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Ideal theory of semigroups based on the bipolar valued fuzzy set theory

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Abstract. The notions of bipolar fuzzy sub-semigroups, bipolar fuzzy left (right) ideals, bipolar fuzzy bi-ideals, bipolar fuzzy $(1, 2)$-ideals and bipolar fuzzy left (right) duo semigroups are introduced, and their relations are investigated. Characterizations of bipolar fuzzy sub-semigroups, bipolar fuzzy left (right) ideals, bipolar fuzzy bi-ideals and bipolar fuzzy $(1, 2)$-ideals are provided. It is shown that every bipolar fuzzy bi-ideal is constant in a group. Conditions for a bipolar fuzzy $(1, 2)$-ideal to be a bipolar fuzzy bi-ideal are given, and conditions for a bipolar fuzzy bi-ideal to be a bipolar fuzzy right ideal are discussed.

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Keywords: Bipolar fuzzy sub-semigroups, Bipolar fuzzy left (right) ideals, Bipolar fuzzy bi-ideals, Bipolar fuzzy $(1, 2)$-ideals, Bipolar fuzzy left (right) duo semigroups.

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1. Introduction

Hong et al. [2] and Kuroki [6, 7] have studied several properties of fuzzy left (right) ideals, fuzzy bi-ideals and fuzzy interior ideals in semigroups. For more other study on the fuzzy theory in semigroups, we refer to papers [8, 14, 16, 17]. In the paper [3], by using a set Ω, the authors defined Ω-fuzzy subsemigroups, Ω-fuzzy left (right) ideals, Ω-fuzzy bi-ideals and Ω-fuzzy interior ideals in semigroups. They described an Ω-fuzzy subsemigroup by using a fuzzy subsemigroup and vice versa, and stated how the homomorphic images and inverse images of Ω-fuzzy subsemigroups become Ω-fuzzy subsemigroups. They dealt with the notion of an Ω-fuzzy left (right) ideal (an Ω-fuzzy bi-ideal, an Ω-fuzzy interior ideal) generated by an Ω-fuzzy set in semigroups, and examined the depictions of them.

In the traditional fuzzy sets, the membership degrees of elements range over the interval $[0, 1]$. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely

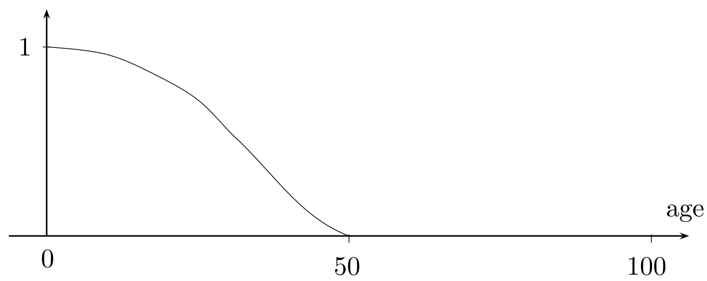

FIGURE 1. A fuzzy set ''*young*''

belongs to its corresponding fuzzy set, and the membership degree 0 indicates that an element does not belong to the fuzzy set. The membership degrees on the interval $(0, 1)$ indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set (see [1, 18]) In the viewpoint of satisfaction degree, the membership degree 0 is assigned to elements which do not satisfy some property. The elements with membership degree 0 are usually regarded as having the same characteristics in the fuzzy set representation. By the way, among such elements, some have irrelevant characteristics to the property corresponding to a fuzzy set and the others have contrary characteristics to the property. The traditional fuzzy set representation cannot tell apart contrary elements from irrelevant elements. Consider a fuzzy set "*young*" defined on the *age* domain $[0, 100]$ (see Figure 1) Now consider two ages 50 and 95 with membership degree 0. Although both of them do not satisfy the property "*young*", we may say that age 95 is more apart from the property rather than age 50 (see [12]). Only with the membership degrees ranged on the interval [0, 1], it is difficult to express the difference of the irrelevant elements from the contrary elements in fuzzy sets. If a set representation could express this kind of difference, it would be more informative than the traditional fuzzy set representation. Based on these observations, Lee [12] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets. Lee [11] applied the notion of bipolar-valued fuzzy set to BCK/BCI-algebras. She introduced the concept of bipolar fuzzy subalgebras/ideals of a BCK/BCI-algebra, and investigated several properties. She gave relations between a bipolar fuzzy subalgebra and a bipolar fuzzy ideal. She provided conditions for a bipolar fuzzy subalgebra to be a bipolar fuzzy ideal. She also gave characterizations of a bipolar fuzzy ideal, and considered the concept of equivalence relations on the family of all bipolar fuzzy ideals of a BCK/BCI-algebra. Jun et al. [4] applied the notion of bipolar-valued fuzzy set to finite state machines, and they introduced the notion of bipolar fuzzy finite state machines. Jun and Park [5] introduced the notion of a bipolar fuzzy filter and a bipolar fuzzy closed quasi filter in BCH-algebras, and investigated several properties.

In this paper, we apply the notion of bipolar-valued fuzzy set to sub-semigroups, left (right) ideals, bi-ideals, $(1, 2)$-ideals and left (right) duo semigroups in semigroups. We introduce the notion of bipolar fuzzy sub-semigroups, bipolar fuzzy left (right) ideals, bipolar fuzzy bi-ideals, bipolar fuzzy $(1,2)$-ideals and bipolar fuzzy left (right) duo semigroups. We consider the relation between these notions, and provide characterizations. We show that every bipolar fuzzy bi-ideal is constant in a group. We give conditions for a bipolar fuzzy $(1,2)$-ideal to be a bipolar fuzzy bi-ideal. We also provide conditions for a bipolar fuzzy bi-ideal to be a bipolar fuzzy right ideal.

2. Preliminaries

2.1. **Basic results on semigroups.** Let S be a semigroup. By a subsemigroup of S we mean a nonempty subset I of S such that $I^2 \subseteq I$. By a left (resp. right) ideal of S we mean a nonempty subset I of S such that $SI \subseteq I$ (resp. $IS \subseteq I$). By two-sided ideal or simply ideal, we mean a nonempty subset of S which is both a left and a right ideal of S. A subsemigroup I of S is called a bi-ideal of S if $ISI \subseteq I$. A subsemigroup I of S is called a $(1,2)$-ideal of S if $ISI^2 \subseteq I$. A semigroup S is said to be $(2,2)$-regular if $x \in x^2Sx^2$ for any $x \in S$. A semigroup S is said to be regular if, for each $x \in S$, there exists $y \in S$ such that $x = xyx$ and $xy = yx$. A semigroup S is said to be left (resp. right) duo if every left (resp. right) ideal of S is a two-sided ideal of S.

2.2. **Basic results on bipolar fuzzy sets.** Fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets etc. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0,1]$ to $[-1,1]$. Bipolar-valued fuzzy sets have membership degrees that represent the degree of satisfaction to the property corresponding to a fuzzy set and its counter-property. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on $(0,1]$ indicate that elements somewhat satisfy the property, and the membership degrees on $[-1,0)$ indicate that elements somewhat satisfy the implicit counter-property (see [12]). Figure 2 shows a bipolar-valued fuzzy set redefined for the fuzzy set "*young*" of Figure 1. The negative membership degrees indicate the satisfaction extent of elements to an implicit counter-property (e.g., old against the property young). This kind of bipolar-valued fuzzy set representation enables the elements with membership degree 0 in traditional fuzzy sets, to be expressed into the elements with membership degree 0 (irrelevant elements) and the elements with negative membership degrees (contrary elements). The age elements 50 and 95, with membership degree 0 in the fuzzy sets of Figure 1, have 0 and a negative membership degree in the bipolar-valued fuzzy set of Figure 2, respectively. Now it is manifested that 50 is an irrelevant age to the property young and 95 is more apart from the property young than 50, i.e., 95 is a contrary age to the property young (see [12]). Let S be the universe of discourse. A bipolar-valued fuzzy set f in S is an object having the form

$$f = \{(x, f_n(x), f_p(x)) \mid x \in S\}$$

where $f_n : S \to [-1,0]$ and $f_p : S \to [0,1]$ are mappings. The positive membership degree $f_p(x)$ denotes the satisfaction degree of an element x to the property

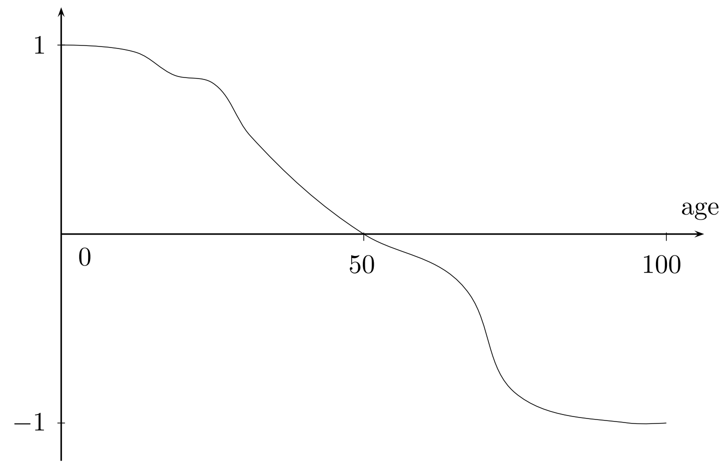

FIGURE 2. A bipolar fuzzy set ''*young*''

corresponding to a bipolar-valued fuzzy set $f = \{(x, f_n(x), f_p(x)) \mid x \in S\}$, and the negative membership degree $f_n(x)$ denotes the satisfaction degree of x to some implicit counter-property of $f = \{(x, f_n(x), f_p(x)) \mid x \in S\}$. If $f_p(x) \neq 0$ and $f_n(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $f = \{(x, f_n(x), f_p(x)) \mid x \in S\}$. If $f_p(x) = 0$ and $f_n(x) \neq 0$, it is the situation that x does not satisfy the property of $f = \{(x, f_n(x), f_p(x)) \mid x \in S\}$ but somewhat satisfies the counter-property of $f = \{(x, f_n(x), f_p(x)) \mid x \in S\}$. It is possible for an element x to be $f_p(x) \neq 0$ and $f_n(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain (see [13]). For the sake of simplicity, we shall use the symbol $f = (S; f_n, f_p)$ for the bipolar-valued fuzzy set $f = \{(x, f_n(x), f_p(x)) \mid x \in S\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

3. Bipolar fuzzy ideals

In what follows, let S denote a semigroup unless otherwise specified.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise,} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

For a bipolar fuzzy set $f = (S; f_n, f_p)$ and $(\alpha, \beta) \in [-1, 0) \times (0, 1]$, we define

$$\begin{aligned} N(f;\alpha) &:= \{x \in S \mid f_n(x) \leq \alpha\}, \\ P(f;\beta) &:= \{x \in S \mid f_p(x) \geq \beta\} \end{aligned} \tag{3.1}$$

which are called the *negative α-cut* of $f = (S; f_n, f_p)$ and the *positive β-cut* of $f = (S; f_n, f_p)$, respectively. The set

$$C(f; (\alpha, \beta)) := N(f; \alpha) \cap P(f; \beta)$$

is called the *bipolar (α, β)-cut set* of $f = (S; f_n, f_p)$. For every $k \in (0, 1)$, if $(\alpha, \beta) = (-k, k)$ then the set

$$C(f; k) := N(f; -k) \cap P(f; k)$$

is called the *bipolar k-cut set* of $f = (S; f_n, f_p)$.

Definition 3.1. A bipolar fuzzy set $f = (S; f_n, f_p)$ in S is called a *bipolar fuzzy sub-semigroup* of S if it satisfies the following condition:

$$(\forall x, y \in S) \begin{pmatrix} f_n(xy) \leq \bigvee \{f_n(x), f_n(y)\}, \\ f_p(xy) \geq \bigwedge \{f_p(x), f_p(y)\} \end{pmatrix}. \tag{3.2}$$

Let $f = (S; f_n, f_p)$ be a bipolar fuzzy sub-semigroup of S and let $(\alpha, \beta) \in [-1, 0] \times [0, 1]$ be such that $C(f; (\alpha, \beta)) \neq \varnothing$. If $x, y \in C(f; (\alpha, \beta))$, then $f_n(x) \leq \alpha$, $f_n(y) \leq \alpha$, $f_p(x) \geq \beta$ and $f_p(y) \geq \beta$. It follows from (3.2) that

$$f_n(xy) \leq \bigvee \{f_n(x), f_n(y)\} \leq \alpha$$

and

$$f_p(xy) \geq \bigwedge \{f_p(x), f_p(y)\} \geq \beta$$

so that $xy \in C(f; (\alpha, \beta))$. Therefore we have the following theorem.

Theorem 3.2. *If a bipolar fuzzy set $f = (S; f_n, f_p)$ in S is a bipolar fuzzy sub-semigroup of S, then every nonempty bipolar (α, β)-cut set of $f = (S; f_n, f_p)$ is a sub-semigroup of S where $(\alpha, \beta) \in [-1, 0] \times [0, 1]$.*

We now consider the converse of Theorem 3.2.

Theorem 3.3. *For any $(\alpha, \beta) \in [-1, 0] \times [0, 1]$, if a nonempty bipolar (α, β)-cut set of $f = (S; f_n, f_p)$ is a sub-semigroup of S, then $f = (S; f_n, f_p)$ is a bipolar fuzzy sub-semigroup of S.*

Proof. Suppose (3.2) is false. Then there exist $a, b \in S$ such that

$$f_n(ab) > \bigvee \{f_n(a), f_n(b)\}, \; f_p(ab) \geq \bigwedge \{f_p(a), f_p(b)\}, \tag{3.3}$$

$$f_n(ab) \leq \bigvee \{f_n(a), f_n(b)\}, \; f_p(ab) < \bigwedge \{f_p(a), f_p(b)\}, \tag{3.4}$$

or

$$f_n(ab) > \bigvee \{f_n(a), f_n(b)\}, \; f_p(ab) < \bigwedge \{f_p(a), f_p(b)\}. \tag{3.5}$$

Let $\alpha = \bigvee \{f_n(a), f_n(b)\}$ and $\beta = \bigwedge \{f_p(a), f_p(b)\}$. For the case (3.3), we get $a, b \in C(f; (\alpha, \beta))$ and $ab \notin C(f; (\alpha, \beta))$. This is a contradiction. Similarly, the conditions (3.4) and (3.5) induce a contradiction. Therefore (3.2) is valid, i.e., $f = (S; f_n, f_p)$ is a bipolar fuzzy sub-semigroup of S. □

Definition 3.4. A bipolar fuzzy set $f = (S; f_n, f_p)$ in S is called a bipolar fuzzy left ideal of S if it satisfies the following condition:

$$(\forall x, y \in S)\left(f_n(xy) \leq f_n(y),\ f_p(xy) \geq f_p(y)\right). \tag{3.6}$$

A bipolar fuzzy right ideal of S is defined in an analogous. A bipolar fuzzy set $f = (S; f_n, f_p)$ in S is called a bipolar fuzzy ideal of S if it is both a bipolar fuzzy left ideal and a bipolar fuzzy right ideal of S. It is clear that any bipolar fuzzy left (right) ideal is a bipolar fuzzy sub-semigroup.

Proposition 3.5. *Let $f = (S; f_n, f_p)$ be a bipolar fuzzy set in S and let (α_1, β_1), $(\alpha_2, \beta_2) \in [-1, 0]\times[0, 1]$. If $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$, then $C(f; (\alpha_1, \beta_1)) \subseteq C(f; (\alpha_2, \beta_2))$.*

Proof. Straightforward. □

Theorem 3.6. *For any bipolar fuzzy set $f = (S; f_n, f_p)$ in S and $(\alpha, \beta) \in [-1, 0] \times [0, 1]$, the following are equivalent:*

(1) *$f = (S; f_n, f_p)$ is a bipolar fuzzy left (right) ideal of S.*
(2) *Every bipolar (α, β)-cut set of $f = (S; f_n, f_p)$ is a left (right) ideal of S when it is nonempty.*

Proof. Assume that $f = (S; f_n, f_p)$ is a bipolar fuzzy left ideal of S. Let $(\alpha, \beta) \in [-1, 0] \times [0, 1]$ be such that $C(f; (\alpha, \beta)) \neq \varnothing$. Let $x \in S$ and $y \in C(f; (\alpha, \beta))$. Then $f_n(y) \leq \alpha$ and $f_p(y) \geq \beta$. It follows from (3.6) that $f_n(xy) \leq f_n(y) \leq \alpha$ and $f_p(xy) \geq f_p(y) \geq \beta$ so that $xy \in C(f; (\alpha, \beta))$. Hence $C(f; (\alpha, \beta))$ is a left ideal of S. Similarly, if $f = (S; f_n, f_p)$ is a bipolar fuzzy right ideal of S, then the nonempty bipolar (α, β)-cut set of $f = (S; f_n, f_p)$ is a right ideal of S.

Conversely, suppose that any nonempty bipolar (α, β)-cut set of $f = (S; f_n, f_p)$ is a left ideal of S. Assume that there exist $a, b \in S$ such that $f_n(ab) > f_n(b)$ or $f_p(ab) < f_p(b)$. If $f_n(ab) > f_n(b)$ and $f_p(ab) \geq f_p(b)$, then we take $\alpha = \frac{f_n(ab)+f_n(b)}{2}$. Then $f_n(ab) > \alpha > f_n(b)$, and so $b \in C\left(f; (\alpha, f_p(b))\right)$ and $ab \notin C\left(f; (\alpha, f_p(b))\right)$. Hence $C\left(f; (\alpha, f_p(b))\right)$ is not a left ideal of S. If $f_n(ab) \leq f_n(b)$ and $f_p(ab) < f_p(b)$, then $f_p(ab) < \beta < f_p(b)$ where $\beta = \frac{f_p(ab)+f_p(b)}{2}$. It follows that $b \in C\left(f; (f_n(b), \beta)\right)$ but $ab \notin C\left(f; (f_n(b), \beta)\right)$. Thus $C\left(f; (f_n(b), \beta)\right)$ is not a left ideal of S. Suppose that $f_n(ab) > f_n(b)$ and $f_p(ab) < f_p(b)$. Then $f_n(ab) > \alpha \geq f_n(b)$ and $f_p(ab) < \beta \leq f_p(b)$ for some $(\alpha, \beta) \in [-1, 0) \times (0, 1]$, and so $b \in C(f; (\alpha, \beta))$ and $ab \notin C(f; (\alpha, \beta))$. Thus $C(f; (\alpha, \beta))$ is not a left ideal of S. This is a contradiction, and therefore $f_n(xy) \leq f_n(y)$ and $f_p(xy) \geq f_p(y)$ for all $x, y \in S$. Consequently, $f = (S; f_n, f_p)$ is a bipolar fuzzy left ideal of S. For the right case, we can prove the theorem by the analogous way. □

Definition 3.7. A bipolar fuzzy set $f = (S; f_n, f_p)$ in S is called a bipolar fuzzy bi-ideal of S if it is a bipolar fuzzy sub-semigroup of S that satisfies the following condition:

$$(\forall a, x, y \in S)\begin{pmatrix} f_n(xay) \leq \bigvee\{f_n(x), f_n(y)\}, \\ f_p(xay) \geq \bigwedge\{f_p(x), f_p(y)\} \end{pmatrix}. \tag{3.7}$$

Example 3.8. Let $S := \{a, b, c, d, e\}$ be a semigroup with the multiplication table which is given by Table 1. Let $f = (S; f_n, f_p)$ be a bipolar fuzzy set in S given by

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	c	c	e
d	a	a	c	d	e
e	a	a	c	c	e

TABLE 1. Multiplication table

	a	b	c	d	e
f_n	-0.7	-0.7	-0.6	-0.5	-0.4
f_p	0.6	0.5	0.4	0.3	0.3

By routine calculations, we know that $f = (S; f_n, f_p)$ is a bipolar fuzzy bi-ideal of S.

Theorem 3.9. *Every bipolar fuzzy left (right) ideal is a bipolar fuzzy bi-ideal.*

Proof. Let $f = (S; f_n, f_p)$ be a bipolar fuzzy left ideal of S. For any $a, x, y \in S$, we have

$$f_n(xay) = f_n((xa)y) \leq f_n(y) \leq \bigvee\{f_n(x), f_n(y)\}$$

and

$$f_p(xay) = f_p((xa)y) \geq f_p(y) \geq \bigwedge\{f_p(x), f_p(y)\}.$$

Hence $f = (S; f_n, f_p)$ is a bipolar fuzzy bi-ideal of S. The right case is proved in an analogous way. □

Theorem 3.10. *For any bipolar fuzzy set $f = (S; f_n, f_p)$ in S and $(\alpha, \beta) \in [-1, 0] \times [0, 1]$, the following are equivalent:*

(1) *$f = (S; f_n, f_p)$ is a bipolar fuzzy bi-ideal of S.*
(2) *Every bipolar (α, β)-cut set of $f = (S; f_n, f_p)$ is a bi-ideal of S when it is nonempty.*

Proof. Suppose that $f = (S; f_n, f_p)$ is a bipolar fuzzy bi-ideal of S. Let $(\alpha, \beta) \in [-1, 0] \times [0, 1]$ be such that $C(f; (\alpha, \beta)) \neq \varnothing$. Since $f = (S; f_n, f_p)$ is a bipolar fuzzy sub-semigroup of S, it follows from Theorem 3.2 that $C(f; (\alpha, \beta))$ is a sub-semigroup of S. Let $x, y \in C(f; (\alpha, \beta))$ and $a \in S$. Then $f_n(x) \leq \alpha$, $f_n(y) \leq \alpha$, $f_p(x) \geq \beta$ and $f_p(y) \geq \beta$. Using (3.7), we have

$$f_n(xay) \leq \bigvee \{f_n(x), f_n(y)\} \leq \alpha$$

and

$$f_p(xay) \geq \bigwedge \{f_p(x), f_p(y)\} \geq \beta.$$

Thus $xay \in C(f; (\alpha, \beta))$, and so $C(f; (\alpha, \beta))$ is a bi-ideal of S.

Conversely, assume that any nonempty bipolar (α, β)-cut set of $f = (S; f_n, f_p)$ is a bi-ideal of S. Then it is a sub-semigroup of S, and thus $f = (S; f_n, f_p)$ is a bipolar fuzzy sub-semigroup of S by Theorem 3.3. Finally suppose that (3.7) is false. Then

$$f_n(xay) > \bigvee \{f_n(x), f_n(y)\} \text{ or } f_p(xay) < \bigwedge \{f_p(x), f_p(y)\}$$

for some $a, x, y \in S$. We can consider the following three cases:

$$\begin{aligned} f_n(xay) &> \bigvee \{f_n(x), f_n(y)\} \\ f_p(xay) &\geq \bigwedge \{f_p(x), f_p(y)\} \end{aligned} \tag{3.8}$$

$$\begin{aligned} f_n(xay) &\leq \bigvee \{f_n(x), f_n(y)\} \\ f_p(xay) &< \bigwedge \{f_p(x), f_p(y)\} \end{aligned} \tag{3.9}$$

$$\begin{aligned} f_n(xay) &> \bigvee \{f_n(x), f_n(y)\} \\ f_p(xay) &< \bigwedge \{f_p(x), f_p(y)\}. \end{aligned} \tag{3.10}$$

For the case (3.8), we take

$$\alpha := \tfrac{1}{2} \left(f_n(xay) + \bigvee \{f_n(x), f_n(y)\} \right)$$

and $\beta := \bigwedge \{f_p(x), f_p(y)\}$. Then $x, y \in C(f; (\alpha, \beta))$, but $xay \notin N(f; \alpha)$ and so $xay \notin C(f; (\alpha, \beta))$. This is a contradiction. Similarly, the case (3.9) induce a contradiction. For the case (3.10), we take

$$\alpha := \tfrac{1}{2} \left(f_n(xay) + \bigvee \{f_n(x), f_n(y)\} \right)$$

and

$$\beta := \tfrac{1}{2} \left(f_p(xay) + \bigwedge \{f_p(x), f_p(y)\} \right).$$

Then $x, y \in N(f; \alpha) \cap P(f; \beta) = C(f; (\alpha, \beta))$ but $xay \notin N(f; \alpha) \cap P(f; \beta) = C(f; (\alpha, \beta))$. This is a contradiction, and consequently (3.7) is valid. Therefore $f = (S; f_n, f_p)$ is a bipolar fuzzy bi-ideal of S. □

Theorem 3.11. *Let S be a regular semigroup in which every bi-ideal is a right ideal. Then every bipolar fuzzy bi-ideal of S is a bipolar fuzzy right ideal of S.*

Proof. Let $f = (S; f_n, f_p)$ be a bipolar fuzzy bi-ideal of S and let $x, y \in S$. Then xSx is a bi-ideal of S, and so xSx is a right ideal of S. Since S is regular, $xy \in (xSx)S \subseteq xSx$. Hence $xy = xax$ for some $a \in S$. Since $f = (S; f_n, f_p)$ is a bipolar fuzzy bi-ideal of S, it follows that

$$f_n(xy) = f_n(xax) \leq \bigvee \{f_n(x), f_n(x)\} = f_n(x)$$

and

$$f_p(xy) = f_p(xax) \geq \bigwedge \{f_p(x), f_p(x)\} = f_p(x).$$

Therefore $f = (S; f_n, f_p)$ is a bipolar fuzzy right ideal of S. □

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	b	c
c	a	b	c	a	a
d	a	a	a	d	e
e	a	d	e	a	a

TABLE 2. Multiplication table

Definition 3.12. A bipolar fuzzy set $f = (S; f_n, f_p)$ in S is called a bipolar fuzzy $(1, 2)$-ideal of S if it is a bipolar fuzzy sub-semigroup of S that satisfies the following condition:

$$(\forall a, x, y, z \in S) \left(\begin{array}{l} f_n(xa(yz)) \leq \bigvee \{f_n(x), f_n(y), f_n(z)\} \\ f_p(xa(yz)) \geq \bigwedge \{f_p(x), f_p(y), f_p(z)\} \end{array} \right). \tag{3.11}$$

Example 3.13. Let $S := \{a, b, c, d, e\}$ be a semigroup with the multiplication table which is given by Table 2. Let $f = (S; f_n, f_p)$ be a bipolar fuzzy set in S given by

	a	b	c	d	e
f_n	-0.7	-0.7	-0.7	-0.4	-0.4
f_p	1	1	1	0	0

By routine calculations, we know that $f = (S; f_n, f_p)$ is a bipolar fuzzy $(1, 2)$-ideal of S. We note that $M := \{a, b, c\}$ is a $(1, 2)$-ideal of S, hence $f = (S; f_n, f_p)$ can be redefined as follows:

$$f_n(x) = \begin{cases} -0.7 & \text{if } x \in M, \\ -0.4 & \text{otherwise,} \end{cases} \qquad f_p(x) = \begin{cases} 1 & \text{if } x \in M, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.14. *Every bipolar fuzzy bi-ideal is a bipolar fuzzy $(1, 2)$-ideal.*

Proof. Let $f = (S; f_n, f_p)$ be a bipolar fuzzy bi-ideal of S. For any $a, x, y \in S$, we have

$$\begin{aligned} f_n(xa(yz)) &= f_n((xay)z) \leq \bigvee \{f_n(xay), f_n(z)\} \\ &\leq \bigvee \left\{ \bigvee \{f_n(x), f_n(y)\}, f_n(z) \right\} \\ &= \bigvee \{f_n(x), f_n(y), f_n(z)\} \end{aligned}$$

and

$$\begin{aligned} f_p(xa(yz)) &= f_p((xay)z) \geq \bigwedge \{f_p(xay), f_p(z)\} \\ &\geq \bigwedge \left\{ \bigwedge \{f_p(x), f_p(y)\}, f_p(z) \right\} \\ &= \bigwedge \{f_p(x), f_p(y), f_p(z)\}. \end{aligned}$$

Therefore $f = (S; f_n, f_p)$ is a bipolar fuzzy $(1, 2)$-ideal of S. □

$\cdot$	a	b	c	x	y	z
a	a	a	a	a	a	a
b	a	a	a	a	a	a
c	a	a	a	a	a	a
x	a	a	a	a	a	b
y	a	a	a	a	b	c
z	a	a	b	a	x	a

TABLE 3. Multiplication table

Corollary 3.15. *Every bipolar fuzzy left (right) ideal is a bipolar fuzzy* $(1, 2)$*-ideal.*

Proof. Straightforward. □

The following example shows that the converse of Theorem 3.14 is not true in general.

Example 3.16. Consider a semigroup $S = \{a, b, c, x, y, z\}$ with the multiplication table which is given by Table 3. Let $f = (S; f_n, f_p)$ be a bipolar fuzzy set in S given by

	a	b	c	x	y	z
f_n	-0.3	-0.9	-0.9	-0.9	-0.9	-0.3
f_p	0.6	0	0	0	0	0.6

Then $f = (S; f_n, f_p)$ is a bipolar fuzzy $(1, 2)$-ideal of S but it is not a bipolar fuzzy bi-ideal of S since

$$f_n(zyz) = f_n(b) = -0.9 > -0.3 = \bigvee \{f_n(z), f_n(z)\}$$

and/or

$$f_p(zyz) = f_p(b) = 0 < 0.6 = \bigwedge \{f_p(z), f_p(z)\} .$$

In order to consider the converse of Theorem 3.14, we need to strengthen the condition of a semigroup S.

Theorem 3.17. *In a regular semigroup, every bipolar fuzzy* $(1, 2)$*-ideal is a bipolar fuzzy bi-ideal.*

Proof. Let $f = (S; f_n, f_p)$ be a bipolar fuzzy $(1, 2)$-ideal of a regular semigroup S. Since S is regular, $xa \in (xSx)S \subseteq xSx$ for all $a, x \in S$. Hence $xa = xbx$ for some $b \in S$. Thus

$$f_n(xay) = f_n((xbx)y) = f_n(xb(xy)) \leq \bigvee \{f_n(x), f_n(x), f_n(y)\} = \bigvee \{f_n(x), f_n(y)\}$$

and

$$f_p(xay) = f_p((xbx)y) = f_p(xb(xy)) \geq \bigwedge \{f_p(x), f_p(x), f_p(y)\} = \bigwedge \{f_p(x), f_p(y)\} .$$

Therefore $f = (S; f_n, f_p)$ is a bipolar fuzzy bi-ideal of S. □

Theorem 3.18. *For any bipolar fuzzy set $f = (S; f_n, f_p)$ in S and $(\alpha, \beta) \in [-1, 0] \times [0, 1]$, the following are equivalent:*

(1) *$f = (S; f_n, f_p)$ is a bipolar fuzzy $(1, 2)$-ideal of S.*
(2) *Every bipolar (α, β)-cut set of $f = (S; f_n, f_p)$ is a $(1, 2)$-ideal of S when it is nonempty.*

Proof. Suppose that $f = (S; f_n, f_p)$ is a bipolar fuzzy $(1, 2)$-ideal of S. Let $(\alpha, \beta) \in [-1, 0] \times [0, 1]$ be such that $C(f; (\alpha, \beta)) \neq \varnothing$. Then $f = (S; f_n, f_p)$ is a bipolar fuzzy sub-semigroup of S, and so $C(f; (\alpha, \beta))$ is a sub-semigroup of S by Theorem 3.2. Let $x, y, z \in C(f; (\alpha, \beta))$ and $a \in S$. Then $f_n(x) \leq \alpha$, $f_n(y) \leq \alpha$, $f_n(z) \leq \alpha$, $f_p(x) \geq \beta$, $f_p(y) \geq \beta$ and $f_p(z) \geq \beta$. Using (3.11), we have

$$f_n(xa(yz)) \leq \bigvee \{f_n(x), f_n(y), f_n(z)\} \leq \alpha$$

and

$$f_p(xa(yz)) \geq \bigwedge \{f_p(x), f_p(y), f_p(z)\} \geq \beta,$$

which imply that $xa(yz) \in C(f; (\alpha, \beta))$. Hence $C(f; (\alpha, \beta))$ is a $(1, 2)$-ideal of S.

Conversely, assume that any nonempty bipolar (α, β)-cut set of $f = (S; f_n, f_p)$ is a $(1, 2)$-ideal of S. Then it is a sub-semigroup of S, and thus $f = (S; f_n, f_p)$ is a bipolar fuzzy sub-semigroup of S by Theorem 3.3. If $f = (S; f_n, f_p)$ does not satisfy the condition (3.11), then there exist $a, x, y, z \in S$ such that

$$\begin{aligned} f_n(xa(yz)) &> \bigvee \{f_n(x), f_n(y), f_n(z)\}, \\ f_p(xa(yz)) &\geq \bigwedge \{f_p(x), f_p(y), f_p(z)\}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} f_n(xa(yz)) &\leq \bigvee \{f_n(x), f_n(y), f_n(z)\}, \\ f_p(xa(yz)) &< \bigwedge \{f_p(x), f_p(y), f_p(z)\}, \end{aligned} \tag{3.13}$$

or

$$\begin{aligned} f_n(xa(yz)) &> \bigvee \{f_n(x), f_n(y), f_n(z)\}, \\ f_p(xa(yz)) &< \bigwedge \{f_p(x), f_p(y), f_p(z)\}. \end{aligned} \tag{3.14}$$

Take $\alpha = \bigvee \{f_n(x), f_n(y), f_n(z)\}$ and $\beta = \bigwedge \{f_p(x), f_p(y), f_p(z)\}$. It follows from (3.12) that $x, y, z \in C(f; (\alpha, \beta))$ and $xa(yz) \notin C(f; (\alpha, \beta))$. This is a contradiction. Same contradictions are induced from conditions (3.13) and (3.14). Hence $f = (S; f_n, f_p)$ satisfies the condition (3.11), and therefore $f = (S; f_n, f_p)$ is a bipolar fuzzy $(1, 2)$-ideal of S. □

Theorem 3.19. *In a group, every bipolar fuzzy bi-ideal is constant.*

Proof. Let S be a group and let $f = (S; f_n, f_p)$ be a bipolar fuzzy bi-ideal of S. For any $x \in S$, we have

$$\begin{aligned} f_n(x) &= f_n(exe) \leq \bigvee \{f_n(e), f_n(e)\} = f_n(e) \\ &= f_n(ee) = f_n\left((xx^{-1})(x^{-1}x)\right) = f_n\left(x\left(x^{-1}x^{-1}\right)x\right) \\ &\leq \bigvee \{f_n(x), f_n(x)\} = f_n(x) \end{aligned}$$

and

$$\begin{aligned} f_p(x) &= f_p(exe) \geq \bigwedge \{f_p(e), f_p(e)\} = f_p(e) \\ &= f_p(ee) = f_p\left((xx^{-1})(x^{-1}x)\right) = f_p\left(x\left(x^{-1}x^{-1}\right)x\right) \\ &\geq \bigwedge \{f_p(x), f_p(x)\} = f_p(x) \end{aligned}$$

where e is the identity of S. It follows that $f_n(x) = f_n(e)$ and $f_p(x) = f_p(e)$ so that $f = (S; f_n, f_p)$ is constant. □

For every $x \in S$, denote by $\langle x \rangle$ the principle bi-ideal of S generated by x. Note that $\langle x \rangle = \{x\} \cup \{x^2\} \cup xSx$.

Theorem 3.20. *Let $f = (S; f_n, f_p)$ be a bipolar fuzzy bi-ideal of S. If S is $(2,2)$-regular, then $f_n(x) = f_n\left(x^2\right)$ and $f_p(x) = f_p\left(x^2\right)$ for all $x \in S$.*

Proof. For any $x \in S$ there exists $a \in S$ such that $x = x^2ax^2$. Hence

$$f_n(x) = f_n\left(x^2ax^2\right) \leq \bigvee \left\{f_n\left(x^2\right), f_n\left(x^2\right)\right\} = f_n\left(x^2\right) \leq \bigvee \{f_n(x), f_n(x)\} = f_n(x)$$

and

$$f_p(x) = f_p\left(x^2ax^2\right) \geq \bigwedge \left\{f_p\left(x^2\right), f_p\left(x^2\right)\right\} = f_p\left(x^2\right) \geq \bigwedge \{f_p(x), f_p(x)\} = f_p(x).$$

It follows that $f_n(x) = f_n\left(x^2\right)$ and $f_p(x) = f_p\left(x^2\right)$. □

Theorem 3.21. *Let $f = (S; f_n, f_p)$ be a bipolar fuzzy bi-ideal of S. If S is a semilattice of groups, then $f_n(x) = f_n\left(x^2\right)$, $f_p(x) = f_p\left(x^2\right)$, $f_n(xy) = f_n(yx)$ and $f_p(xy) = f_p(yx)$ for all $x, y \in S$.*

Proof. Since S is a semilattice of groups, S is a union of groups. Hence S is $(2,2)$-regular. It follows from Theorem 3.20 that $f_n(x) = f_n\left(x^2\right)$ and $f_p(x) = f_p\left(x^2\right)$ for all $x \in S$. Let $x, y \in S$. Note that S is a semilattice of groups if and only if the set of all bi-ideals of S is a semilattice under the multiplication of subsets. Hence

$$\begin{aligned} (xy)^3 = (xyx)(yxy) &\in \langle xyx \rangle \langle yxy \rangle = \langle yxy \rangle (\langle xyx \rangle)^2 \\ &\subseteq \langle yxy \rangle S \langle xyx \rangle \subseteq yxySxyx \subseteq yxSyx. \end{aligned}$$

It follows that there exists $a \in S$ such that $(xy)^3 = (yx)a(yx)$. Hence

$$f_n(xy) = f_n\left((xy)^3\right) = f_n((yx)a(yx)) \leq \bigvee \{f_n(yx), f_n(yx)\} = f_n(yx)$$

and

$$f_p(xy) = f_p\left((xy)^3\right) = f_p((yx)a(yx)) \geq \bigwedge \{f_p(yx), f_p(yx)\} = f_p(yx).$$

Similarly, we have $f_n(yx) \leq f_n(xy)$ and $f_p(yx) \geq f_p(xy)$. This completes the proof. □

Definition 3.22. A semigroup S is said to be bipolar fuzzy left (right) duo if every bipolar fuzzy left (right) ideal of S is a bipolar fuzzy ideal of S.

Theorem 3.23. *Let S be a regular semigroup. If S is left (right) duo, then S is bipolar fuzzy left (right) duo.*

Proof. Assume that S is left duo and let $f = (S; f_n, f_p)$ be a bipolar fuzzy left ideal of S. Let $x, y \in S$. Since the left ideal Sx is a two-sided ideal of S and S is regular, we have $xy \in (xSx)y \subseteq (Sx)S \subseteq Sx$. It follows that there exists $a \in S$ such that $xy = ax$. As $f = (S; f_n, f_p)$ is a bipolar fuzzy left ideal of S, we get $f_n(xy) = f_n(ax) \leq f_n(x)$ and $f_p(xy) = f_p(ax) \geq f_p(x)$. This means that $f = (S; f_n, f_p)$ is a bipolar fuzzy right ideal of S and so S is bipolar fuzzy left duo. For the right case, we can prove the theorem by the analogous way. □

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References

[1] D. Dubois and H. Prade, Fuzzy sets and systems: Theory and Applications, Academic Press, 1980.

[2] S. M. Hong, Y. B. Jun and J. Meng, Fuzzy interior ideals in semigroups, Indian J. Pure Appl. Math. 26 (1995) 859–863.

[3] K. Hur, Y. B. Jun and H. S. Kim, Fuzzy subsemigroups and fuzzy ideals with operators in semigroups, Ann. Fuzzy Math. Inf. 1 (2011) 1–12.

[4] Y. B. Jun and J. Kavikumar, Bipolar fuzzy finite state machines, Bull. Malays. Math. Sci. Soc. 34 (2011) 181–188.

[5] Y. B. Jun and C. H. Park, Filters of BCH-algebras based on bipolar-valued fuzzy sets, Int. Math. Forum 4 (2009) 631–643.

[6] N. Kuroki, Fuzzy semiprime ideals in semigroups, Fuzzy Sets and Systems 8 (1982) 71–79.

[7] N. Kuroki, On fuzzy semigroups, Inform. Sci. 53 (1991) 203–236.

[8] N. Kuroki, Fuzzy generalized bi-ideals in semigroups, Inform. Sci. 66 (1992) 235–243.

[9] S. Lajos, A note on semilattice of groups, Acta. Sci. Math. (Szeged) 33 (1972) 315–317.

[10] S. Lajos, On generalized bi-ideals in semigroups, Coll. Math. Soc. Janos Bolyai, 20, Algebraic Theory of Semigroups, (G. Pollak, Ed.) North-Holland (1979) 335–340.

[11] K. J. Lee, Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras, Bull. Malays. Math. Sci. Soc. 32 (2009) 361–373.

[12] K. M. Lee, Bipolar-valued fuzzy sets and their operations, Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand (2000) 307–312.

[13] K. M. Lee, Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets, J. Fuzzy Logic Intelligent Systems 14 (2004) 125–129.

[14] Z. W. Mo and X. P. Wang, On pointwise depiction of fuzzy regularity of semigroups, Inform. Sci. 74 (1993) 265–274.

[15] M. Petrich, Introduction to Semigroups, Columbus, Ohio 1973.

[16] M. A. Samhan, Fuzzy congruences on semigroups, Inform. Sci. 74 (1993) 165–175.

[17] X. P. Wang and W. J. Liu, Fuzzy regular subsemigroups in semigroups, Inform. Sci. 68 (1993) 225–231.

[18] H. -J. Zimmermann, Fuzzy Set Theory and its Applications, Kluwer-Nijhoff Publishing, 1985.

[19] L. A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965) 338–353.

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