FP-soft set theory and its applications

NAIM ÇAĞMAN, FILIZ ÇITAK, SERDAR ENGİNOĞLU

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Abstract. In this work, we first introduce fuzzy parameterized (FP) soft sets and their related properties. We then propose a decision making method based on FP-soft set theory. We finally give an example which shows that the method can be successfully applied to the problems that contain uncertainties.

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Corresponding Author: Naim Çağman (ncagman@gop.edu.tr)

1. Introduction

Many fields deal daily with the uncertain data that may not be successfully modeled by the classical mathematics. The probability theory, fuzzy sets [20], rough sets [16], and other mathematical tools are well-known and often useful approaches to describe uncertainty. However, all of these theories have their own difficulties which are pointed out in [15] by Molodtsov who then proposed a completely new approach for modeling vagueness and uncertainty, that is free from the difficulties. This so-called soft set theory has potential applications in many different fields. Maji et al. [12] firstly worked on detailed theoretical study of soft sets. After than, the properties and applications on the soft set theory have been studied by many authors (e.g. [3, 4, 5, 9, 13, 18, 22]). The algebraic structure of soft set theory has also been studied in more detail (e.g. [1, 2, 7, 10]). Many interesting applications of soft set theory have been expanded by embedding the ideas of fuzzy sets (e.g. [6, 8, 11, 14, 17, 19, 21]).

In this paper, in the next section, most of the fundamental definitions of the operations of soft sets are presented. In Section 3, we have defined fuzzy parameterized soft sets, in short written FP-soft sets, whose parameters sets are fuzzy sets and have improved several results. In Section 4, we have defined the fuzzy decision set of a FP-soft set to construct a decision method by which approximate functions of a soft set are combined to produce a single fuzzy set that can be used to evaluate
2. Soft sets

In this section, for subsequent discussions of this work, we have presented the basic definitions and results of soft set theory that may be found in earlier studies [1, 12, 15].

Throughout this work, $U$ refers to an initial universe, $E$ is a set of parameters, $P(U)$ is the power set of $U$, and $A \subseteq E$.

**Definition 2.1.** A soft set $F_A$ on the universe $U$ is defined by the set of ordered pairs

$$F_A = \{(x, f_A(x)) : x \in E, f_A(x) \in P(U)\}$$

where $f_A : E \to P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$.

Here, $f_A$ is called the approximate function of the soft set $F_A$. The value of $f_A(x)$ is a set called $x$-element of the soft set for all $x \in E$. It is worth noting that the sets $f_A(x)$ may be empty. Some of them may be empty, some may have nonempty intersection.

The subscript $A$ in the notation $f_A$ indicates that $f_A$ is the approximate function of $F_A$. From now on, we will use $S(U)$ instead of the set of all soft sets over $U$.

**Definition 2.2.** Let $F_A \in S(U)$. If $f_A(x) = \emptyset$ for all $x \in E$, then $F_A$ is called an empty set, denoted by $F_{\emptyset}$. $f_A(x) = \emptyset$ means that there is no element in $U$ related to the parameter $x \in E$. Therefore, we do not display such elements in the soft sets, as it is meaningless to consider such parameters.

**Definition 2.3.** Let $F_A \in S(U)$. If $f_A(x) = U$ for all $x \in A$, then $F_A$ is called an $A$-universal set, denoted by $F_A^U$. If $A = E$, then the $A$-universal set is called universal soft set denoted by $F_E^U$.

**Definition 2.4.** Let $F_A, F_B \in S(U)$. Then, $F_A$ is a soft subset of $F_B$, denoted by $F_A \subseteq F_B$, if $f_A(x) \subseteq f_B(x)$ for every $x \in E$.

**Definition 2.5.** Let $F_A, F_B \in S(U)$. Then, $F_A$ and $F_B$ are soft equal, denoted by $F_A = F_B$, if and only if $f_A(x) = f_B(x)$ for every $x \in E$.

**Definition 2.6.** Let $F_A \in S(U)$. Then, complement of $F_A$, denoted by $F_A^c$, is a soft set defined by the approximate function

$$f_{A^c}(x) = U \setminus f_A(x).$$

**Definition 2.7.** Let $F_A, F_B \in S(U)$. Then, union of $F_A$ and $F_B$, denoted by $F_A \cup F_B$, is a soft set defined by the approximate function

$$f_{A \cup B}(x) = f_A(x) \cup f_B(x).$$

**Definition 2.8.** Let $F_A, F_B \in S(U)$. Then, intersection of $F_A$ and $F_B$, denoted by $F_A \cap F_B$, is a soft set defined by the approximate function

$$f_{A \cap B}(x) = f_A(x) \cap f_B(x).$$

each alternative. In Section 5, we have given an application that shows that these methods work successfully. In the final section, some concluding comments have been presented.
Proposition 2.9. Let $F_A, F_B, F_C \in S(U)$. Then

(1) $F_A \cup F_A = F_A$, $F_A \cap F_A = F_A$.

(2) $F_A \cup F_\emptyset = F_A$, $F_A \cap F_\emptyset = F_\emptyset$.

(3) $F_A \cup F_E = F_E$, $F_A \cap F_E = F_A$.

(4) $F_A \cup F_A^c = F_E$, $F_A \cap F_A^c = F_\emptyset$.

(5) $F_A \cup F_B = F_B \cup F_A$, $F_A \cap F_B = F_B \cap F_A$.

(6) $(F_A \cup F_B) \cup F_C = F_A \cup (F_B \cup F_C)$, $(F_A \cap F_B) \cap F_C = F_A \cap (F_B \cap F_C)$.

3. FP-soft sets

In this section, we give definition of fuzzy parameterized soft sets (FP-soft sets) and their operations. In Section 2, the subsets of $E$ were classical sets, denoted by the letter $A, B, C, ...$, but in this section, the subsets of $E$ will be fuzzy denoted by the letter $X, Y, Z, ...$, to avoid confusion and complexity of the symbols.

Definition 3.1. Let $U$ be an initial universe, $P(U)$ be the power set of $U$, $E$ be the set of all parameters and $X$ be a fuzzy set over $E$. An FP-soft set $F_X$ on the universe $U$ is defined by the set of ordered pairs

$$F_X = \{(\mu_X(x)/x, f_X(x)) : x \in E, f_X(x) \in P(U), \mu_X(x) \in [0, 1]\},$$

where the function $f_X : E \to P(U)$ is called approximate function such that $f_X(x) = \emptyset$ if $\mu_X(x) = 0$, and the function $\mu_X : E \to [0, 1]$ is called membership function of FP-soft set $F_X$. The value of $\mu_X(x)$ is the degree of importance of the parameter $x$, and depends on the decision maker’s requirements.

Note that from now on the sets of all FP-soft sets over $U$ will be denoted by $FPS(U)$.

Definition 3.2. Let $F_X \in FPS(U)$. If $f_X(x) = \emptyset$ for all $x \in E$, then $F_X$ is called an $X$-empty FP-soft set, denoted by $F_{X^e}$.

If $X = \emptyset$, then $F_X$ is called an empty FP-soft set, denoted by $F_\emptyset$.

Definition 3.3. Let $F_X \in FPS(U)$. If $X$ is a crisp subset of $E$ and $f_X(x) = U$ for all $x \in X$, then $F_X$ is called $X$-universal FP-soft set, denoted by $F_{\bar{X}}$.

If $X = E$, then the $E$-universal FP-soft set is called universal FP-soft set, denoted by $F_E$.

Example 3.4. Assume that $U = \{u_1, u_2, u_3, u_4, u_5\}$ is a universal set and $E = \{x_1, x_2, x_3, x_4\}$ is a set of parameters.

If $X = \{0.2/x_2, 0.5/x_3, 1/x_4\}$ and $f_X(x_2) = \{u_2, u_4\}$, $f_X(x_3) = \emptyset$, and $f_X(x_4) = U$, then $F_X = \{(0.2/x_2, \{u_2, u_4\}), (0.5/x_3, \emptyset), (1/x_4, U)\}$.

If $Y = \{0.3/x_2, 0.7/x_3\}$, $f_Y(x_2) = \emptyset$ and $f_Y(x_3) = \emptyset$, then $F_Y = F_{\emptyset}$.

If $Z = \emptyset$, then $F_Z = F_\emptyset$.

If $T = \{1/x_1, 1/x_2\}$, $f_T(x_1) = U$, and $f_T(x_2) = U$, then $F_T = F_{\bar{T}}$.

Definition 3.5. Let $F_X, F_Y \in FPS(U)$. Then, $F_X$ is an FP-soft subset of $F_Y$, denoted by $F_X \subseteq F_Y$, if $\mu_X(x) \leq \mu_Y(x)$ and $f_X(x) \subseteq f_Y(x)$ for all $x \in E$.

Remark 3.6. $F_X \subseteq F_Y$ does not imply that every element of $F_X$ is an element of $F_Y$ as in the definition of the classical subset.
For example, assume that $U = \{u_1, u_2, u_3, u_4\}$ is a universal set of objects and $E = \{x_1, x_2, x_3\}$ is a set of all the parameters. If $X = \{(0.5/x_1)\}$ and $Y = \{0.9/x_1, 0.1/x_3\}$, and $F_X = \{(0.5/x_1, \{u_2, u_4\})\}$, $F_Y = \{(0.9/x_1, \{u_2, u_3, u_4\}), (0.1/x_3, \{u_1, u_3\})\}$, then for all $x \in E$, $\mu_X(x) \leq \mu_Y(x)$ and $f_X(x) \subseteq f_Y(x)$ is valid. Hence, $F_X \subseteq F_Y$. It is clear that $(0.5/x_1, \{u_2, u_4\}) \in F_X$, but $(0.5/x_1, \{u_2, u_4\}) \notin F_Y$.

**Proposition 3.7.** Let $F_X, F_Y \in FPS(U)$. Then

1. $F_X \subseteq F_Y$.
2. $F_Y \subseteq F_X$.
3. $F_X \subseteq F_Y$ and $F_Y \subseteq F_Z \Rightarrow F_X \subseteq F_Z$.

**Proof.** They can be proved easily by using the approximate and membership functions of the FP-soft sets.

**Definition 3.8.** Let $F_X, F_Y \in FPS(U)$. Then, $F_X$ and $F_Y$ are FP-soft equal, written as $F_X = F_Y$, if and only if $\mu_X(x) = \mu_Y(x)$ and $f_X(x) = f_Y(x)$ for all $x \in E$.

**Proposition 3.9.** Let $F_X, F_Y, F_Z \in FPS(U)$. Then

1. $F_X = F_Y$ and $F_Y = F_Z \Rightarrow F_X = F_Z$.
2. $F_X \subseteq F_Y$ and $F_Y \subseteq F_X \Leftrightarrow F_X = F_Y$.

**Proof.** The proofs are trivial.

**Definition 3.10.** Let $F_X \in FPS(U)$. Then, complement $F_X$, denoted by $F_X^c$, is an FP-soft set defined by the approximate and membership functions as

$$\mu_X^c(x) = 1 - \mu_X(x) \text{ and } f_X^c(x) = U \setminus f_X(x).$$

**Proposition 3.11.** Let $F_X \in FPS(U)$. Then,

1. $(F_X^c)^c = F_X$.
2. $F_X^c = F_X^c$.

**Proof.** By using the approximate and membership functions of the FP-soft sets, the proofs can be straightforward.

**Definition 3.12.** Let $F_X, F_Y \in FPS(U)$. Then, union $F_X$ and $F_Y$, denoted by $F_X \cup F_Y$, is defined by

$$\mu_{X \cup Y}(x) = \max\{\mu_X(x), \mu_Y(x)\} \text{ and } f_{X \cup Y}(x) = f_X(x) \cup f_Y(x), \text{ for all } x \in E.$$

**Proposition 3.13.** Let $F_X, F_Y, F_Z \in FPS(U)$. Then

1. $F_X \cup F_X = F_X$.
2. $F_X \cup F_Y = F_X$.
3. $F_X \cup F_Z = F_Z$.
4. $F_X \cup F_Y = F_Y \cup F_X$.
5. $(F_X \cup F_Y) \cup F_Z = F_X \cup (F_Y \cup F_Z)$.

**Proof.** The proofs can be easily obtained from Definition 3.12.
Definition 3.14. Let $F_X, F_Y \in FPS(U)$. Then, intersection of $F_X$ and $F_Y$, denoted by $F_X \cap F_Y$, is an FP-soft sets defined by the approximate and membership functions

$$\mu_{X \cap Y}(x) = \min\{\mu_X(x), \mu_Y(x)\} \text{ and } f_{X \cap Y}(x) = f_X(x) \cap f_Y(x).$$

Proposition 3.15. Let $F_X, F_Y, F_Z \in FPS(U)$. Then

1. $F_X \cap F_X = F_X.$
2. $F_X \cap F_\Phi = F_\Phi.$
3. $F_X \cap F_E = F_X.$
4. $F_X \cap F_Y = F_Y \cap F_X.$
5. $(F_X \cap F_Y) \cap F_Z = F_X \cap (F_Y \cap F_Z).$

Proof. The proofs can be easily obtained from Definition 3.14. □

Remark 3.16. Let $F_X \in FPS(U)$. If $F_X \neq F_\Phi$ and $F_X \neq F_E$, then $F_X \cup F_X^c \neq F_E$ and $F_X \cap F_X^c \neq F_\Phi$.

Proposition 3.17. Let $F_X, F_Y \in FPS(U)$. Then De Morgan’s laws are valid

1. $(F_X \cup F_Y)^c = F_X^c \cap F_Y^c.$
2. $(F_X \cap F_Y)^c = F_X^c \cup F_Y^c.$

Proof. For all $x \in E$,

$$(1) \quad \mu_{(X \cup Y)^c}(x) = 1 - \mu_{X \cup Y}(x) = 1 - \max\{\mu_X(x), \mu_Y(x)\} = \min\{1 - \mu_X(x), 1 - \mu_Y(x)\} = \min\{\mu_X^c(x), \mu_Y^c(x)\} = \mu_{X^c \cap Y^c}(x)$$

and

$$(2) \quad f_{(X \cup Y)^c}(x) = U \setminus f_{X \cup Y}(x) = U \setminus (f_X(x) \cup f_Y(x)) = (U \setminus f_X(x)) \cap (U \setminus f_Y(x)) = f_{X^c}(x) \cap f_{Y^c}(x) = f_{X^c \cap Y^c}(x).$$

Likewise, the proof of (2) can be made similarly. □

Proposition 3.18. Let $F_X, F_Y, F_Z \in FPS(U)$. Then

1. $F_X \cap (F_Y \cap F_Z) = (F_X \cap F_Y) \cap (F_X \cap F_Z).$
2. $F_X \cap (F_Y \cup F_Z) = (F_X \cap F_Y) \cup (F_X \cap F_Z).$

Proof. For all $x \in E$,

$$(1) \quad \mu_{X \cap (Y \cap Z)}(x) = \max\{\mu_X(x), \mu_{Y \cap Z}(x)\} = \max\{\mu_X(x), \min\{\mu_Y(x), \mu_Z(x)\}\} = \min\{\max\{\mu_X(x), \mu_Y(x)\}, \mu_X(x), \mu_Z(x)\}\} = \min\{\mu_X \cap_Y(x), \mu_X \cap_Z(x)\} = \mu_{(X \cap Y) \cap (X \cap Z)}(x)$$

$$(2) \quad \mu_{X \cap (Y \cup Z)}(x) = \max\{\mu_X(x), \mu_{Y \cup Z}(x)\} = \max\{\mu_X(x), \max\{\mu_Y(x), \mu_Z(x)\}\} = \min\{\max\{\mu_X(x), \mu_Y(x)\}, \mu_X(x), \mu_Z(x)\}\} = \min\{\mu_X \cap_Y(x), \mu_X \cap_Z(x)\} = \mu_{(X \cap Y) \cup (X \cap Z)}(x)$$
and

\[ f_{X \cup (Y \cap Z)}(x) = f_X(x) \cup f_{Y \cap Z}(x) = f_X(x) \cup (f_Y(x) \cap f_Z(x)) = (f_X(x) \cup f_Y(x)) \cap (f_X(x) \cup f_Z(x)) = f_{X \cup Y}(x) \cap f_{X \cup Z}(x) = f_{(X \cup Y) \cap (X \cup Z)}(x). \]

Likewise, the proof of (2) can be made in a similar way. □

4. Fuzzy decision set of an FP-soft set

In this section, we define fuzzy decision set of an FP-soft set to construct a decision method by which approximate functions of a soft set are combined to produce a single fuzzy set that can be used to evaluate each alternative.

**Definition 4.1.** Let \( F_X \in FPS(U) \). Then a fuzzy decision set of \( F_X \), denoted by \( F^d_X \), is defined by

\[ F^d_X = \{ \mu_{F^d_X}(u) / u \in U \} \]

which is a fuzzy set over \( U \), its membership function \( \mu_{F^d_X} \) is defined by

\[ \mu_{F^d_X} : U \to [0, 1], \quad \mu_{F^d_X}(u) = \frac{1}{|\text{supp}(X)|} \sum_{x \in \text{supp}(X)} \mu_X(x) \chi_{f_X(x)}(u) \]

where \( \text{supp}(X) \) is the support set of \( X \), \( f_X(x) \) is the crisp subset determined by the parameter \( x \) and

\[ \chi_{f_X(x)}(u) = \begin{cases} 1, & u \in f_X(x), \\ 0, & u \notin f_X(x). \end{cases} \]

5. Applications

Once a fuzzy decision set of an FP-soft set has been arrived at, it may be necessary to choose the best single alternative from the alternatives. Therefore, we can make a decision by the following algorithm.

**Step 1:** Construct a soft set \( F_X \) over \( U \).

**Step 2:** Compute the fuzzy decision set \( F^d_X \),

**Step 3:** Select the largest membership grade \( \max \mu_{F^d_X}(u) \).

Let us consider the following example to illustrate the idea.

**Example 5.1.** Let us assume that some one goes to an automobile showroom to buy an automobile. There are eight alternatives \( U = \{ u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8 \} \). There may be five parameters \( E = \{ e_1, e_2, e_3, e_4, e_5 \} \) to evaluate the automobiles. For \( i = 1, 2, 3, 4, 5 \), the parameters \( e_i \) stand for “beautiful”, “expensive”, “large”, “equipped with conveniences”, “high speedy”, respectively. He/She considers only four parameters “expensive”, “large”, “equipped with conveniences” and “high speedy” which are important with degree 0.8, 0.3, 0.5 and 0.6, respectively. That is, the subset of parameters is \( X = \{ 0.8/e_2, 0.3/e_3, 0.5/e_4, 0.6/e_5 \} \). Now we can find a suitable automobile to buy.
Step 1: After a serious discussion, He/She evaluates the alternative from point of view of the goals and the constraint according to a chosen subset $X$ of $E$ to constructs an FP-soft set,

$$FX = \left\{ \left(0.8/e_2, \{u_2, u_5, u_7\}\right), \left(0.3/e_3, \{u_1, u_2, u_3, u_4, u_5, u_8\}\right), \left(0.5/e_4, \{u_1, u_2, u_4, u_7, u_8\}\right), \left(0.6/e_5, \{u_1, u_3, u_7, u_8\}\right) \right\}.$$ 

Step 2: The fuzzy decision set of $F_X$ can be found as,

$$F^d_X = \{0.35/u_1, 0.40/u_2, 0.22/u_3, 0.20/u_4, 0.27/u_5, 0.47/u_7, 0.35/u_8\}.$$ 

Step 3: Finally, the largest membership grade can be chosen by

$$\max \mu_{F^d_X}(u) = 0.47$$

which means that the candidate $u_7$ has the largest membership grade, hence it is selected.

Note that if there are more then one largest membership grades, then it better to reset the degree of parameters.

6. Conclusion

In this paper, we first defined FP-soft sets and their operations. We then presented the decision methods on the FP-soft set theory. Finally, we provided an application that demonstrated that this method can successfully work. It can be applied to many fields to the problems that contain uncertainty, and would be beneficial to extend the proposed method to subsequent studies. However, the approach should be more comprehensive in the future to solve the related problems.

References


NAIM ÇAĞMAN (ncagman@gop.edu.tr) – Department of Mathematics, Gaziosmanpaşa University, Tokat, Turkey

FİLİZ ÇİTAK (filizcitak@gop.edu.tr) – Department of Mathematics, Gaziosmanpaşa University, Tokat, Turkey

SERDAR ENGİNOĞLU (serdarenginoglu@gop.edu.tr) – Department of Mathematics, Gaziosmanpaşa University, Tokat, Turkey