Thus $xyz \in L(f;t)$ and so $L(f;t)$ is a ternary subsemigroup of $S$.

Conversely, assume that $\varnothing \neq L(f;t)$ is a ternary subsemigroup of $S$ for all $t \in [0,1]$. Suppose there exist $x, y, z \in S$ such that $f(xyz) > \max\{f(x), f(y), f(z)\}$. Choose $t \in [0,1]$ such that $f(xyz) > t \geq \max\{f(x), f(y), f(z)\}$. Then $x, y, z \in L(f;t)$ but $xyz \notin L(f;t)$, which is a contradiction. Thus

$$f(xyz) \leq \max\{f(x), f(y), f(z)\}.$$

Hence $f$ is an anti fuzzy ternary subsemigroup of $S$. □

**Theorem 3.7.** *A fuzzy subset $f$ of a ternary semigroup $S$ is an anti fuzzy generalized bi-ideal of $S$ if and only if $\varnothing \neq L(f;t)$ is a generalized bi-ideal of $S$ for all $t \in [0,1]$.*

*Proof.* Let $f$ be an anti fuzzy generalized bi-ideal of $S$ and $x, y, z \in L(f;t)$. Then $f(x) \leq t$, $f(y) \leq t$, $f(z) \leq t$. Hence

$$f(xuyvz) \leq \max\{f(x), f(y), f(z)\} \leq \{t, t, t\} = t.$$

This implies that $xuyvz \in L(f;t)$. Thus $L(f;t)$ is a generalized bi-ideal of $S$.

Conversely, assume that $\varnothing \neq L(f;t)$ is a generalized bi-ideal of $S$ for all $t \in [0,1]$. Suppose that there exist $x, y, z, u, v \in S$ such that

$$f(xuyvz) > \max\{f(x), f(y), f(z)\}.$$

Choose $t \in [0,1]$ such that $f(xuyvz) > t \geq \max\{f(x), f(y), f(z)\}$. Then $x, y, z \in L(f;t)$ but $xuyvz \notin L(f;t)$, which is a contradiction. Hence

$$f(xuyvz) \leq \max\{f(x), f(y), f(z)\}.$$

Therefore $f$ is an anti fuzzy generalized bi-ideal of $S$. □

**Theorem 3.8.** *A fuzzy subset $f$ of a ternary semigroup $S$ is an anti fuzzy bi-ideal of $S$ if and only if $\varnothing \neq L(f;t)$ is a bi-ideal of $S$ for all $t \in [0,1]$.*

*Proof.* Follows from Theorem 3.6 and Theorem 3.7. □

**Theorem 3.9.** *A fuzzy subset $f$ of a ternary semigroup $S$ is an anti fuzzy left (right, lateral) ideal of $S$ if and only if $\varnothing \neq L(f;t)$ is a left (right, lateral) ideal of $S$ for all $t \in [0,1]$.*

*Proof.* The proof is similar to the proof of Theorem 3.6. □

**Theorem 3.10.** *A non-empty subset $L$ of a ternary semigroup $S$ is a left (right, lateral) ideal of $S$ if and only if the fuzzy subset $f$ defined by*

$$f(x) = \begin{cases} t & \text{if } x \in S \setminus L, \\ r & \text{if } x \in L. \end{cases}$$

*is an anti fuzzy left (right, lateral) ideal of $S$, where $t, r \in [0,1]$, $t \geq r$.*

*Proof.* Suppose $L$ is a left ideal of $S$ and $x, y, z \in S$. Let $z \in L$. Then $xyz \in L$. Hence $f(xyz) = r = f(z)$. If $z \notin L$, then $f(z) = t \geq f(xyz)$. Hence $f$ is an anti fuzzy left ideal of $S$.

Conversely, assume that $f$ is an anti fuzzy left ideal of $S$. Let $z \in L$ and $x, y \in S$. Then $f(xyz) \leq f(z) = r$. This implies that $f(xyz) = r$, that is $xyz \in L$. Hence $L$ is a left ideal of $S$. □

The following theorems can be proved in a similar fashion.

**Theorem 3.11.** *A non-empty subset $A$ of a ternary semigroup $S$ is a ternary subsemigroup of $S$ if and only if the fuzzy subset $f$ of $S$ defined by*

$$f(x) = \begin{cases} t & \text{if } x \in S \setminus A, \\ r & \text{if } x \in A. \end{cases}$$

*is an anti fuzzy ternary subsemigroup of $S$, where $t, r \in [0,1]$, $t \geq r$.*

**Theorem 3.12.** *A non-empty subset $A$ of a ternary semigroup $S$ is a generalized bi-ideal (bi-ideal) of $S$ if and only if the fuzzy subset $f$ of $S$ defined by*

$$f(x) = \begin{cases} t & \text{if } x \in S \setminus A, \\ r & \text{if } x \in A. \end{cases}$$

*is an anti fuzzy generalized bi-ideal (bi-ideal) of $S$, where $t, r \in [0,1]$, $t \geq r$.*

**Remark 3.13.** From the above theorems we conclude that a non-empty subset $A$ of a ternary semigroup $S$ is a left (right, lateral, bi, generalized bi) ideal of $S$ if and only if the characteristic function of the complemet of $A$, that is $C_{A^c}$ is an anti fuzzy left (right, lateral, bi-, generalized bi) ideal of $S$.

**Lemma 3.14.** *Union of any family of anti fuzzy left (right, lateral) ideals of a ternary semigroup $S$ is an anti fuzzy left (right, lateral) ideal of $S$.*

*Proof.* Let $\{f_i\}_{i \in I}$ be a family of anti fuzzy left ideals of a ternary semigroup $S$. Then

$$\left(\bigvee_{i \in I} f_i\right)(xyz) = \bigvee_{i \in I} (f_i(xyz)) \leq \bigvee_{i \in I} (f_i(z)) = \left(\bigvee_{i \in I} f_i\right)(z).$$

Hence $\bigvee_{i \in I} f_i$ is an anti fuzzy left ideal of $S$. □

**Lemma 3.15.** *Intersection of any family of anti fuzzy left (right, lateral) ideals of a ternary semigroup $S$ is an anti fuzzy left (right, lateral) ideal of $S$.*

*Proof.* Let $\{f_i\}_{i \in I}$ be a family of anti fuzzy left ideals of a ternary semigroup $S$. Then

$$\left(\bigwedge_{i \in I} f_i\right)(xyz) = \bigwedge_{i \in I} (f_i(xyz)) \leq \bigwedge_{i \in I} (f_i(z)) = \left(\bigwedge_{i \in I} f_i\right)(z).$$

Hence $\bigwedge\limits_{i \in I} f_i$ is an anti fuzzy left ideal of $S$. □

**Definition 3.16.** Let $f, g, h$ be any three fuzzy subsets of a ternary semigroup $S$. Then their product $f * g * h$ is defined by

$$(f * g * h)(a) = \begin{cases} \bigwedge_{a=pqr} (f(p) \vee g(q) \vee h(r)) & \text{if there exist } p, q, r \in S \\ & \text{such that } a = pqr, \\ 1 & \text{otherwise,} \end{cases}$$

for all $a \in S$.

**Theorem 3.17.** *The product of three anti fuzzy left (right, lateral) ideals of a ternary semigroup $S$ is again an anti fuzzy left (right, lateral) ideal of $S$.*

*Proof.* Let $f, g, h$ be three anti fuzzy left ideals of a ternary semigroup $S$. Then

$$(f * g * h)(z) = \bigwedge_{z=pqr} \{f(p) \vee g(q) \vee h(r)\}$$

If $z = pqr$, then $xy(pqr) = (xyp)qr$, since $f$ is an anti fuzzy left ideal of $S$. Hence $f(xyp) \leq f(p)$, and thus

$$\begin{aligned} (f * g * h)(z) &= \bigwedge_{z=pqr} \{f(p) \vee g(q) \vee h(r)\} \\ &\geq \bigwedge_{z=pqr} \{f(xyp) \vee g(q) \vee h(r)\} \\ &= \bigwedge_{xyz=abc} \{f(a) \vee g(b) \vee h(c)\} \\ &= (f * g * h)(xyz). \end{aligned}$$

Hence $(f * g * h)(z) \geq (f * g * h)(xyz)$. If $z$ is not expressible as $z = pqr$, then $(f * g * h)(z) = 1 \geq (f * g * h)(xyz)$. Hence $f * g * h$ is an anti fuzzy left ideal of $S$. □

**Theorem 3.18.** *Let $f$ be an anti fuzzy right ideal, $g$ an anti fuzzy lateral ideal and $h$ an anti fuzzy left ideal of a ternary semigroup $S$. Then*

$$f * g * h \geq f \vee g \vee h.$$

*Proof.* Let $f$ be an anti fuzzy right ideal of $S$, $g$ an anti fuzzy lateral ideal and $h$ an anti fuzzy left ideal of $S$. If $x$ is not expressible as $x = abc$, then

$$(f * g * h)(x) = 1 \geq (f \vee g \vee h)(x).$$

If $x$ is expressible as $x = abc$, then

$$\begin{aligned} (f * g * h)(x) &= \bigwedge_{x=abc} \{f(a) \vee g(b) \vee h(c)\} \\ &\geq \bigwedge_{x=abc} \{f(abc) \vee g(abc) \vee h(abc)\} \\ &= \bigwedge_{x=abc} \{f(x) \vee g(x) \vee h(x)\} \\ &= (f \vee g \vee h)(x). \end{aligned}$$

Thus $f * g * h \geq f \vee g \vee h$. □

**Theorem 3.19.** *Let $f$ be an anti fuzzy right ideal and $g$ an anti fuzzy left ideal of a ternary semigroup $S$. Then $f * \mathcal{O} * g \geq f \vee g$.*

Where $\mathcal{O}$ is the fuzzy subset of $S$ mapping every element of $S$ on 0.

*Proof.* Let $f$ be an anti fuzzy right ideal and $g$ an anti fuzzy left ideal of $S$. Let $a \in S$. If $a$ is not expressible as $a = xyz$ for some $x, y, z \in S$, then

$$(f * \mathcal{O} * g)(a) = 1 \geq (f \vee g)(a).$$

If $a$ is expressible as $a = xyz$ for some $x, y, z \in S$, then

$$\begin{aligned}(f * \mathcal{O} * g)(a) &= \bigwedge_{a=xyz} \{f(x) \vee \mathcal{O}(y) \vee g(z)\} \\ &\geq \bigwedge_{a=xyz} \{f(xyz) \vee g(xyz)\} \\ &= \bigwedge_{a=xyz} (f(a) \vee g(a)) \\ &= (f \vee g)(a).\end{aligned}$$

Thus $f * \mathcal{O} * g \geq f \vee g$. □

**Theorem 3.20.** *A fuzzy subset $f$ of a ternary semigroup $S$ is an anti fuzzy ternary subsemigroup of $S$ if and only if*

$$f * f * f \geq f.$$

*Proof.* Suppose $f$ is an anti fuzzy ternary subsemigroup of $S$ and $x \in S$. Then

$$\begin{aligned}(f * f * f)(x) &= \bigwedge_{x=abc} \{f(a) \vee f(b) \vee f(c)\} \\ &\geq \bigwedge_{x=abc} f(abc) = f(x).\end{aligned}$$

If $x$ is not expressible as $x = abc$ for all $a, b, c \in S$, then $(f * f * f)(x) = 1 \geq f(x)$. Hence $f * f * f \geq f$.

Conversely, assume that $f * f * f \geq f$. Then for all $x, y, z \in S$, we have

$$\begin{aligned}f(xyz) &\leq (f * f * f)(xyz) \\ &= \bigwedge_{xyz=abc} \{f(a) \vee f(b) \vee f(c)\} \\ &\leq f(x) \vee f(y) \vee f(z).\end{aligned}$$

Hence $f$ is an anti fuzzy ternary subsemigroup of $S$. □

**Theorem 3.21.** *A fuzzy subset $f$ of a ternary semigroup $S$ is an anti fuzzy left (right, lateral) ideal of $S$ if and only if $\mathcal{O} * \mathcal{O} * f \geq f$ ( $f * \mathcal{O} * \mathcal{O} \geq f$, $\mathcal{O} * f * \mathcal{O} \geq f$ ).*

Where $\mathcal{O}$ is the fuzzy subset of $S$ mapping every element of $S$ on 0.

*Proof.* Suppose $f$ is an anti fuzzy left ideal of $S$ and $x \in S$. Then

$$\begin{aligned}(\mathcal{O} * \mathcal{O} * f)(x) &= \bigwedge_{x=abc} \{\mathcal{O}(a) \vee \mathcal{O}(b) \vee f(c)\} = \bigwedge_{x=abc} f(c) \\ &\geq \bigwedge_{x=abc} f(abc) = f(x).\end{aligned}$$

If $x$ is not expressible as $x = abc$ for all $a, b, c \in S$, then

$$(\mathcal{O} * \mathcal{O} * f)(x) = 1 \geq f(x).$$

Hence $\mathcal{O} * \mathcal{O} * f \geq f$.

Conversely, assume that $\mathcal{O} * \mathcal{O} * f \geq f$. Let $x, y, z \in S$. Then

$$\begin{aligned}f(xyz) &\leq (\mathcal{O} * \mathcal{O} * f)(xyz) \\ &= \bigwedge_{xyz=abc} \{\mathcal{O}(a) \vee \mathcal{O}(b) \vee f(c)\} \\ &= \bigwedge_{xyz=abc} f(c) \leq f(z).\end{aligned}$$

This completes the proof. □

**Definition 3.22.** A fuzzy subset $f$ of a ternary semigroup $S$ is called an anti fuzzy quasi-ideal of $S$ if it satisfies

(i) $f(x) \leq \max\{(f * \mathcal{O} * \mathcal{O})(x),\ (\mathcal{O} * f * \mathcal{O})(x),\ (\mathcal{O} * \mathcal{O} * f)(x)\}$,
(ii) $f(x) \leq \max\{(f * \mathcal{O} * \mathcal{O})(x),\ (\mathcal{O} * \mathcal{O} * f * \mathcal{O} * \mathcal{O})(x),\ (\mathcal{O} * \mathcal{O} * f)(x)\}$.

**Theorem 3.23.** *A fuzzy subset $f$ of a ternary semigroup $S$ is an anti fuzzy quasi-ideal of $S$ if and only if $\varnothing \neq L(f;t)$ is a quasi-ideal of $S$ for all $t \in [0,1]$.*

*Proof.* Suppose $f$ is an anti fuzzy quasi-ideal of $S$ and $a \in SSL(f;t) \cap SL(f;t)S \cap L(f;t)SS$. Then there exist $x, y, z \in L(f;t)$ and $r, s, p, q, u, v \in S$ such that $a = pqx = rys = zuv$. Thus $f(x) \leq t$, $f(y) \leq t$, $f(z) \leq t$.

$$\begin{aligned}(\mathcal{O} * \mathcal{O} * f)(a) &= \bigwedge_{a=cde} \{\mathcal{O}(c) \vee \mathcal{O}(d) \vee f(e)\} \\ &\leq \mathcal{O}(p) \vee \mathcal{O}(q) \vee f(x) = f(x) \leq t,\end{aligned}$$

$$\begin{aligned}(f * \mathcal{O} * \mathcal{O})(a) &= \bigwedge_{a=lmn} \{f(l) \vee \mathcal{O}(m) \vee \mathcal{O}(n)\} \\ &\leq \{f(z) \vee \mathcal{O}(u) \vee \mathcal{O}(v)\} = f(z) \leq t\end{aligned}$$

and

$$\begin{aligned}(\mathcal{O} * f * \mathcal{O})(a) &= \bigwedge_{a=ijk} \{\mathcal{O}(i) \vee f(j) \vee \mathcal{O}(k)\} \\ &\leq \{\mathcal{O}(r) \vee f(y) \vee \mathcal{O}(s)\} = f(y) \leq t.\end{aligned}$$

Hence $f(a) \leq \max\{(f * \mathcal{O} * \mathcal{O})(x),\ (\mathcal{O} * f * \mathcal{O})(x),\ (\mathcal{O} * \mathcal{O} * f)(x)\} \leq t$. Thus $a \in L(f;t)$, so $SSL(f;t) \cap SL(f;t)S \cap L(f;t)SS \subseteq L(f;t)$.

Now, let $a \in (SSL(f;t)) \cap (SSL(f;t)SS) \cap (L(f;t)SS)$. Then there exist $x, y, z \in L(f;t)$ and $p, q, r, s, t, u, v, w \in S$ such that $a = pqx = zuv = rsylw$. Now,

$$\begin{aligned}(\mathcal{O} * \mathcal{O} * f * \mathcal{O} * \mathcal{O})(a) &= \bigwedge_{a=bcdem} \{\mathcal{O}(b) \vee \mathcal{O}(c) \vee f(d) \vee \mathcal{O}(e) \vee \mathcal{O}(m)\} \\ &\leq \{\mathcal{O}(r) \vee \mathcal{O}(s) \vee f(y) \vee \mathcal{O}(l) \vee \mathcal{O}(w)\} = f(y) \leq t\end{aligned}$$

Hence$f(a) \leq \max\{(f * \mathcal{O} * \mathcal{O})(a),\ (\mathcal{O} * \mathcal{O} * f * \mathcal{O} * \mathcal{O})(a),\ (\mathcal{O} * \mathcal{O} * f)(a)\} \leq t$. Thus $a \in L(f;t)$. Hence $(SSL(f;t)) \cap (SSL(f;t)SS) \cap (L(f;t)SS) \subseteq L(f;t)$. Therefore $L(f;t)$ is a quasi-ideal of $S$.

Conversely, assume that $\varnothing \neq L(f;t)$ is a quasi-ideal of $S$ for all $t \in [0,1]$. Let $a \in S$ be such that $\max\{(f * \mathcal{O} * \mathcal{O})(a),\ (\mathcal{O} * f * \mathcal{O})(a),\ (\mathcal{O} * \mathcal{O} * f)(a)\} < f(a)$. Choose $t \in [0,1]$ such that

$$\max\{(f * \mathcal{O} * \mathcal{O})(a),\ (\mathcal{O} * f * \mathcal{O})(a),\ (\mathcal{O} * \mathcal{O} * f)(a)\} \leq t < f(a).$$

Then $(f * \mathcal{O} * \mathcal{O})(a) \leq t$ implies $a \in L(f;t)SS$, $(\mathcal{O} * f * \mathcal{O})(a) \leq t$ implies $a \in SL(f;t)S$ and $(\mathcal{O} * \mathcal{O} * f)(a) \leq t$ implies $a \in SSL(f;t)$. This implies that $a \in (SSL(f;t) \cap SL(f;t)S \cap L(f;t)SS \subseteq L(f;t)$. This implies that $f(a) \leq t$. Which is a contradiction. Hence $\max\{(f * \mathcal{O} * \mathcal{O})(a),\ (\mathcal{O} * f * \mathcal{O})(a),\ (\mathcal{O} * \mathcal{O} * f)(a)\} \geq f(a)$. Thus $f$ is an anti fuzzy quasi-ideal of $S$. □

**Theorem 3.24.** *A non-empty subset $Q$ of a ternary semigroup $S$ is a quasi-ideal of $S$ if and only if the fuzzy subset $f$ defined by*

$$f(x) = \begin{cases} t & \text{if } x \in S \setminus Q, \\ r & \text{if } x \in Q. \end{cases}$$

*is an anti fuzzy quasi-ideal of $S$, where $r, t \in [0,1]$, such that $t \geq r$.*

*Proof.* Similar to the proof of Theorem 3.10. □

**Remark 3.25.** From the above theorems we conclude that a non-empty subset $Q$ of a ternary semigroup $S$ is a quasi-ideal of $S$ if and only if the characteristic function of the complement of $Q$, that is, $C_{Q^c}$ is an anti fuzzy quasi-ideal of $S$.

**Theorem 3.26.** *Every anti fuzzy left (right, lateral) ideal of a ternary semigroup $S$ is an anti fuzzy quasi-ideal of $S$.*

*Proof.* Let $f$ be an anti fuzzy left ideal of $S$. Then

$$\begin{aligned}&\max\{(f * \mathcal{O} * \mathcal{O})(x),\ (\mathcal{O} * f * \mathcal{O})(x),\ (\mathcal{O} * \mathcal{O} * f)(x)\} \\ &\geq (\mathcal{O} * \mathcal{O} * f)(x) \geq f(x).\end{aligned}$$

Similarly

$$\begin{aligned}&\max\{(f * \mathcal{O} * \mathcal{O})(x),\ (\mathcal{O} * \mathcal{O} * f * \mathcal{O} * \mathcal{O})(x),\ (\mathcal{O} * \mathcal{O} * f)(x)\} \\ &\geq (\mathcal{O} * \mathcal{O} * f)(x) \geq f(x).\end{aligned}$$

Thus $f$ is an anti fuzzy quasi-ideal of $S$. □

**Theorem 3.27.** *Every anti fuzzy quasi-ideal of a ternary semigroup $S$ is an anti fuzzy bi-ideal of $S$.*

*Proof.* Suppose $f$ is an anti fuzzy quasi-ideal of $S$. Then

$$f(xyz) \leq \{(f*\mathcal{O}*\mathcal{O})(xyz) \vee (\mathcal{O}*f*\mathcal{O})(xyz) \vee (\mathcal{O}*\mathcal{O}*f)(xyz)\}$$

$$= \left[\bigwedge_{xyz=abc}\{f(a) \vee \mathcal{O}(b) \vee \mathcal{O}(c)\}\right] \vee \left[\bigwedge_{xyz=pqr}\{\mathcal{O}(p) \vee f(q) \vee \mathcal{O}(r)\}\right]$$

$$\vee \left[\bigwedge_{xyz=lmn}\{\mathcal{O}(l) \vee \mathcal{O}(m) \vee f(n)\}\right]$$

$$\leq \{f(x) \vee \mathcal{O}(y) \vee \mathcal{O}(z)\} \vee \{\mathcal{O}(x) \vee f(y) \vee \mathcal{O}(z)\} \vee \{\mathcal{O}(x) \vee \mathcal{O}(y) \vee f(z)\}$$

$$= f(x) \vee f(y) \vee f(z).$$

This implies that $f(xyz) \leq \max\{f(x), f(y), f(z)\}$. Also,

$$f(xuyvz) \leq \{(f*\mathcal{O}*\mathcal{O})(xuyvz) \vee (\mathcal{O}*\mathcal{O}*f*\mathcal{O}*\mathcal{O})(xuyvz) \vee (\mathcal{O}*\mathcal{O}*f)(xuyvz)\}$$

$$= \left[\bigwedge_{xuyvz=abcde}\{f(a) \vee \mathcal{O}(bcd) \vee \mathcal{O}(e)\}\right]$$

$$\vee \left[\bigwedge_{xuyvz=pqrst}\{\mathcal{O}(p) \vee \mathcal{O}(q) \vee f(r) \vee \mathcal{O}(s) \vee \mathcal{O}(t)\}\right]$$

$$\vee \left[\bigwedge_{xuyvz=ijklm}\{\mathcal{O}(ijk) \vee \mathcal{O}(l) \vee f(m)\}\right]$$

$$\leq \{f(x) \vee \mathcal{O}(uyv) \vee \mathcal{O}(z)\} \vee \{\mathcal{O}(x) \vee \mathcal{O}(u) \vee f(y) \vee \mathcal{O}(v) \vee \mathcal{O}(z)\}$$

$$\vee \{\mathcal{O}(x) \vee \mathcal{O}(uyv) \vee f(z)\}$$

$$= f(x) \vee f(y) \vee f(z).$$

This implies that $f(xuyvz) \leq \max\{f(x), f(y), f(z)\}$. Thus $f$ is an anti fuzzy bi-ideal of $S$. □

**Corollary 3.28.** *Every anti fuzzy left (right, lateral) ideal of a ternary semigroup $S$ is an anti fuzzy bi-ideal of $S$.*

*Proof.* Follows from Theorem 3.26 and Theorem 3.27. □

## 4. Regular Ternary Semigroups

In this section we are characterizing the regular ternary semigroups by the properties of their anti fuzzy ideals, anti fuzzy quasi-ideals, anti fuzzy bi-ideals and anti fuzzy generalized bi-ideals.

**Theorem 4.1.** *For a ternary semigroup $S$, the following conditions are equivalent.*

(1) *$S$ is regular.*
(2) *$(f \vee g \vee h) = f*g*h$ for every anti fuzzy right ideal $f$ and every anti fuzzy lateral ideal $g$ and every anti fuzzy left ideal $h$ of $S$.*

*Proof.* (1) ⇒ (2) Let $f$ be an anti fuzzy right ideal, $g$ an anti fuzzy lateral ideal and $h$ an anti fuzzy left ideal of $S$. Then by Theorem 3.18 $f * g * h \geq f \vee g \vee h$. Let $a \in S$. Then there exists $x \in S$ such that $a = axa = a\,(xax)\,a$. Thus we have

$$\begin{aligned}(f * g * h) &= \bigwedge_{a=pqr} \{f\,(p) \vee g\,(q) \vee h\,(r)\} \\ &\leq \{f\,(a) \vee g\,(xax) \vee h\,(a)\} \\ &\leq \{f\,(a) \vee g\,(a) \vee h\,(a)\} \\ &= (f \vee g \vee h)\,(a)\,.\end{aligned}$$

Thus $f * g * h \leq f \vee g \vee h$. Hence $f * g * h = f \vee g \vee h$.

(2) ⇒ (1) In order to show that $S$ is regular, we shall show that $R \cap M \cap L = RML$ for every right ideal $R$, every lateral ideal $M$ and every left ideal $L$ of $S$. Since $RML \subseteq R \cap M \cap L$ always holds. We only show that $R \cap M \cap L \subseteq RML$. Suppose on contrary that there exist $a \in R \cap M \cap L$ such that $a \notin RML$. Then by Remark 3.13, the characteristic functions $C_{R^c}$, $C_{M^c}$ and $C_{L^c}$ are anti fuzzy right ideal, anti fuzzy lateral ideal and anti fuzzy left ideal of $S$, respectively. Since $a \in R \cap M \cap L$. This implies that $a \in R$, $a \in M$ and $a \in L$. Thus $C_{R^c}\,(a) = C_{M^c}\,(a) = C_{L^c}\,(a) = 0$, also since $a \notin RML$. This implies that there does not exist $x, y, z \in S$ such that $a = xyz$. Thus $(C_{R^c} * C_{M^c} * C_{L^c})\,(a) = 1$. Which is a contradiction to the hypothesis. Thus $R \cap M \cap L \subseteq RML$. Hence $R \cap M \cap L = RML$. Thus by Theorem 2.1 $S$ is regular. □

**Theorem 4.2.** *For a ternary semigroup $S$, the following conditions are equivalent.*

(1) *$S$ is regular.*
(2) *$f = f * \mathcal{O} * f * \mathcal{O} * f$ for every anti fuzzy generalized bi-ideal $f$ of $S$.*
(3) *$f = f * \mathcal{O} * f * \mathcal{O} * f$ for every anti fuzzy bi-ideal $f$ of $S$.*
(4) *$f = f * \mathcal{O} * f * \mathcal{O} * f$ for every anti fuzzy quasi-ideal $f$ of $S$.*

*Proof.* (1) ⇒ (2) Let $f$ be an anti fuzzy generalized bi-ideal of $S$ and $a \in S$. Since $S$ is regular, so there exist $x \in S$ such that $a = axa = axaxa$

$$\begin{aligned}(f * \mathcal{O} * f * \mathcal{O} * f)\,(a) &= \bigwedge_{a=pqrst} \{f\,(p) \vee \mathcal{O}\,(q) \vee f\,(r) \vee \mathcal{O}\,(s) \vee f\,(t)\} \\ &\leq \{f\,(a) \vee \mathcal{O}\,(x) \vee f\,(a) \vee \mathcal{O}\,(x) \vee f\,(a)\} = f\,(a)\,.\end{aligned}$$

Thus $f * \mathcal{O} * f * \mathcal{O} * f \leq f$. Since $f$ is an anti fuzzy generalized bi-ideal of $S$, so $f * \mathcal{O} * f * \mathcal{O} * f \geq f$. Hence $f = f * \mathcal{O} * f * \mathcal{O} * f$.

It is clear that (2) ⇒ (3) ⇒ (4).

(4) ⇒ (1) Let $A$ be any quasi-ideal of $S$. Then

$$ASASA \subseteq (ASS) \cap (SSASS) \cap (SSA) \subseteq A$$

because $A$ is a quasi-ideal of $S$. By Remark 3.25, $C_{A^c}$ is an anti fuzzy quasi-ideal of $S$. Let $a \in A$. Then $C_{A^c}\,(a) = 0$. If $a \notin ASASA$, then there does not exist $p, q, r, s, t \in A$ such that $a = pqrst$. Thus $((C_{A^c} * \mathcal{O} * C_{A^c}) * \mathcal{O} * C_{A^c})\,(a) = 1$, which is a contradiction. So $ASASA \subseteq A$. Thus $ASASA = A$. By Theorem 2.2, $S$ is regular. □

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M. Shabir (mshabirbhatti@yahoo.co.uk) – Department of Mathematics Quaid-i-Azam University 45320 Islamabad 44000, Pakistan

N. Rehman (noorrehman82@yahoo.com) – Department of Mathematics Quaid-i-Azam University 45320 Islamabad 44000, Pakistan