Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 2, No. 2, (October 2011), pp. 207- 217
ISSN 2093–9310
http://www.afmi.or.kr

$(L, \odot)$-smooth topogenous spaces and $(L, \odot)$-smooth quasi-uniform spaces

M. A. Abdel-Sattar

Received 18 January 2011; Revised 16 March 2011; Accepted 11 April 2011

Abstract. In this paper we study the concept of $(L, \odot)$-smooth topogenous structures in the framework of $(L, \odot)$-smooth topologies and $(L, \odot)$-smooth quasi-proximities. Some fundamental properties of them are studied. The relationship between $(L, \odot)$-smooth topogenous structures and $(L, \odot)$-smooth quasi-uniformities is established.

2010 AMS Classification: 54A40, 06D72, 03B52

Keywords: $(L, \odot)$ -smooth topological spaces, $(L, \odot)$-smooth topogenous spaces, $(L, \odot)$-smooth quasi-uniform spaces.

Corresponding Author: M. A. Abdel-Sattar (mabdelsattar2002@yahoo.com)

1. Introduction

Katsaras and Petalas [10] introduced the concept of fuzzy syntopogenous structures as the foundations of the theories of Chang fuzzy topological spaces [3], Hutton fuzzy uniform spaces [5], Katsaras fuzzy proximity spaces [6, 7]. Katsaras [8, 9] developed the notions of a fuzzy syntopogenous structures in the sense of Lowen fuzzy topological spaces [13], Lowen fuzzy uniform spaces [14] and Artico-Moresco fuzzy proximity spaces [1, 2]. Moreover, Šostak [21] and Ramadan [16] expanded a fuzzy syntopogenous structures in the sense of Šostak fuzzy topological spaces [20]. Šostak [20] introduced the notion of (L, $\wedge$)-fuzzy topological spaces as a generalization of L-topological spaces [13]. Höhle and Šostak substitute a complete quasi-monoidal lattice (or GL -monoid) instead of a completely distributive lattice or an unit interval. Ramadan et al [18] introduce the concept of L-fuzzy topogenous spaces and L-fuzzy quasi- uniform spaces.

In this paper, we introduce the $(L, \odot)$-smooth topogenous spaces in the sense of Šostak fuzzy topological spaces [20], Samanta fuzzy proximities and unformities [19], and Yue et al. fuzzy quasi-uniform spaces [22]. It is different from the definitions of L-fuzzy topogenous structures [8, 9, 16, 17, 18, 19]. We study a natural relationship between $(L, \odot)$-fuzzy topogenous structures and $(L, \odot)$-fuzzy quasi-uniformities..

2. Preliminaries

Throughout this paper, let X be a nonempty set. $L = (L, \leq, \vee, \odot, {}', 0, 1)$ denotes a completely distributive lattice with order-reversing involution $'$ which has the least and greatest elements,say 0 and 1, respectively. Let L^X be the family of all L-fuzzy subsets of X. For $\alpha \in L$, $\overline{\alpha}(x) = \alpha$ for all $x \in X$.

Definition 2.1 ([4])**.** A triple $(L\ , \leq, \odot)$ is called a strictly two-sided, commutative quantal (stsc-quantale, for short) iff it satisfies the following properties:
(L1) $(L, \odot)$ is a commutative semigroup.
(L2) $a = a \odot 1$, for each $a \in L$.
(L3) $\odot$ is distributive over arbitrary joins, i.e., $\left(\bigvee_{i\in\Gamma} a_i\right) \odot b = \bigvee_{i\in\Gamma} (a_i \odot b)$.

Remark 2.2 ([4])**.** (1) Each frame is a stsc-quantale. In particular, the unit interval $([0, 1],\ \leq,\ \wedge, 0, 1)$ is a stsc-quantales.
(2) Every continuous t-norm T on $([0, 1], \leq, t)$ with $\odot = t$ is a stsc-quantales.
(3) Every GL-monoid is a stsc-quantale.
(4) Let $(L, \leq, \odot)$ be a stsc-quantale. For each $x, y \in L$, we define

$$x \to y = \bigvee\{z \in L \mid x \odot z \leq y\}.$$

Then it satisfies Galois correspondence, that is,

$$(x \odot y) \leq z \Leftrightarrow x \leq (y \to z).$$

(5) $(L, \leq, \odot, \oplus, {}^*)$ is a stsc-quantale with an order-reversing involution * defined by $x \oplus y = (x^* \odot y^*)^*$ unless otherwise specified.

Definition 2.3 ([4])**.** A stsc-quantale $(L, \leq, \odot, {}^*)$ is called a complete MV-algebra iff it satisfies the following property:
(MV) $(x \to y) \to y = x \vee y, \quad \forall x, y \in L$ which is defined as
$x \to y = \vee\{z \in L \mid x \odot z \leq y\}, \quad x^* = x \to 0.$

Lemma 2.4 ([4])**.** *Let $(L, \leq, \odot, \oplus, {}^*)$ be a stsc-quantale with an order-reversing involution *. For each $x, y, z \in L, \{y_i \mid i \in \Gamma\} \subset L$, we have the following properties:*
(1) *If $y \leq z$ then $(x \odot y) \leq (x \odot z)$ and $(x \oplus y) \leq (x \oplus z)$.*
(2) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$.
(3) $\wedge_{i\in\Gamma}\ y_i^* = (\vee_{i\in\Gamma}\ y_i)^*$ *and* $\vee_{i\in\Gamma} y_i^* = (\wedge_{i\in\Gamma} y_i)^*$.
(4) $x \oplus (\wedge_{i\in\Gamma} y_i) = \wedge_{i\in\Gamma}\ (x \oplus y_i)$.
(5) $(x \vee y) \odot (z \vee w) \leq (x \vee z) \vee (y \odot w) \leq (x \oplus z) \vee (y \odot w)$.
(6) $x \odot (x \to y) \leq y$ *and* $x \to y \leq (y \to z) \to (x \to z)$.
(7) *If $x^* = x \to 0$, then $x \to y = y^* \to x^*$.*
(8) *If $x^* = x \to 0$, then $x \odot (x^* \oplus y^*) \leq y^*$.*
(9) *If L is a complete MV-algebra, then*
$x \odot y = (x \to y^*)^*, \quad (x \oplus y) = x^* \to y,$
$(x \oplus z) \odot y \leq x \oplus (y \odot z),$
$(x \odot y) \odot (z \oplus w) \leq (x \odot z) \oplus (y \odot w),$
$x \oplus (\vee_{i\in\Gamma}\ y_i) = \vee_{i\in\Gamma}(x \oplus y_i)$ *and*
$x \odot (\wedge_{i\in\Gamma}\ y_i) = \wedge_{i\in\Gamma}(x \odot y_i).$

All algebraic operations on L can be extended pointwise to the set L^X as follows:

(1) $\lambda \leq \mu$ *iff* $\lambda(x) \leq \mu(x), \forall x \in X$.
(2) $(\lambda \odot \mu)(x) = \lambda(x) \odot \mu(x), \forall x \in X$.
(3) $(\lambda \to \mu)(x) = \lambda(x) \to \mu(x), \forall x \in X$.

Definition 2.5 ([4, 12, 15])**.** A map $\tau : L^X \to L$ is called an $(L, \odot)$-smooth topology if it satisfies the following conditions:
(o1) $\tau(\underline{0}) = \tau(\underline{1}) = 1$,
(o2) $\tau(\mu_1 \odot \mu_2) \geq \tau(\mu_1) \odot \tau(\mu_2),\ \forall\ \mu_1, \mu_2 \in L^X$.
(o3) $\tau(\vee_{i\in\Gamma}\mu_i) \geq \wedge_{i\in\Gamma}\tau(\mu_i)$ for any $\{\mu_i\}_{i\in\Gamma} \subset L^X$.

The pair (X, τ) is called an $(L, \odot)$-smooth topological spaces.

Let τ_1 and τ_2 be $(L, \odot)$-smooth topologies on X. We say that τ_1 is finer than τ_2 (τ_2 is coarser than τ_1), denoted by $\tau_2 \leq \tau_1$, if $\tau_2(\lambda) \leq \tau_1(\lambda)$ for all $\lambda \in L^X$.

Let (X, τ_1) and (Y, τ_2) be $(L, \odot)$-smooth topological spaces.

A function $f : (X, \tau_1) \to (Y, \tau_2)$ is called $(L, \odot)$-smooth continuous map if $\tau_2(\lambda) \leq \tau_1(f^{-1}(\lambda))$ for all $\lambda \in L^Y$.

From [4, 5, 10, 11], let Ω_X denote the family of all functions $f :\ L^X \to L^X$ with the following properties:
(a) $f(\overline{0}) = \overline{0}$ and $\mu \leq f(\mu)$, for every $\mu \in L^X$.
(b) $f(\bigvee_{i\in\Gamma} \mu_i) = \bigvee_{i\in\Gamma} f(\mu_i)$, for $\{\mu_i\}_{i\in\Gamma} \subset L^X$.

For $f \in \Omega_X$, the function $f^{-1} \in \Omega_X$ is defined by

$$f^{-1}(\mu) = \bigwedge\{\rho \mid f(\rho') \leq \mu'\}.$$

For $f, g \in \Omega_X$, we define, for all $\mu \in L^X$,

$$(f \odot g)(\mu) = \bigwedge\{f(\mu_1) \vee g(\mu_2) \mid \mu_1 \vee \mu_2 = \mu\}, \quad f \circ g(\mu) = f(g(\mu)).$$

Then $f \odot g, f \circ g \in \Omega_X$.

Lemma 2.6. *For every $f, g, h, f_1, g_1 \in \Omega_X$, the following properties hold:*

(1) *If $f \leq f_1, g \leq g_1$, then $f \odot g \leq f_1 \odot g_1$.*
(2) *$f \odot g \leq f$, $f \odot g \leq g$ and $f \odot f = f$.*
(3) *$(f^{-1})^{-1} = f$.*
(4) *$f \leq g$ iff $f^{-1} \leq g^{-1}$.*
(5) *$f(\mu) \leq \rho$ iff $f^{-1}(\rho') \leq \mu'$.*
(6) *Let a function $f_{\overline{1},\overline{1}} :\ L^X \to L^X$ be defined by*

$$f_{\overline{1},\overline{1}}(\mu) = \begin{cases} \overline{1} & \text{if } \mu \neq \overline{0}, \\ \overline{0} & \text{if } \mu = \overline{0}. \end{cases}$$

Then $f_{\overline{1},\overline{1}} = f^{-1}_{\overline{1},\overline{1}} \in \Omega_X$ and $f \odot f_{\overline{1},\overline{1}} = f$.

(7) *$(f \circ g)^{-1} = f^{-1} \circ g^{-1}$.*
(8) *$(f \odot g)^{-1} = f^{-1} \odot g^{-1}$.*
(9) *$(f \odot g) \odot h = f \odot (g \odot h)$.*

When $L = (L, \leq, \vee, \odot, ', 0, 1) = (L, \leq, \vee, \wedge, ', 0, 1)$ we have the definition

Definition 2.7 ([4, 19]). A function $\mathcal{U} : \Omega_X \to L$ is said to be an $(L, \wedge)$-*smooth quasi-uniformity* on X if it satisfies the following conditions:
(FQU1) If $f \leq g$, then $\mathcal{U}(f) \leq \mathcal{U}(g)$.
(FQU2) $\mathcal{U}(f \wedge g) \geq \mathcal{U}(f) \wedge \mathcal{U}(g)$, for each $f, g \in \Omega_X$.
(FQU3) For each $f \in \Omega_X$, $\bigvee\{\mathcal{U}(g) \mid g \circ g \leq f\} \geq \mathcal{U}(f)$.
(FQU4) There exists $f \in \Omega_X$ such that $\mathcal{U}(f) = 1$.

The pair $(X, \mathcal{U})$ is said to be $(L, \wedge)$-*smooth quasi-uniform space*. The $(L, \wedge)$-smooth quasi-uniform space $(X, \mathcal{U})$ is called an $(L, \wedge)$-*smooth uniform space* if it satisfies
(FU) For each $f \in \Omega_X$, $\bigvee\{\mathcal{U}(g) \mid g \leq f^{-1}\} \geq \mathcal{U}(f)$.

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two $(L, \wedge)$-*smooth* (quasi-)uniformities on X. $\mathcal{U}_1$ is *finer* than $\mathcal{U}_2$ (or $\mathcal{U}_2$ is *coarser* than $\mathcal{U}_1$) , denoted by $\mathcal{U}_2 \leq \mathcal{U}_1$, iff for any $f \in \Omega_X, \mathcal{U}_2(f) \leq \mathcal{U}_1(f)$.

Remark 2.8. (1) Let $(X, \mathcal{U})$ be an $(L, \wedge)$-*smooth* quasi-uniform space. Put

$$\mathcal{U}_r = \{f \in \Omega_X \mid \mathcal{U}(f) > r\}$$

for each $r \in L - \{1\}$, where $L = [0, 1]$, then $\mathcal{U}_r$ is a Hunton fuzzy uniformity on X (see [5]).

(2) Let $(X, \mathcal{U})$ be an $(L, \wedge)$-smooth quasi-uniform space. By (FQU1), (FQU2) and Lemma 2.6(2), we have $\mathcal{U}(f \wedge g) = \mathcal{U}(f) \wedge \mathcal{U}(g)$, where $L = [0, 1]$.

(3) If $(X, \mathcal{U})$ is an $(L, \wedge)$-smooth uniform space, then, by (FU), (FQU1) and Lemma 2.6(3), we have $\mathcal{U}(f) = \mathcal{U}(f^{-1})$.

(4) Let $(X, \mathcal{U})$ be an $(L, \wedge)$-smooth quasi-uniform space. By Lemma 2.6(6) and (FQU4), since $f \leq f_{\overline{1},\overline{1}}$ for all $f \in \Omega_X$, we have $\mathcal{U}(f_{\overline{1},\overline{1}}) = 1$.

3. $(L, \odot)$-smooth topogenous spaces and $(L, \odot)$-smooth quasi-uniform spaces

Definition 3.1. A function $\eta : L^X \times L^X \to L$ is called an $(L, \odot)$-*smooth topogenous structure* on X if it satisfies the following axioms for any $\lambda, \lambda_1, \lambda_2, \mu, \mu_1, \mu_2 \in L^X$:
(T1) $\eta(\overline{1}, \overline{1}) = \eta(\overline{0}, \overline{0}) = 1$.
(T2) If $\eta(\lambda, \mu) \neq 0$, then $\lambda \leq \mu$.
(T3) If $\lambda \leq \lambda_1$ and $\mu_1 \leq \mu$, then $\eta(\lambda_1, \mu_1) \leq \eta(\lambda, \mu)$.
(T4) $\eta(\lambda_1 \vee \lambda_2, \mu) \geq \eta(\lambda_1, \mu) \odot \eta(\lambda_2, \mu)$.
(T5) $\eta(\lambda, \mu_1 \odot \mu_2) \geq \eta(\lambda, \mu_1) \odot \eta(\lambda, \mu_2)$.
(T6) $\eta \leq \eta \circ \eta$ where, for any $\lambda, \mu \in L^X$,

$$\eta \circ \eta(\lambda, \mu) = \bigvee_{\nu \in L^X} (\eta(\lambda, \nu) \odot \eta(\nu, \mu)).$$

The pair (X, η) is called the $(L, \odot)$-*smooth topogenous space*.

The $(L, \odot)$-smooth topogenous structure η is called *symmetric* if $\eta = \eta^s$ where

$$\eta^s(\lambda, \mu) = \eta(\mu', \lambda'), \ \ \forall \lambda, \mu \in L^X.$$

Let η_1 and η_2 be two $(L,\odot)$-smooth topogenous structures on X. η_1 is finer than η_2 (η_2 is coarser than η_1), denoted by $\eta_2 \leq \eta_1$, if $\eta_2(\lambda,\mu) \leq \eta_1(\lambda,\mu)$ for each $\lambda,\mu \in L^X$.

Remark 3.2. Let (X,η) be an $(L,\odot)$-smooth topogenous space. Then
(1) From (T3),(T4) and (T5), we have the following conditions:
(T4)′ $\eta(\lambda_1 \vee \lambda_2,\mu) = \eta(\lambda_1,\mu) \odot \eta(\lambda_2,\mu)$.
(T5)′ $\eta(\lambda,\mu_1 \odot \mu_2) = \eta(\lambda,\mu_1) \odot \eta(\lambda,\mu_2)$.
(2) If $L = [0,1]$ and $\odot = \wedge$ we put $\eta_r = \{(\lambda,\mu) \in L^X \times L^X \mid \eta(\lambda,\mu) > r\}$ for each $r \in L - \{1\}$. Define $(\lambda,\mu) \in \eta_r$ iff $\lambda \preceq \mu$. Then η_r is a Katsaras fuzzy topogenous structure on X.

Example 3.3. We define a function $\eta : L^X \times L^X \to L$, where $L = [0,1]$ and $\odot = \wedge$ as follows

$$\eta(\lambda,\mu) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \mu = \overline{1}, \\ \frac{2}{3}, & \text{if } \overline{0} \neq \lambda \leq \mu \neq \overline{1}, \\ 0, & \text{otherwise.} \end{cases}$$

Then η is an $(L,\odot)$-smooth topogenous structure on X because for $\overline{0} \neq \lambda \leq \mu \neq \overline{1}$, $\eta \circ \eta(\lambda,\mu) \geq \eta(\lambda,\mu) \wedge \eta(\mu,\mu) = \frac{2}{3}$, other cases and other conditions are easily proved.

Theorem 3.4. *Let $(X,\mathcal{U})$ be an $(L,\odot)$-smooth quasi-uniform space. Define*

$$\eta_{\mathcal{U}}(\mu,\rho) = \bigvee \{\mathcal{U}(f) \mid f(\mu) \leq \rho\}.$$

Then $(X,\eta_{\mathcal{U}})$ is an $(L,\odot)$-smooth topogenous space. If $(X,\mathcal{U})$ is an $(L,\odot)$-smooth uniform space, $(X,\eta_{\mathcal{U}})$ is a symmetric $(L,\odot)$-smooth topogenous space.

Proof. (T1) There exists $f \in \Omega_X$ such that $\mathcal{U}(f) = 1$. Since $f(\overline{1}) = \overline{1}$ and $f(\overline{0}) = \overline{0}$, $\eta_{\mathcal{U}}(\overline{1},\overline{1}) = \eta_{\mathcal{U}}(\overline{0},\overline{0}) = 1$.
(T2) If $\eta_{\mathcal{U}}(\mu,\rho) \neq 0$, then there exists $f \in \Omega_X$ such that $\mathcal{U}(f) \neq 0$ and $f(\mu) \leq \rho$. It implies $\mu \leq \rho$.
(T3) If $\lambda \leq \lambda_1$ and $\mu_1 \leq \mu$, then for each $f \in \Omega_X$ with $f(\lambda_1) \leq \mu_1$, we have $f(\lambda) \leq f(\lambda_1) \leq \mu_1 \leq \mu$. Thus, $\eta_{\mathcal{U}}(\lambda_1,\mu_1) \leq \eta_{\mathcal{U}}(\lambda,\mu)$.
(T4) Suppose there exist $\lambda_1,\lambda_2,\mu \in L^X$ such that

$$\eta_{\mathcal{U}}(\lambda_1 \vee \lambda_2,\mu) \ngeq \eta_{\mathcal{U}}(\lambda_1,\mu) \odot \eta_{\mathcal{U}}(\lambda_1,\mu).$$

From the definition of $\eta_{\mathcal{U}}(\lambda_i,\mu)$ for $i \in \{1,2\}$, there exists $f_i \in \Omega_X$ with $f_i(\lambda_i) \leq \mu$ such that

$$\eta_{\mathcal{U}}(\lambda_1 \vee \lambda_2,\mu) \ngeq \mathcal{U}(f_1) \odot \mathcal{U}(f_2).$$

On the other hand, since $(f_1 \odot f_2)(\lambda_1 \vee \lambda_2) \leq f_1(\lambda_1) \vee f_2(\lambda_2) \leq \mu$,

$$\eta_{\mathcal{U}}(\lambda_1 \vee \lambda_2,\mu) \geq \mathcal{U}(f_1 \odot f_2).$$

Since $\mathcal{U}(f_1 \odot f_2) \geq \mathcal{U}(f_1) \odot \mathcal{U}(f_2)$, it is a contradiction. Thus, $\eta_{\mathcal{U}}(\lambda_1 \vee \lambda_2,\mu) \geq \eta_{\mathcal{U}}(\lambda_1,\mu) \odot \eta_{\mathcal{U}}(\lambda_1,\mu)$.
(T5) Suppose there exist $\lambda,\mu_1,\mu_2 \in L^X$ such that

$$\eta_{\mathcal{U}}(\lambda,\mu_1 \odot \mu_2) \ngeq \eta_{\mathcal{U}}(\lambda,\mu_1) \odot \eta_{\mathcal{U}}(\lambda,\mu_2).$$

From the definition of $\eta_{\mathcal{U}}(\lambda,\mu_i)$ for $i \in \{1,2\}$, there exists $f_i \in \Omega_X$ with $f_i(\lambda) \le \mu_i$ such that

$$\eta_{\mathcal{U}}(\lambda,\mu_1 \odot \mu_2) \not\geq \mathcal{U}(f_1) \odot \mathcal{U}(f_2).$$

Since $(f_1 \odot f_2)(\lambda) \le f_1(\lambda) \odot f_2(\lambda) \le \mu_1 \odot \mu_2$, $\eta_{\mathcal{U}}(\lambda,\mu_1 \odot \mu_2) \ge \mathcal{U}(f_1 \odot f_2)$. Since $\mathcal{U}(f_1 \odot f_2) \ge \mathcal{U}(f_1) \odot \mathcal{U}(f_2)$, it is a contradiction.

(T6) Suppose there exist $\mu, \rho \in L^X$ such that

$$\eta_{\mathcal{U}} \circ \eta_{\mathcal{U}}(\mu,\rho) \not\geq \eta_{\mathcal{U}}(\mu,\rho).$$

From the definition of $\eta_{\mathcal{U}}(\mu,\rho)$, there exists $f \in \Omega_X$ with $f(\mu) \le \rho$ such that

$$\eta_{\mathcal{U}} \circ \eta_{\mathcal{U}}(\mu,\rho) \not\geq \mathcal{U}(f).$$

Since $\bigvee\{\mathcal{U}(g) \mid g \circ g \le f\} \ge \mathcal{U}(f)$, there exists $g \in \Omega_X$ with $g \circ g(\mu) \le f(\mu) \le \rho$ such that

$$\eta_{\mathcal{U}} \circ \eta_{\mathcal{U}}(\mu,\rho) \not\geq \mathcal{U}(g).$$

On the other hand, since $g(\mu) = f(\mu)$ and $g \circ g(\mu) \le \rho$, we have

$$\eta_{\mathcal{U}}(\mu, g(\mu)) \ge \mathcal{U}(g), \quad \eta_{\mathcal{U}}(g(\mu),\rho) \ge \mathcal{U}(g).$$

Hence $\eta_{\mathcal{U}} \circ \eta_{\mathcal{U}}(\mu,\rho) \ge \mathcal{U}(g)$. It is a contradiction.

Let $(X,\mathcal{U})$ be an $(L,\odot)$-smooth uniform space. From Lemma 2.6(5) , since $f(\mu) \le \rho$ iff $f^{-1}(\rho') \le \mu'$ also $\mathcal{U}(f) = \mathcal{U}(f^{-1})$, we have $\eta_{\mathcal{U}} = \eta^s_{\mathcal{U}}$. Hence $(X,\eta_{\mathcal{U}})$ is a symmetric $(L,\odot)$-smooth topogenous space. □

Lemma 3.5. *Let (X,η) be an $(L,\odot)$-smooth topogenous space. Let*

$$\eta^0 = \{(\mu,\rho) \in L^X \times L^X \mid \eta(\mu,\rho) \ne 0\}.$$

For every $(\mu,\rho) \in \eta^0$, we define $f_{\mu,\rho} : L^X \to L^X$ as follows:

$$f_{\mu,\rho}(\lambda) = \begin{cases} \overline{0} & \text{if } \lambda = \overline{0}, \\ \rho & \text{if } \overline{0} \ne \lambda \le \mu, \\ \overline{1} & \text{otherwise.} \end{cases}$$

Then we have the following statements.

(1) *$f_{\mu,\rho} \in \Omega_X$.*

(2) *If $\lambda \le \mu$, $\nu \le \rho$ and $f_{\mu,\nu} \in \Omega_X$, then $f_{\mu,\nu} \le f_{\lambda,\rho}$.*

(3) *For each $f_{\mu,\rho}$, there exists $\nu \in L^X$ such that $f_{\nu,\rho} \circ f_{\mu,\nu} = f_{\mu,\rho}$.*

(4) *If (X,η) is a symmetric $(L,\odot)$-smooth topogenous space and $f_{\mu,\rho} \in \Omega_X$, then $(f_{\mu,\rho})^{-1} = f_{\rho',\mu'}$.*

(5) *For each $i = 1,...,n$, f_{μ_i,ρ_i} with $(\mu_i,\rho_i) \in \eta^0$, denote*

$$\Gamma = \left\{ J \subseteq \{1,...,n\} \mid \lambda \le \bigvee_{j \in J} \mu_j \right\}$$

and put $\tau_J = \bigvee_{j\in J} \rho_j$ for any nonempty subset J of $\{1,...,n\}$. Then

$$\wedge_{i=1}^n f_{\mu_i,\rho_i}(\lambda) = \begin{cases} \overline{0} & \text{if } \lambda = \overline{0}, \\ \bigwedge_{J\in\Gamma} \tau_J & \text{if } \Gamma \ne \varnothing, \\ \overline{1} & \text{if } \Gamma = \varnothing. \end{cases}$$

Proof. (1) From the definition of $f_{\mu,\rho}$, we have $f_{\mu,\rho}(\overline{0}) = \overline{0}$. If $\overline{0} \neq \lambda \leq \mu$, then $f_{\mu,\rho}(\lambda) = \rho$. Since $(\mu,\rho) \in \eta^0$, that is, $\eta(\mu,\rho) \neq 0$, by (T2), $\mu \leq \rho$. Hence $\lambda \leq f_{\mu,\rho}(\lambda)$. If $\lambda \not\leq \mu$, then $\lambda \leq f_{\mu,\rho}(\lambda) = \overline{1}$. It follows that $\lambda \leq f_{\mu,\rho}(\lambda)$. Finally, we easily show that $f_{\mu,\rho}(\bigvee_{i\in\Gamma} \nu_i) = \bigvee_{i\in\Gamma} f_{\mu,\rho}(\nu_i)$ from the following two conditions:

(a) $\bigvee_{i\in\Gamma} \nu_i \leq \mu$ iff $\nu_i \leq \mu$ for all $i \in \Gamma$,

(b) $\bigvee_{i\in\Gamma} \nu_i \not\leq \mu$ iff $\nu_i \not\leq \mu$ for some $i \in \Gamma$.

Hence $f_{\mu,\rho} \in \Omega_X$.

(2) From definitions of $f_{\mu,\nu}$ and $f_{\lambda,\rho}$, it is trivial.

(3) From (T6), since $\eta \circ \eta(\mu,\rho) = \bigvee_{\nu\in L^X}(\eta(\mu,\nu) \odot \eta(\nu,\rho)) \geq \eta(\mu,\rho) \neq 0$, there exists $\nu \in L^X$ such that $\eta(\mu,\nu) \neq 0$ and $\eta(\nu,\rho) \neq 0$. Hence $f_{\mu,\nu}, f_{\nu,\rho} \in \Omega_X$. Moreover, it is easily proved $f_{\nu,\rho} \circ f_{\mu,\nu}(\lambda) = f_{\mu,\rho}(\lambda)$ for any $\lambda \in L^X$.

(4) Since (X,η) is a symmetric $(L,\odot)$-smooth topogenous space and $f_{\mu,\rho} \in \Omega_X$, then $\eta(\mu,\rho) = \eta(\rho',\mu') \neq 0$. It follows that $f_{\rho',\mu'} \in \Omega_X$. We show that $(f_{\mu,\rho})^{-1}(\lambda) = f_{\rho',\mu'}(\lambda)$ for all $\lambda \in L^X$ from the following statements (a), (b) and (c):

(a) If $\lambda = \overline{0}$,then $(f_{\mu,\rho})^{-1}(\overline{0}) = \bigwedge\{\nu \mid f_{\mu,\rho}(\nu') \leq \overline{1}\} = \overline{0} = f_{\rho',\mu'}(\overline{0})$.

(b) If $\overline{0} \neq \lambda \leq \rho'$, then, by the definition of $f_{\mu,\rho}$, we have

$$f_{\mu,\rho}(\nu') \leq \lambda' \text{ iff } f_{\mu,\rho}(\nu') \leq \rho \text{ iff } \nu' \leq \mu.$$

Hence

$$\begin{aligned}(f_{\mu,\rho})^{-1}(\lambda) &= \bigwedge\{\nu \in L^X \mid f_{\mu,\rho}(\nu') \leq \lambda'\} \\ &= \bigwedge\{\nu \in L^X \mid \nu \geq \mu'\} \\ &= \mu' \\ &= f_{\rho',\mu'}(\lambda).\end{aligned}$$

(c) If $\lambda \not\leq \rho'$ and $f_{\mu,\rho}(\nu') \leq \lambda'$, then , by the definition of $f_{\mu,\rho}$, we only have $f_{\mu,\rho}(\nu') = \overline{0}$. It implies that $\nu = \overline{1}$. Hence $(f_{\mu,\rho})^{-1}(\lambda) = f_{\rho',\mu'}(\lambda) = \overline{1}$.

(5) If $\lambda = \overline{0}$ or $\Gamma = \varnothing$, then it is trivial. We only show that for $\Gamma \neq \varnothing$, $\wedge_{i=1}^n f_{\mu_i,\rho_i}(\lambda) = \bigwedge_{J\in\Gamma} \tau_J$.

Suppose $\wedge_{i=1}^n f_{\mu_i,\rho_i}(\lambda) \not\leq \bigwedge_{J\in\Gamma} \tau_J$. Since $\Gamma \neq \varnothing$, there exist $J \in \Gamma$ with $\lambda \leq \bigvee_{j\in J} \mu_j$ such that

$$\bigwedge_{i=1}^n f_{\mu_i,\rho_i}(\lambda) \not\leq \tau_J.$$

Put for $i \in \{1,...,n\}$,

$$\lambda_i = \begin{cases} \lambda \odot \mu_i & \text{if } i \in J, \\ \overline{0} & \text{otherwise}. \end{cases}$$

Since $\lambda = \bigvee_{i\in J} \lambda_i$ and $\lambda_i \leq \mu_i$ for all $i \in J$, we obtain

$$\bigwedge_{i=1}^n f_{\mu_i,\rho_i}(\lambda) \leq \bigvee_{i=1}^n f_{\mu_i,\rho_i}(\lambda_i) \leq \bigvee_{i\in J} \rho_i = \tau_J.$$

It is a contradiction. Hence $\wedge_{i=1}^n f_{\mu_i,\rho_i}(\lambda) \leq \bigwedge_{J\in\Gamma} \tau_J$.

Suppose $\bigwedge_{i=1}^n f_{\mu_i,\rho_i}(\lambda) \not\geq \bigwedge_{J\in\Gamma} \tau_J$. There exist $\lambda_i \in L^X$ with $\lambda = \bigvee_{i=1}^n \lambda_i$ such that

$$\left(\bigvee_{i=1}^n f_{\mu_i,\rho_i}(\lambda_i)\right) \not\geq \bigwedge_{J\in\Gamma} \tau_J.$$

Put $\nu = \bigvee_{i=1}^{n} f_{\mu_i,\rho_i}(\lambda_i)$ and $K = \{k \in \{1,...,n\} \mid \rho_k \leq \nu\}$. We obtain $\tau_K \leq \nu$. If $i \notin K$, then $\rho_i \not\leq \nu$. Hence $f_{\mu_i,\rho_i}(\lambda_i) = \overline{0}$, which implies $\lambda_i = \overline{0}$.

If $k \in K$, then $\lambda_k \leq \mu_k$ because $f_{\mu_k,\rho_k}(\lambda_k) \neq \overline{1}$. It implies that

$$\lambda = \bigvee_{i=1}^{n} \lambda_i = \bigvee_{k\in K} \lambda_k \leq \bigvee_{k\in K} \mu_k.$$

Then there exists $K \in \Gamma$ such that

$$\bigvee_{i=1}^{n} f_{\mu_i,\rho_i}(\lambda_i) = \nu \geq \tau_K \geq \bigwedge_{K\in\Gamma} \tau_K.$$

It is a contradiction. Hence $\bigwedge_{i=1}^{n} f_{\mu_i,\rho_i}(\lambda) \geq \bigwedge_{J\in\Gamma} \tau_J$. □

Example 3.6. For each $i = 1, 2$, f_{μ_i,ρ_i} with $(\mu_i, \rho_i) \in \eta^0$, we have

$$f_{\mu_1,\rho_1} \odot f_{\mu_2,\rho_2}(\lambda) = \begin{cases} \overline{0} & \text{if } \lambda = \overline{0}, \\ \rho_1 \odot \rho_2 & \text{if } \overline{0} \neq \lambda \leq \mu_1 \wedge \mu_2, \\ \rho_1 & \text{if } \lambda \leq \mu_1,\ \lambda \not\leq \mu_2, \\ \rho_2 & \text{if } \lambda \leq \mu_2,\ \lambda \not\leq \mu_1 \\ \rho_1 \vee \rho_2 & \text{if } \lambda \leq \mu_1 \vee \mu_2,\ \lambda \not\leq \mu_1,\ \lambda \not\leq \mu_2, \\ \overline{1} & \text{otherwise.} \end{cases}$$

Remark 3.7. Let (X, η) be an $(L, \odot)$-smooth-fuzzy topogenous space. Define a function $\mathcal{U}_\eta : \Omega_X \to L$ by

$$\mathcal{U}_\eta(f) = \bigvee \left\{ \bigwedge_{i=1}^{n} \eta(\mu_i, \rho_i) \mid \wedge_{i=1}^{n} f_{\mu_i,\rho_i} \leq f \right\}.$$

where the $\bigvee$ is taken over every finite family $\{f_{\mu_i,\rho_i} \mid i = 1, ..., n\}$. Then $\mathcal{U}_\eta$ is an $(L, \odot)$-smooth quasi-uniformity on X. If (X, η) is a symmetric $(L, \odot)$-smooth topogenous space, $\mathcal{U}_\eta$ is an $(L, \odot)$-smooth uniformity on X.

Definition 3.8. The $(L, \odot)$-smooth quasi-uniform space $(X, \mathcal{U})$ is said to be compatible with $(L, \odot)$-smooth topogenous space (X, η) if $\eta_{\mathcal{U}} = \eta$.

The class $\Pi(\eta)$ denotes the family of all $(L, \odot)$-smooth quasi-uniformities which are compatible with a given $(L, \odot)$-smooth topogenous structure η.

Theorem 3.9. *Let (X, η) be an $(L, \odot)$-smooth topogenous space and the $(L, \odot)$-smooth topogenous structure $\eta_{\mathcal{U}_\eta}$ induced by $\mathcal{U}_\eta$. Then we have:*

(1) *$\eta_{\mathcal{U}_\eta} = \eta$, that is, $\mathcal{U}_\eta \in \Pi(\eta)$.*
(2) *$\mathcal{U}_\eta$ is the coarsest member of $\Pi(\eta)$.*

Proof. (1) First, we will show that $\eta_{\mathcal{U}_\eta} \geq \eta$. If $\eta(\mu, \rho) = 0$, then it is trivial. If $\eta(\mu, \rho) \neq 0$, then by Lemma 3.5(1), there exists $f_{\mu,\rho} \in \Omega_X$ such that $\mathcal{U}_\eta(f_{\mu,\rho}) \geq \eta(\mu, \rho)$ from Remark 3.7. It follows that $f_{\mu,\rho}(\mu) = \rho$, from Theorem 3.4, $\eta_{\mathcal{U}_\eta}(\mu, \rho) \geq \mathcal{U}_\eta(f_{\mu,\rho})$. Hence $\eta_{\mathcal{U}_\eta} \geq \eta$.

Suppose that $\eta_{\mathcal{U}_\eta} \not\leq \eta$. Then there exist $\mu, \rho \in L^X$ such that

$$\eta_{\mathcal{U}_\eta}(\mu, \rho) \not\leq \eta(\mu, \rho). \tag{3.1}$$

From the definition of $\eta_{\mathcal{U}_\eta}(\mu,\rho)$, there exists $f \in \Omega_X$ with $f(\mu) \le \rho$ such that

$$\mathcal{U}_\eta(f) \not\le \eta(\mu,\rho).$$

From the definition of $\mathcal{U}_\eta$, there exists a finite family $\{f_{\mu_i,\rho_i} \mid \bigwedge_{i=1}^m f_{\mu_i,\rho_i} \le f\}$ such that

$$\bigwedge_{i=1}^{m} \eta(\mu_i,\rho_i) \not\le \eta(\mu,\rho). \tag{3.2}$$

On the other hand, put $\Gamma = \{J \subseteq \{1,...,m\} \mid \mu \le \bigvee_{j\in J}\mu_j\}$. If $\Gamma = \varnothing$, then $\bigwedge_{i=1}^m f_{\mu_i,\rho_i}(\mu) = \overline{1} \le \rho$. Thus, $\rho = \overline{1}$, and $\eta(\mu,\rho) \ge \eta(\overline{1},\overline{1}) = 1$. It is a contradiction for the equation (3.1). If $\rho = \overline{0}$, by $\eta_{f_\eta}(\mu,\rho) \ne 0$ and (T2), $\mu = \overline{0}$. Hence $\eta(\overline{0},\overline{0}) = 1$. It is a contradiction for the equation (3.1). If $\Gamma \ne \varnothing$ and $\rho \ne \overline{0}$, by Lemma 3.5(5), then there exists $\Gamma = \{J \subseteq \{1,...,m\} \mid \mu \le \bigvee_{j\in J}\mu_j\}$ such that

$$\bigwedge_{i=1}^{m} f_{\mu_i,\rho_i}(\mu) = \bigwedge_{J\in\Gamma} \tau_J \le \rho.$$

Hence $\rho \ge \bigwedge_{J\in\Gamma}(\vee_{j\in J}\rho_j)$. Moreover, we have $\mu \le \wedge_{J\in\Gamma}(\vee_{j\in J}\mu_j)$. Since

$$\eta(\vee_{j\in J}\mu_j, \vee_{j\in J}\rho_j) \ge \bigwedge_{i=1}^{m} \eta(\mu_i,\rho_i),$$

we have

$$\eta(\mu,\rho) \ge \eta(\bigwedge_{J\in\Gamma}(\vee_{j\in J}\mu_j), \bigwedge_{J\in\Gamma}(\vee_{j\in J}\rho_j)) \ge \bigwedge_{i=1}^{m} \eta(\mu_i,\rho_i).$$

It is a contradiction for the equation (3.2). Therefore $\eta \ge \eta_{\mathcal{U}_\eta}$.

(2) By (1), we have that $\mathcal{U}_\eta$ is compatible with η. Let $\mathcal{U}$ be an arbitrary member of $\Pi(\eta)$. We will show that $\mathcal{U}_\eta(f) \le \mathcal{U}(f)$ for all $f \in \Omega_X$.

Suppose that there exists $f \in \Omega_X$ such that

$$\mathcal{U}_\eta(f) \not\le \mathcal{U}(f).$$

There exists a finite family $\{f_{\mu_i,\rho_i} \mid \bigwedge_{i=1}^m f_{\mu_i,\rho_i} \le f\}$ such that

$$\bigwedge_{i=1}^{m} \eta(\mu_i,\rho_i) \not\le \mathcal{U}(f).$$

Since $\mathcal{U} \in \Pi(\eta)$, that is, $\eta(\mu_i,\rho_i) = \eta_{\mathcal{U}}(\mu_i,\rho_i)$ for $i = 1,...,m$, there exists $g_i \in \Omega_X$ with $g_i(\mu_i) \le \rho_i$ such that

$$\bigwedge_{i=1}^{m} \mathcal{U}(g_i) \not\le \mathcal{U}(f). \tag{3.3}$$

On the other hand, put $g = \bigwedge_{i=1}^m g_i$. Since $g_i(\mu_i) \le \rho_i$, by the definition of f_{μ_i,ρ_i}, we have $g_i \le f_{\mu_i,\rho_i}$. It follows that

$$g = \bigwedge_{i=1}^{m} g_i \le \bigwedge_{i=1}^{m} f_{\mu_i,\rho_i} \le f.$$

Hence $\mathcal{U}(f) \ge \mathcal{U}(g) \ge \bigwedge_{i=1}^m \mathcal{U}(g_i)$. It is a contradiction for the equation (3.3). □

Example 3.10. Define a function $\eta : L^X \times L^X \to L$, where $L = [0,1]$ and $\odot = \wedge$ as follows:

$$\eta(\lambda,\mu) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \mu = \overline{1}, \\ \frac{2}{3} & \text{if } \overline{0} \neq \lambda \leq \chi_{\{x\}},\ \overline{1} \neq \mu \geq \chi_{\{x\}}, \\ 0 & \text{otherwise} \end{cases}$$

where χ_A is a characteristic function of A. Then (X, η) is L-fuzzy topogenous space. From Remark 3.7, we can obtain a quasi-uniformity $\mathcal{U}_\eta : \Omega_X \to L$ on X as follows:

$$\mathcal{U}_\eta(f) = \begin{cases} 1 & \text{if } f = f_{\overline{1},\overline{1}}, \\ \frac{2}{3} & \text{if } f_{\chi_{\{x\}},\chi_{\{x\}}} \leq f \neq f_{\overline{1},\overline{1}}, \\ 0 & \text{otherwise.} \end{cases}$$

If $\overline{0} \neq \lambda \leq \chi_{\{x\}}$ and $\overline{1} \neq \mu \geq \chi_{\{x\}}$, then, by Lemma 3.5(2), $f_{\chi_{\{x\}},\chi_{\{x\}}} \leq f_{\lambda,\mu}$. Hence $\eta_{\mathcal{U}_\eta}(\lambda,\mu) = \frac{2}{3}$. By a similar method, we have $\eta_{\mathcal{U}_\eta} = \eta$.

References

[1] G. Artico and R. Moresco, Fuzzy proximities and totally bounded fuzzy uniformities, J. Math. Anal. Appl. 99 (1984) 320–337.

[2] G. Artico and R. Moresco, Fuzzy proximities compatible with Lowen fuzzy uniformities, Fuzzy Sets and Systems 21 (1987) 85–98.

[3] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.

[4] U. Höhle and A. P. Šostak, Axiomatic foundation of fixed -basis fuzzy topology, Chapter 3, in Mathematics of fuzzy sets, Logic, Topology and Measure Theory, Handbook of fuzzy set series, Kluwer Academic Publisher, Dordrecht 1999.

[5] B. Hutton , Uniformities on fuzzy topological spaces, J. Math. Anal. Appl. 58 (1997) 559–571.

[6] A. K. Katsaras, On fuzzy proximity spaces, J. Math. Anal.Appl. 75 (1980) 571–583.

[7] A. K. Katsaras, Fuzzy proximity spaces, J. Math. Anal. Appl. 68 (1979) 100–110.

[8] A. K. Katsaras, Fuzzy syntopogenous structures compatible with Lowen fuzzy uniformities and Artico-Moresco fuzzy proximities, Fuzzy Sets and Systems 36 (1990) 375–393.

[9] A. K. Katsaras, Operations on fuzzy syntopogenous structures, Fuzzy sets and Systems 43 (1991) 199–217.

[10] A. K. Katsaras and C. G. Petalas, A unified theory of fuzzy topologies, fuzzy proximities and fuzzy uniformities, Rev. Roumaine. Math. Pures Appl. 28 (1983) 845–856.

[11] A. K. Katsaras and C. G. Petalas, On fuzzy syntopogenous structures, J. Math. Anal. Appl. 99 (1984) 219–236.

[12] Y. C. Kim and J. W. Park, $(L, \odot)$-filters and $(L, \odot)$-fuzzy topologies, Int. J. Contemp. Math. Sci. 4 (2009) 865–872.

[13] R. Lowen, Fuzzy topological spaces and fuzzy compactness, J. Math. Anal. Appl. 56 (1976) 621–633.

[14] R. Lowen, Fuzzy uniform spaces, J. Math. Anal. Appl. 82 (1981) 370–385.

[15] A. A. Ramadan, Smooth topological spaces, Fuzzy Sets and Systems 48 (1992) 371–375.

[16] A. A. Ramadan, Smooth pretopogenous structures, Fuzzy Sets and Systems 86 (1997) 381–389.

[17] A. A. Ramadan, M. El-dardery and M. A. Abdel-Sattar, L-fuzzy syntopogeneous structures, Int. J. Math. Comput. 4 (2009) 81–91.

[18] A. A. Ramadan, M. A. Abdel-Sattar, M. El-dardery and Y. C. Kim, L-fuzzy topogenous spaces and L-fuzzy quasi-uniform spaces, J. Fuzzy Math. 11 (2003) 125–138.

[19] S. K. Samanta, Fuzzy proximities and fuzzy uniformities, Fuzzy Sets and Systems 70 (1995) 97–105.

[20] A. P. Šostak, On a fuzzy topological structure, Suppl. Rend. Circ. Mat Palermo Ser. II, 11 (1985) 89–103.

[21] A. P. Šostak, Fuzzy syntopogenous structures, Quaest. Math. 20 (1997) 431–461.

[22] Y. Yue and Fu- Guishi, On (L, M)-fuzzy quasi-uniform spaces, Fuzzy Sets and Systems 158 (2007) 1472–1485.

M. A. Abdel-Sattar (mabdelsattar2002@yahooo.com) – Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef, EGYPT.
Current address : King Khaled University, Faculty of Education of Girls of ABHA, K. S. A