Characterizations of ternary semigroups by their anti fuzzy ideals

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Received 8 March 2011; Revised 2 April 2011; Accepted 5 April 2011

ABSTRACT. In this paper we define anti fuzzy left (right, lateral) ideals, anti fuzzy quasi-ideals, anti fuzzy bi-ideals and anti fuzzy generalized bi-ideals in ternary semigroups. We also characterize different classes of ternary semigroups by the properties of these ideals.

2010 AMS Classification: 03E72, 20N10

Keywords: Ternary semigroup, Anti fuzzy ideal.

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1. Introduction


In this paper we introduce anti fuzzy left (right, lateral) ideals, anti fuzzy quasi-ideals, anti fuzzy bi-ideals and anti fuzzy generalized bi-ideals in ternary semigroups and discuss some basic properties of such ideals.
2. Preliminaries

A ternary semigroup is an algebraic structure \((S, f)\) such that \(S\) is a non-empty set and \(f : S^3 \to S\) is a ternary operation satisfying the following associative law: 
f(f(a, b, c), d, e) = f(a, f(b, c, d), e) = f(a, b, f(c, d, e))\) for all \(a, b, c, d, e \in S\). \(c.f [7]\).

A non-empty subset \(A\) of a ternary semigroup \(S\) is called a ternary subsemigroup of \(S\) if \(AAA \subseteq A\). By a left (right, lateral) ideal of a ternary semigroup \(S\) we mean a non-empty subset \(A\) of \(S\) such that \(SSA \subseteq A\) (\(ASS \subseteq A\), \(SAS \subseteq A\)). If a non-empty subset \(A\) of \(S\) is a left and right ideal of \(S\), then it is called a two sided ideal of \(S\).

If a non-empty subset \(A\) of a ternary semigroup \(S\) is a left, right and lateral ideal of \(S\), then it is called an ideal of \(S\). A non-empty subset \(A\) of a ternary semigroup \(S\) is called a quasi-ideal of \(S\) if \(ASS \cap SAS \cap SSA \subseteq A\) and \(ASS \cap SSA \cap SAS \subseteq A\). \(A\) is called a bi-ideal of \(S\) if it is a ternary subsemigroup of \(S\) and \(ASASA \subseteq A\). A non-empty subset \(A\) of a ternary semigroup \(S\) is called a generalized bi-ideal of \(S\) if \(ASASA \subseteq A\). It is clear that every left (right, lateral) ideal of \(S\) is a quasi-ideal, every quasi-ideal is a bi-ideal and every bi-ideal is a generalized bi-ideal. An element \(a\) of a ternary semigroup \(S\) is called regular if there exists an element \(x \in S\) such that \(axa = a\). A ternary semigroup \(S\) is called regular if every element of \(S\) is regular. It is obvious that for a regular ternary semigroup the concept of quasi-ideal, bi-ideal and generalized bi-ideal coincide.

A fuzzy subset \(f\) of a universe \(X\) is a function from \(X\) into the unit closed interval \([0, 1]\), that is, \(f : X \to [0, 1]\). For any two fuzzy subsets \(f\) and \(g\), \(f \leq g\) means \(f(x) \leq g(x)\) for all \(x \in S\). The symbols \(f \wedge g\) and \(f \vee g\) have the following meanings 

\[(f \wedge g)(x) = f(x) \wedge g(x)\] and \[(f \vee g)(x) = f(x) \vee g(x)\] for all \(x \in S\).

Let \(f, g\) and \(h\) be three fuzzy subsets of a ternary semigroup \(S\). The product \(f \circ g \circ h\) is a fuzzy subset of \(S\) defined by:

\[(f \circ g \circ h)(a) = \begin{cases} \bigvee_{a = xyz} (f(x) \wedge g(y) \wedge h(z)) & \text{if there exist } x, y, z \in S \\ 0 & \text{otherwise,} \end{cases}\]

A fuzzy subset \(f\) of a ternary semigroup \(S\) is called a fuzzy ternary subsemigroup of \(S\) if \(f(xyz) \geq \min\{f(x), f(y), f(z)\}\) and is called fuzzy left (right, lateral) ideal of \(S\) if \(f(xyz) \geq f(z) f(xy) \geq f(x), f(xy) \geq f(y)\) for all \(x, y, z \in S\). A fuzzy subset \(f\) of a ternary semigroup \(S\) is called a fuzzy ideal of \(S\) if it is a fuzzy left ideal, fuzzy right ideal and fuzzy lateral ideal of \(S\). A fuzzy subset \(f\) of a ternary semigroup \(S\) is called a fuzzy quasi-ideal of \(S\) if

\[(f \circ S \circ S) \wedge (S \circ f \circ S) \wedge (S \circ S \circ f) \leq f,\]

where \(S\) is the fuzzy subset of \(S\) mapping every element of \(S\) on 1.

A fuzzy subset \(f\) of a ternary semigroup \(S\) is called a fuzzy bi-ideal of \(S\) if it is a fuzzy ternary subsemigroup of \(S\) and \(f(xuvyz) \geq \min\{f(x), f(y), f(z)\}\). A fuzzy subset \(f\) of a ternary semigroup \(S\) is called a fuzzy generalized bi-ideal of \(S\) if \(f(xuvyz) \geq \min\{f(x), f(y), f(z)\}\) for all \(x, y, z, u, v \in S\).
It is obvious that every fuzzy left (right, lateral) ideal of a ternary semigroup $S$ is a fuzzy quasi-ideal and every fuzzy quasi-ideal of $S$ is a fuzzy bi-ideal and every fuzzy bi-ideal of $S$ is a fuzzy generalized bi-ideal of $S$, but the converse need not be true.

Theorem 2.1 ([12]). The following conditions on a ternary semigroup $S$ are equivalent.

1. $S$ is regular.
2. $R \cap M \cap L = RML$ for every right ideal $R$, every lateral ideal $M$ and every left ideal $L$ of $S$.

Theorem 2.2 ([11]). The following conditions on a ternary semigroup $S$ are equivalent.

1. $S$ is regular.
2. $BSBSB = B$ for every bi-ideal $B$ of $S$.
3. $QSQSQ = Q$ for every quasi-ideal $Q$ of $S$.

3. Major Section

In this section we define anti fuzzy left (right, lateral) ideal, anti fuzzy quasi-ideal, anti fuzzy bi-ideal and anti fuzzy generalized bi-ideal in ternary semigroups and study some basic properties of these ideals.

Definition 3.1. A fuzzy subset $f$ of a ternary semigroup $S$ is called an anti fuzzy ternary subsemigroup of $S$ if

$$f(xyz) \leq \max\{f(x), f(y), f(z)\}$$

for all $x, y, z \in S$.

Definition 3.2. A fuzzy subset $f$ of a ternary semigroup $S$ is called an anti fuzzy left (right, lateral) ideal of $S$ if

$$f(xyz) \leq f(z) \quad (f(xyz) \leq f(x), f(xyz) \leq f(y))$$

for all $x, y, z \in S$.

A fuzzy subset $f$ of a ternary semigroup $S$ is called an anti fuzzy ideal of $S$ if it is an anti fuzzy left ideal, anti fuzzy right ideal and anti fuzzy lateral ideal of $S$.

Definition 3.3. An anti fuzzy ternary subsemigroup $f$ of a ternary semigroup $S$ is called an anti fuzzy bi-ideal of $S$ if it satisfies

$$f(xuyvz) \leq \max\{f(x), f(y), f(z)\}$$

for all $x, y, z, u, v \in S$.

Definition 3.4. A fuzzy subset $f$ of a ternary semigroup $S$ is called an anti fuzzy generalized bi-ideal of $S$ if

$$f(xuyvz) \leq \max\{f(x), f(y), f(z)\}$$

for all $x, y, z, u, v \in S$.

Definition 3.5. For any fuzzy subset $f$ of a ternary semigroup $S$ and $t \in [0, 1]$, the set

$$L(f; t) = \{x \in S : f(x) \leq t\}$$

is called the lower level cut of $f$.

Next we characterize anti fuzzy ternary subsemigroups left (right, lateral, bi) ideal of $S$ by its lower level cuts.
Theorem 3.6. A fuzzy subset $f$ of a ternary semigroup $S$ is an anti fuzzy ternary subsemigroup of $S$ if and only if $\emptyset \neq L(f; t)$ is a ternary subsemigroup of $S$ for all $t \in [0, 1]$.

Proof. Let $f$ be an anti fuzzy ternary subsemigroup of $S$ and $x, y, z \in L(f; t)$. Then $f(x) \leq t$, $f(y) \leq t$, and $f(z) \leq t$. Since $f(xyz) \leq \max\{f(x), f(y), f(z)\}$, so $f(xyz) \leq \max\{t, t, t\} = t$. This implies that $f(xyz) \leq t$. Thus $xyz \in L(f; t)$ and so $L(f; t)$ is a ternary subsemigroup of $S$.

Conversely, assume that $\emptyset \neq L(f; t)$ is a ternary subsemigroup of $S$ for all $t \in [0, 1]$. Suppose there exist $x, y, z \in S$ such that $f(xyz) > \max\{f(x), f(y), f(z)\}$. Choose $t \in [0, 1]$ such that $f(xyz) > t \geq \max\{f(x), f(y), f(z)\}$. Then $x, y, z \in L(f; t)$ but $xyz \notin L(f; t)$, which is a contradiction. Thus

$$f(xyz) \leq \max\{f(x), f(y), f(z)\}.$$ 

Hence $f$ is an anti fuzzy ternary subsemigroup of $S$. \hfill \Box

Theorem 3.7. A fuzzy subset $f$ of a ternary semigroup $S$ is an anti fuzzy generalized bi-ideal of $S$ if and only if $\emptyset \neq L(f; t)$ is a generalized bi-ideal of $S$ for all $t \in [0, 1]$.

Proof. Let $f$ be an anti fuzzy generalized bi-ideal of $S$ and $x, y, z \in L(f; t)$. Then $f(x) \leq t$, $f(y) \leq t$, and $f(z) \leq t$. Hence

$$f(xuyvz) \leq \max\{f(x), f(y), f(z)\} \leq \{t, t, t\} = t.$$ 

This implies that $xuyvz \in L(f; t)$. Thus $L(f; t)$ is a generalized bi-ideal of $S$.

Conversely, assume that $\emptyset \neq L(f; t)$ is a generalized bi-ideal of $S$ for all $t \in [0, 1]$. Suppose that there exist $x, y, z, u, v \in S$ such that

$$f(xuyvz) > \max\{f(x), f(y), f(z)\}.$$ 

Choose $t \in [0, 1]$ such that $f(xuyvz) > t \geq \max\{f(x), f(y), f(z)\}$. Then $x, y, z \in L(f; t)$ but $xuyvz \notin L(f; t)$, which is a contradiction. Hence

$$f(xuyvz) \leq \max\{f(x), f(y), f(z)\}.$$ 

Therefore $f$ is an anti fuzzy generalized bi-ideal of $S$. \hfill \Box

Theorem 3.8. A fuzzy subset $f$ of a ternary semigroup $S$ is an anti fuzzy bi-ideal of $S$ if and only if $\emptyset \neq L(f; t)$ is a bi-ideal of $S$ for all $t \in [0, 1]$.

Proof. Follows from Theorem 3.6 and Theorem 3.7. \hfill \Box

Theorem 3.9. A fuzzy subset $f$ of a ternary semigroup $S$ is an anti fuzzy left (right, lateral) ideal of $S$ if and only if $\emptyset \neq L(f; t)$ is a left (right, lateral) ideal of $S$ for all $t \in [0, 1]$.

Proof. The proof is similar to the proof of Theorem 3.6. \hfill \Box

Theorem 3.10. A non-empty subset $L$ of a ternary semigroup $S$ is a left (right, lateral) ideal of $S$ if and only if the fuzzy subset $f$ defined by

$$f(x) = \begin{cases} 
 t & \text{if } x \in S \setminus L, \\
 r & \text{if } x \in L.
\end{cases}$$

is an anti fuzzy left (right, lateral) ideal of $S$, where $t, r \in [0, 1]$, $t \geq r$.  

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Proof. Suppose $L$ is a left ideal of $S$ and $x, y, z \in S$. Let $z \in L$. Then $xyz \in L$. Hence $f(xyz) = r = f(z)$. If $z \notin L$, then $f(z) = t \geq f(xyz)$. Hence $f$ is an anti fuzzy left ideal of $S$.

 Conversely, assume that $f$ is an anti fuzzy left ideal of $S$. Let $z \in L$ and $x, y \in S$. Then $f(xyz) \leq f(z) = r$. This implies that $f(xyz) = r$, that is $xyz \in L$. Hence $L$ is a left ideal of $S$. □

The following theorems can be proved in a similar fashion.

**Theorem 3.11.** A non-empty subset $A$ of a ternary semigroup $S$ is a ternary subsemigroup of $S$ if and only if the fuzzy subset $f$ of $S$ defined by

$$f(x) = \begin{cases} t & \text{if } x \in S \setminus A, \\ r & \text{if } x \in A. \end{cases}$$

is an anti fuzzy ternary subsemigroup of $S$, where $t, r \in [0, 1]$, $t \geq r$.

**Theorem 3.12.** A non-empty subset $A$ of a ternary semigroup $S$ is a generalized bi-ideal (bi-ideal) of $S$ if and only if the fuzzy subset $f$ of $S$ defined by

$$f(x) = \begin{cases} t & \text{if } x \in S \setminus A, \\ r & \text{if } x \in A. \end{cases}$$

is an anti fuzzy generalized bi-ideal (bi-ideal) of $S$, where $t, r \in [0, 1]$, $t \geq r$.

**Remark 3.13.** From the above theorems we conclude that a non-empty subset $A$ of a ternary semigroup $S$ is a left (right, lateral, bi, generalized bi) ideal of $S$ if and only if the characteristic function of the complement of $A$, that is $C_A$ is an anti fuzzy left (right, lateral, bi-, generalized bi) ideal of $S$.

**Lemma 3.14.** Union of any family of anti fuzzy left (right, lateral) ideals of a ternary semigroup $S$ is an anti fuzzy left (right, lateral) ideal of $S$.

**Proof.** Let $\{f_i\}_{i \in I}$ be a family of anti fuzzy left ideals of a ternary semigroup $S$. Then

$$\left( \bigvee_{i \in I} f_i \right) (xyz) = \bigvee_{i \in I} (f_i(xyz)) \leq \bigvee_{i \in I} (f_i(z)) = \left( \bigvee_{i \in I} f_i \right) (z).$$

Hence $\bigvee_{i \in I} f_i$ is an anti fuzzy left ideal of $S$. □

**Lemma 3.15.** Intersection of any family of anti fuzzy left (right, lateral) ideals of a ternary semigroup $S$ is an anti fuzzy left (right, lateral) ideal of $S$.

**Proof.** Let $\{f_i\}_{i \in I}$ be a family of anti fuzzy left ideals of a ternary semigroup $S$. Then

$$\left( \bigwedge_{i \in I} f_i \right) (xyz) = \bigwedge_{i \in I} (f_i(xyz)) \leq \bigwedge_{i \in I} (f_i(z)) = \left( \bigwedge_{i \in I} f_i \right) (z).$$

Hence $\bigwedge_{i \in I} f_i$ is an anti fuzzy left ideal of $S$. □
Definition 3.16. Let \( f, g, h \) be any three fuzzy subsets of a ternary semigroup \( S \). Then their product \( f \ast g \ast h \) is defined by

\[
(f \ast g \ast h)(a) = \begin{cases} 
\bigwedge_{a=pqr} (f(p) \lor g(q) \lor h(r)) & \text{if there exist } p, q, r \in S \text{ such that } a = pqr, \\
1 & \text{otherwise,}
\end{cases}
\]

for all \( a \in S \).

Theorem 3.17. The product of three anti fuzzy left (right, lateral) ideals of a ternary semigroup \( S \) is again an anti fuzzy left (right, lateral) ideal of \( S \).

Proof. Let \( f, g, h \) be three anti fuzzy left ideals of a ternary semigroup \( S \). Then

\[
(f \ast g \ast h)(z) = \bigwedge_{z=pqr} \{f(p) \lor g(q) \lor h(r)\}
\]

If \( z = pqr \), then \( xy(pqr) = (xyp)qr \), since \( f \) is an anti fuzzy left ideal of \( S \). Hence \( f(xyp) \leq f(p) \), and thus

\[
(f \ast g \ast h)(z) = \bigwedge_{z=pqr} \{f(p) \lor g(q) \lor h(r)\} \\
\geq \bigwedge_{z=pqr} \{f(xyp) \lor g(q) \lor h(r)\} \\
= \bigwedge_{xyz=abc} \{f(a) \lor g(b) \lor h(c)\} \\
= (f \ast g \ast h)(xyz).
\]

Hence \( (f \ast g \ast h)(z) \geq (f \ast g \ast h)(xyz) \). If \( z \) is not expressible as \( z = pqr \), then \( (f \ast g \ast h)(z) = 1 \geq (f \ast g \ast h)(xyz) \). Hence \( f \ast g \ast h \) is an anti fuzzy left ideal of \( S \). \( \square \)

Theorem 3.18. Let \( f \) be an anti fuzzy right ideal, \( g \) an anti fuzzy lateral ideal and \( h \) an anti fuzzy left ideal of a ternary semigroup \( S \). Then

\[
f \ast g \ast h \geq f \lor g \lor h.
\]

Proof. Let \( f \) be an anti fuzzy right ideal of \( S \), \( g \) an anti fuzzy lateral ideal and \( h \) an anti fuzzy left ideal of \( S \). If \( x \) is not expressible as \( x = abc \), then

\[
(f \ast g \ast h)(x) = 1 \geq (f \lor g \lor h)(x).
\]

If \( x \) is expressible as \( x = abc \), then

\[
(f \ast g \ast h)(x) = \bigwedge_{x=abc} \{f(a) \lor g(b) \lor h(c)\} \\
\geq \bigwedge_{x=abc} \{f(abc) \lor g(abc) \lor h(abc)\} \\
= \bigwedge_{x=abc} \{f(x) \lor g(x) \lor h(x)\} \\
= (f \lor g \lor h)(x).
\]

Thus \( f \ast g \ast h \geq f \lor g \lor h \). \( \square \)
Theorem 3.19. Let \( f \) be an anti fuzzy right ideal and \( g \) an anti fuzzy left ideal of a ternary semigroup \( S \). Then \( f \ast O \ast g \geq f \lor g \).

Where \( O \) is the fuzzy subset of \( S \) mapping every element of \( S \) on 0.

Proof. Let \( f \) be an anti fuzzy right ideal and \( g \) an anti fuzzy left ideal of \( S \). Let \( a \in S \). If \( a \) is not expressible as \( a = xyz \) for some \( x, y, z \in S \), then
\[
(f \ast O \ast g)(a) = 1 \geq (f \lor g)(a).
\]
If \( a \) is expressible as \( a = xyz \) for some \( x, y, z \in S \), then
\[
(f \ast O \ast g)(a) = \bigwedge_{a=xyz} \{ f(x) \lor O(y) \lor g(z) \}
\]
\[
\geq \bigwedge_{a=xyz} \{ f(xyz) \lor g(xyz) \}
\]
\[
= \bigwedge_{a=xyz} (f(a) \lor g(a))
\]
\[
= (f \lor g)(a).
\]
Thus \( f \ast O \ast g \geq f \lor g \).

\( \square \)

Theorem 3.20. A fuzzy subset \( f \) of a ternary semigroup \( S \) is an anti fuzzy ternary subsemigroup of \( S \) if and only if
\[
f \ast f \ast f \geq f.
\]

Proof. Suppose \( f \) is an anti fuzzy ternary subsemigroup of \( S \) and \( x \in S \). Then
\[
(f \ast f \ast f)(x) = \bigwedge_{x=abc} \{ f(a) \lor f(b) \lor f(c) \}
\]
\[
\geq \bigwedge_{x=abc} f(abc) = f(x).
\]
If \( x \) is not expressible as \( x = abc \) for all \( a, b, c \in S \), then \( (f \ast f \ast f)(x) = 1 \geq f(x) \). Hence \( f \ast f \ast f \geq f \).
Conversely, assume that \( f \ast f \ast f \geq f \). Then for all \( x, y, z \in S \), we have
\[
f(xyz) \leq (f \ast f \ast f)(xyz)
\]
\[
= \bigwedge_{xyz=abc} \{ f(a) \lor f(b) \lor f(c) \}
\]
\[
\leq f(x) \lor f(y) \lor f(z).
\]
Hence \( f \) is an anti fuzzy ternary subsemigroup of \( S \).

\( \square \)

Theorem 3.21. A fuzzy subset \( f \) of a ternary semigroup \( S \) is an anti fuzzy left (right, lateral) ideal of \( S \) if and only if \( O \ast O \ast f \geq f \) (\( f \ast O \ast O \geq f \), \( O \ast f \ast O \geq f \)).

Where \( O \) is the fuzzy subset of \( S \) mapping every element of \( S \) on 0.
Proof. Suppose \( f \) is an anti fuzzy left ideal of \( S \) and \( x \in S \). Then
\[
(O \ast O \ast f)(x) = \bigwedge_{x=abc} \{O(a) \lor O(b) \lor f(c)\} = \bigwedge_{x=abc} f(c)
\]
\[
\geq \bigwedge_{x=abc} f(abc) = f(x).
\]
If \( x \) is not expressible as \( x = abc \) for all \( a, b, c \in S \), then
\[
(O \ast O \ast f)(x) = 1 \geq f(x).
\]
Hence \( O \ast O \ast f \geq f \).

Conversely, assume that \( O \ast O \ast f \geq f \). Let \( x, y, z \in S \). Then
\[
f(xyz) \leq (O \ast O \ast f)(xyz)
\]
\[
= \bigwedge_{xyz=abc} \{O(a) \lor O(b) \lor f(c)\}
\]
\[
\leq \bigwedge_{xyz=abc} f(c) \leq f(z).
\]
This completes the proof. \( \square \)

**Definition 3.22.** A fuzzy subset \( f \) of a ternary semigroup \( S \) is called an anti fuzzy quasi-ideal of \( S \) if it satisfies
\begin{enumerate}[(i)]  
  \item \( f(x) \leq \max\{ (f \ast O \ast O)(x), (O \ast f \ast O)(x), (O \ast O \ast f)(x) \} \),
  \item \( f(x) \leq \max\{ (f \ast O \ast O)(x), (O \ast O \ast f \ast O \ast O)(x), (O \ast O \ast f)(x) \} \).
\end{enumerate}

**Theorem 3.23.** A fuzzy subset \( f \) of a ternary semigroup \( S \) is an anti fuzzy quasi-ideal of \( S \) if and only if \( \emptyset \neq L(f; t) \) is a quasi-ideal of \( S \) for all \( t \in [0, 1] \).

Proof. Suppose \( f \) is an anti fuzzy quasi-ideal of \( S \) and \( a \in SSL(f; t) \cap SL(f; t) \cap L(f; t)SS \). Then there exist \( x, y, z \in L(f; t) \) and \( r, s, p, q, u, v \in S \) such that \( a = pqx = rys = zwv \). Thus \( f(x) \leq t, f(y) \leq t, f(z) \leq t \).
\[
(O \ast O \ast f)(a) = \bigwedge_{a=abc} \{O(c) \lor O(d) \lor f(e)\}
\]
\[
\leq O(p) \lor O(q) \lor f(x) = f(x) \leq t,
\]
\[
(f \ast O \ast O)(a) = \bigwedge_{a=abc} \{f(l) \lor O(m) \lor O(n)\}
\]
\[
\leq \{f(z) \lor O(u) \lor O(v)\} = f(z) \leq t
\]
and
\[
(O \ast f \ast O)(a) = \bigwedge_{a=ijk} \{O(i) \lor f(j) \lor O(k)\}
\]
\[
\leq \{O(r) \lor f(y) \lor O(s)\} = f(y) \leq t.
\]
Hence \( f(a) \leq \max\{ (f \ast O \ast O)(x), (O \ast f \ast O)(x), (O \ast O \ast f)(x) \} \leq t \). Thus \( a \in L(f; t) \), so \( SSL(f; t) \cap SL(f; t) \cap L(f; t)SS \subseteq L(f; t) \).
Now, let \( a \in (SSL(f; t) \cap (SSL(f; t) SS) \cap (L(f; t) SS) \cap (L(f; t) SS)). \) Then there exist \( x, y, z \in L(f; t) \) and \( p, q, r, s, t, u, v, w \in S \) such that \( a = pqx = zwv = rsytw. \) Now,
\[
(O \ast O \ast f \ast O \ast O)(a) = \bigwedge_{a \in \text{bedem}} \{O(h) \vee O(e) \vee f(d) \vee O(e) \vee O(m)\}
\]
\[
\leq \{O(r) \vee O(s) \vee f(y) \vee O(l) \vee O(w)\} = f(y) \leq t
\]
Hence \( f(a) \leq \max\{(f \ast O \ast O)(a), (O \ast f \ast f \ast O \ast O)(a), (O \ast f \ast f)(a)\} \leq t. \) Thus \( a \in L(f; t). \) Hence \( (SSL(f; t)) \cap (SSL(f; t) SS) \cap (L(f; t) SS) \subseteq L(f; t). \) Therefore \( L(f; t) \) is a quasi-ideal of \( S. \)

Conversely, assume that \( \emptyset \neq L(f; t) \) is a quasi-ideal of \( S \) for all \( t \in [0, 1]. \) Let \( a \in S \) be such that \( \max\{(f \ast O \ast O)(a), (O \ast f \ast O)(a), (O \ast f \ast f)(a)\} < f(a). \) Choose \( t \in [0, 1] \) such that
\[
\max\{(f \ast O \ast O)(a), (O \ast f \ast O)(a), (O \ast f \ast f)(a)\} \leq t < f(a).
\]
Then \( (f \ast O \ast O)(a) \leq t \) implies \( a \in L(f; t) SS, \) \( (O \ast f \ast O)(a) \leq t \) implies \( a \in SSL(f; t) \) and \( (O \ast f \ast f)(a) \leq t \) implies \( a \in SSL(f; t). \) This implies that \( a \in (SSL(f; t) \cap SSL(f; t) SS) \cap (L(f; t) SS) \subseteq L(f; t). \) This implies that \( f(a) \leq t. \) Which is a contradiction. Hence \( \max\{(f \ast O \ast O)(a), (O \ast f \ast O)(a), (O \ast f \ast f)(a)\} \geq f(a). \) Thus \( f \) is an anti fuzzy quasi-ideal of \( S. \)

**Theorem 3.24.** A non-empty subset \( Q \) of a ternary semigroup \( S \) is a quasi-ideal of \( S \) if and only if the fuzzy subset \( f \) defined by
\[
f(x) = \begin{cases} t & \text{if } x \in S \setminus Q, \\ r & \text{if } x \in Q. \end{cases}
\]
is an anti fuzzy quasi-ideal of \( S, \) where \( r, t \in [0, 1], \) such that \( t \geq r. \)

**Proof.** Similar to the proof of Theorem 3.10. \( \square \)

**Remark 3.25.** From the above theorems we conclude that a non-empty subset \( Q \) of a ternary semigroup \( S \) is a quasi-ideal of \( S \) if and only if the characteristic function of the complement of \( Q, \) that is, \( C_Q, \) is an anti fuzzy quasi-ideal of \( S. \)

**Theorem 3.26.** Every anti fuzzy left (right, lateral) ideal of a ternary semigroup \( S \) is an anti fuzzy quasi-ideal of \( S. \)

**Proof.** Let \( f \) be an anti fuzzy left ideal of \( S. \) Then
\[
\max\{(f \ast O \ast O)(x), (O \ast f \ast O)(x), (O \ast O \ast f)(x)\} \geq (O \ast O \ast f)(x) \geq f(x).
\]
Similarly
\[
\max\{(f \ast O \ast O)(x), (O \ast O \ast f \ast O \ast O)(x), (O \ast O \ast f)(x)\} \geq (O \ast O \ast f)(x) \geq f(x).
\]
Thus \( f \) is an anti fuzzy quasi-ideal of \( S. \) \( \square \)

**Theorem 3.27.** Every anti fuzzy quasi-ideal of a ternary semigroup \( S \) is an anti fuzzy bi-ideal of \( S. \)
Proof. Suppose \( f \) is an anti fuzzy quasi-ideal of \( S \). Then
\[
\begin{align*}
 f(\text{xyz}) &\leq \big\{ (f \ast O \ast O)(\text{xyz}) \vee (O \ast f \ast O)(\text{xyz}) \vee (O \ast O \ast f)(\text{xyz}) \big\} \\
 &= \bigwedge_{\text{xyz}=abc} \{ f(a) \vee O(b) \vee O(c) \} \vee \bigwedge_{\text{xyz}=pqr} \{ O(p) \vee f(q) \vee O(r) \} \\
 &\quad \vee \bigwedge_{\text{xyz}=lmn} \{ O(l) \vee O(m) \vee f(n) \} \\
 &\leq \{ f(x) \vee O(y) \vee O(z) \} \vee \{ O(x) \vee f(y) \vee O(z) \} \vee \{ O(x) \vee O(y) \vee f(z) \} \\
 &= f(x) \vee f(y) \vee f(z).
\end{align*}
\]
This implies that \( f(\text{xyz}) \leq \max \{ f(x), f(y), f(z) \} \). Also,
\[
\begin{align*}
 f(\text{xuyvz}) &\leq \big\{ (f \ast O \ast O)(\text{xuyvz}) \vee (O \ast f \ast O \ast O)(\text{xuyvz}) \vee (O \ast O \ast f)(\text{xuyvz}) \big\} \\
 &= \bigwedge_{\text{xuyvz}=abcde} \{ f(a) \vee O(bcd) \vee O(e) \} \vee \bigwedge_{\text{xuyvz}=pqrst} \{ O(p) \vee O(q) \vee f(r) \vee O(s) \vee O(t) \} \\
 &\quad \vee \bigwedge_{\text{xuyvz}=ijklm} \{ O(ijk) \vee O(lm) \vee f(m) \} \\
 &\leq \{ f(x) \vee O(uyv) \vee O(z) \} \vee \{ O(x) \vee O(u) \vee f(y) \vee O(v) \vee O(z) \} \\
 &\quad \vee \{ O(x) \vee O(uyv) \vee f(z) \} \\
 &= f(x) \vee f(y) \vee f(z).
\end{align*}
\]
This implies that \( f(\text{xuyvz}) \leq \max \{ f(x), f(y), f(z) \} \). Thus \( f \) is an anti fuzzy bi-ideal of \( S \). \( \square \)

Corollary 3.28. Every anti fuzzy left (right, lateral) ideal of a ternary semigroup \( S \) is an anti fuzzy bi-ideal of \( S \).

Proof. Follows from Theorem 3.26 and Theorem 3.27. \( \square \)

4. Regular Ternary Semigroups

In this section we are characterizing the regular ternary semigroups by the properties of their anti fuzzy ideals, anti fuzzy quasi-ideals, anti fuzzy bi-ideals and anti fuzzy generalized bi-ideals.

Theorem 4.1. For a ternary semigroup \( S \), the following conditions are equivalent.

1. \( S \) is regular.
2. \( (f \vee g \vee h) = f \ast g \ast h \) for every anti fuzzy right ideal \( f \) and every anti fuzzy lateral ideal \( g \) and every anti fuzzy left ideal \( h \) of \( S \).
Proof. (1) ⇒ (2) Let \( f \) be an anti fuzzy right ideal, \( g \) an anti fuzzy lateral ideal and \( h \) an anti fuzzy left ideal of \( S \). Then by Theorem 3.18 \( f \ast g \ast h \geq f \vee g \vee h \). Let \( a \in S \). Then there exists \( x \in S \) such that \( a = axa = (axa)a \). Thus we have

\[
(f \ast g \ast h) = \bigcap_{a=pr} \{ f(p) \vee g(q) \vee h(r) \} \\
\leq \{ f(a) \vee g(axa) \vee h(a) \} \\
\leq \{ f(a) \vee g(a) \vee h(a) \} \\
= (f \vee g \vee h)(a).
\]

Thus \( f \ast g \ast h \leq f \vee g \vee h \). Hence \( f \ast g \ast h = f \vee g \vee h \).

(2) ⇒ (1) In order to show that \( S \) is regular, we shall show that \( R \cap M \cap L = RML \) for every right ideal \( R \), every lateral ideal \( M \) and every left ideal \( L \) of \( S \). Since \( RML \subseteq R \cap M \cap L \) always holds. We only show that \( R \cap M \cap L \subseteq RML \). Suppose on contrary that there exist \( a \in R \cap M \cap L \) such that \( a \notin RML \). Then by Remark 3.13, the characteristic functions \( C_{R}^{c}, C_{M}^{c} \) and \( C_{L}^{c} \) are anti fuzzy right ideal, anti fuzzy lateral ideal and anti fuzzy left ideal of \( S \), respectively. Since \( a \in R \cap M \cap L \). This implies that \( a \in R, a \in M \) and \( a \in L \). Thus \( C_{R}^{c}(a) = C_{M}^{c}(a) = C_{L}^{c}(a) = 0 \), also since \( a \notin RML \). This implies that there does not exist \( x, y, z \in S \) such that \( a = xyz \). Thus \( (C_{R}^{c} \ast C_{M}^{c} \ast C_{L}^{c})(a) = 1 \). Which is a contradiction to the hypothesis. Thus \( R \cap M \cap L \subseteq RML \). Hence \( R \cap M \cap L = RML \). Thus by Theorem 2.1 \( S \) is regular. □

**Theorem 4.2.** For a ternary semigroup \( S \), the following conditions are equivalent.

1. \( S \) is regular.
2. \( f = f \ast O \ast f \ast f \ast f \) for every anti fuzzy generalized bi-ideal \( f \) of \( S \).
3. \( f = f \ast O \ast f \ast O \ast f \ast f \) for every anti fuzzy bi-ideal \( f \) of \( S \).
4. \( f = f \ast O \ast f \ast O \ast f \ast f \) for every anti fuzzy quasi-ideal \( f \) of \( S \).

Proof. (1) ⇒ (2) Let \( f \) be an anti fuzzy generalized bi-ideal of \( S \) and \( a \in S \). Since \( S \) is regular, so there exist \( x \in S \) such that \( a = axa = axaxa \)

\[
(f \ast O \ast f \ast O \ast f)(a) = \bigcap_{a=pr} \{ f(p) \vee O(q) \vee f(r) \vee O(s) \vee f(t) \} \\
\leq \{ f(a) \vee O(x) \vee f(a) \vee O(x) \vee f(a) \} = f(a).
\]

Thus \( f \ast O \ast f \ast O \ast f \leq f \). Since \( f \) is an anti fuzzy generalized bi-ideal of \( S \), so \( f \ast O \ast f \ast O \ast f \geq f \). Hence \( f = f \ast O \ast f \ast O \ast f \).

It is clear that (2) ⇒ (3) ⇒ (4).

(4) ⇒ (1) Let \( A \) be any quasi-ideal of \( S \). Then

\[
ASASA \subseteq (ASS) \cap (SSASS) \cap (SSA) \subseteq A
\]

because \( A \) is a quasi-ideal of \( S \). By Remark 3.25, \( C_{A}^{c} \) is an anti fuzzy quasi-ideal of \( S \). Let \( a \in A \). Then \( C_{A}^{c}(a) = 0 \). If \( a \notin ASASA \), then there does not exist \( p, q, r, s, t \in A \) such that \( a = pqrst \). Thus \(((C_{A}^{c} \ast O \ast C_{A}^{c}) \ast O \ast C_{A}^{c})(a) = 1 \), which is a contradiction. So \( ASASA \subseteq A \). Thus \( ASASA = A \). By Theorem 2.2, \( S \) is regular. □
Acknowledgements. The authors are highly grateful to referees/Editor in Chief Prof. Young Bae Jun for their valuable comments and suggestions for improving the paper.

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