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Measures of compactness in L-topological spaces

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Abstract. The notion of fuzzy compactness degrees is introduced in L-topological spaces by means of the implication operator of L. An L-set G is fuzzy compact if and only if its fuzzy compactness degree $\mathrm{com}(G) = \top$. Some properties of fuzzy compactness degrees are investigated. In particular, when $L = [0, 1]$, it is different from corresponding notions presented by Šostak, E. Lowen and R. Lowen.

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1. Introduction

The concept of compactness of a $[0, 1]$-topological space was first introduced by Chang [1] in terms of open cover. It can be regarded as a successful definition of compactness in poslat topology from the categorical point of view (see [12, 14]). Moreover, Gantner et al. introduced α-compactness [3], Lowen introduced fuzzy compactness, strong fuzzy compactness and ultra-fuzzy compactness [9, 10], Liu introduced Q-compactness [7], Wang and Zhao introduced N-compactness [19, 21], and Shi introduced S^*-compactness [15]. Shi also present a new definition of fuzzy compactness in L-topological spaces when L is a complete De Morgan algebra [16].

For the above notions of compactness, an L-fuzzy set is either compact or not. However, considering measures of compactness, E. Lowen and R. Lowen [11] introduced the notion of the compactness degrees of $[0, 1]$-topological spaces. G. Jäger generalized it to fuzzy convergence spaces in [6]. Moreover A.P. Šostak [17, 18] also introduced a definition of the compactness degree $c(M)$ for a fuzzy set M in a $[0, 1]$-topological space.

In this paper, a new notion of fuzzy compactness degrees is introduced in L-topological spaces by means of the implication operator of L. It is different from those notions in [6, 11, 17].

2. Preliminaries

Throughout this paper, $(L, \bigvee, \bigwedge, ')$ is a complete De Morgan frame [4, 13]. The smallest element and the largest element in L are denoted by $\perp$ and $\top$, respectively.

We say that a is wedge below b in L, denoted by $a \prec b$, if for every subset $D \subseteq L$, $\bigvee D \geq b$ implies $d \geq a$ for some $d \in D$ [2]. A complete lattice L is completely distributive if and only if $b = \bigvee\{a \in L : a \prec b\}$ for each $b \in L$. For any $b \in L$, define $\beta(b) = \{a \in L : a \prec b\}$. Some properties of β can be found in [8, 20].

In a complete De Morgan frame L, there exists a binary operation $\mapsto$. Explicitly the implication is given by

$$a \mapsto b = \bigvee\{c \in L \mid a \wedge c \leq b\}.$$

It is easy to check the following properties of $\mapsto$.

(1) $(a \mapsto b) \geq c \Leftrightarrow a \wedge c \leq b$;
(2) $a \mapsto b = \top \Leftrightarrow a \leq b$;
(3) $a \mapsto (\bigwedge_i b_i) = \bigwedge_i (a \mapsto b_i)$;
(4) $(\bigvee_i a_i) \mapsto b = \bigwedge_i (a_i \mapsto b)$;
(5) $(a \mapsto c) \wedge (c \mapsto b) \leq a \mapsto b$;
(6) $a \leq b \Rightarrow c \mapsto a \leq c \mapsto b$.
(7) $a \leq b \Rightarrow b \mapsto c \leq a \mapsto c$.
(8) $(a \mapsto b) \wedge (c \mapsto d) \leq a \wedge c \mapsto b \wedge d$.

For a nonempty set X, L^X denotes the set of all L-fuzzy sets on X. $\underline{a}$ denotes the constant L-fuzzy sets on X taking the value a. An L-topological space is a pair (X, τ), where τ is a subfamily of L^X which contains $\underline{\perp}$, $\underline{\top}$ and is closed for any suprema and finite infima. Each member of τ is called an open L-set and its quasi-complement is called a closed L-set.

For a subfamily $\Phi \subseteq L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamilies of Φ.

Definition 2.1 ([17, 18])**.** An L-fuzzy inclusion on X is a mapping $\widetilde{\subset} : L^X \times L^X \to L^X$ defined by the equality $\widetilde{\subset}(A, B) = \bigwedge_{x \in X} (A'(x) \vee B(x))$.

In the sequel, we shall write $[A \widetilde{\subset} B]$ instead of $\widetilde{\subset}(A, B)$.

Lemma 2.2 ([16])**.** *Let $f : X \to Y$ be a set map and $f_L^{\to} : L^X \to L^Y$ is the extension of f. Then for any $\mathcal{P} \subseteq L^X$, we have that*

$$\bigwedge_{y \in Y} \left(f_L^{\to}(G)'(y) \vee \bigvee_{B \in \mathcal{P}} B(y) \right) = \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{B \in \mathcal{P}} f_L^{\leftarrow}(B)(x) \right).$$

3. Measures of fuzzy compactness

In order to generalize the notion of compactness to L-topological spaces, we introduced the following notion.

In an L-topological space (X,τ), an L-set $G \in L^X$ is said to be fuzzy compact [16] if for every family $\mathcal{U}$ of open L-sets, it follows that

$$\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right) \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x)\right),$$

i.e.,

$$\left[G \widetilde{\subset} \bigvee \mathcal{U}\right] \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \left[G \widetilde{\subset} \bigvee \mathcal{V}\right].$$

According to the above definition, we know that an L-fuzzy set is either compact or not. But we know that for any $a, b \in L$,

$$a \leq b \Leftrightarrow a \mapsto b = \top.$$

If we define $[a \leq b] = a \mapsto b$, then $a \leq b$ always is true to some degree.

Therefore we can naturally introduce the notion of fuzzy compactness degrees as follows:

Definition 3.1. Let (X,τ) be an L-topological space and $G \in L^X$. The fuzzy compactness degree $\mathrm{com}(G)$ of G is defined as

$$\begin{aligned}\mathrm{com}(G) &= \bigwedge_{\mathcal{U}\in 2^{\tau}}\left(\bigwedge_{x\in X}\left(G' \vee \bigvee_{A\in\mathcal{U}} A\right)(x) \mapsto \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(G' \vee \bigvee_{A\in\mathcal{V}} A\right)(x)\right)\\ &= \bigwedge_{\mathcal{U}\in 2^{\tau}}\left(\left[G \widetilde{\subset} \bigvee \mathcal{U}\right] \mapsto \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \left[G \widetilde{\subset} \bigvee \mathcal{V}\right]\right).\end{aligned}$$

Obviously G is fuzzy compact if and only if $\mathrm{com}(G) = \top$.

The following lemma is obvious.

Lemma 3.2. *Let (X,τ) be an L-topological space and $G \in L^X$. Then $\mathrm{com}(G) \geq a$ if and only if for any $\mathcal{U} \in 2^{\tau}$,*

$$\left[G \widetilde{\subset} \bigvee \mathcal{U}\right] \wedge a \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \left[G \widetilde{\subset} \bigvee \mathcal{V}\right].$$

By Lemma 3.2 we can easily obtain the following result.

Theorem 3.3. *Let (X,τ) be an L-topological space and $G \in L^X$. Then*

$$\mathrm{com}(G) = \bigvee\left\{a \in L : \left[G \widetilde{\subset} \bigvee \mathcal{U}\right] \wedge a \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \left[G \widetilde{\subset} \bigvee \mathcal{V}\right] \ \ \textit{for any} \ \ \mathcal{U} \in 2^{\tau}\right\}.$$

It is easy to see that if an L-topology τ on a set X is finite, then for each $G \in L^X$, $\mathrm{com}(G) = \top$. Moreover if X is a singleton set, then for any L-topology on X and any $G \in L^X$, $\mathrm{com}(G) = \top$.

Theorem 3.4. *For any $G, H \in L^X$, $\mathrm{com}(G \vee H) \geq \mathrm{com}(G) \wedge \mathrm{com}(H)$.*

Proof. By Theorem 3.3 we have

$$\mathrm{com}(G\vee H)=\bigvee\left\{a\in L:\left[G\vee H\widetilde{\subset}\bigvee\mathcal{U}\right]\wedge a\leq\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\left[G\vee H\widetilde{\subset}\bigvee\mathcal{V}\right],\forall\mathcal{U}\in 2^{\tau}\right\}$$

$$=\bigvee\Big\{a\in L:[G\widetilde{\subset}\bigvee\mathcal{U}]\wedge[H\widetilde{\subset}\bigvee\mathcal{U}]\wedge a$$

$$\leq\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}[G\widetilde{\subset}\bigvee\mathcal{V}]\wedge[H\widetilde{\subset}\bigvee\mathcal{V}],\forall\mathcal{U}\in 2^{\tau}\Big\}$$

$$\geq\bigvee\left\{a\in L:\left[G\widetilde{\subset}\bigvee\mathcal{U}\right]\wedge a\leq\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\left[G\widetilde{\subset}\bigvee\mathcal{V}\right]\right\}\wedge$$

$$\bigvee\left\{a\in L:\left[H\widetilde{\subset}\bigvee\mathcal{U}\right]\wedge a\leq\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\left[H\widetilde{\subset}\bigvee\mathcal{V}\right],\forall\mathcal{U}\in 2^{\tau}\right\}$$

$$=\mathrm{com}(G)\wedge\mathrm{com}(H).$$

□

Theorem 3.5. *For any $G\in L^X$ and any closed L-set H, $\mathrm{com}(G\wedge H)\geq\mathrm{com}(G)$.*

Proof. By Theorem 3.3 we have

$$\mathrm{com}(G\wedge H)=\bigvee\left\{a\in L:\left[G\wedge H\widetilde{\subset}\bigvee\mathcal{U}\right]\wedge a\leq\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\left[G\wedge H\widetilde{\subset}\bigvee\mathcal{V}\right],\forall\mathcal{U}\in 2^{\tau}\right\}$$

$$=\bigvee\left\{a\in L:\left[G\widetilde{\subset}H\vee\bigvee\mathcal{U}\right]\wedge a\leq\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\left[G\widetilde{\subset}H\vee\bigvee\mathcal{V}\right],\forall\mathcal{U}\in 2^{\tau}\right\}$$

$$\geq\mathrm{com}(G).$$

□

Theorem 3.6. *Let $f:X\to Y$ be a set map, τ_1 be an L-topology on X, τ_2 be an L-topology on Y, and $f:(X,\tau_1)\to(Y,\tau_2)$ be continuous. Then* $\mathrm{com}\left(f_L^{\rightarrow}(G)\right)\geq\mathrm{com}(G)$.

Proof. This can be proved from the following inequality.

$$\mathrm{com}(f_L^{\rightarrow}(G))$$

$$=\bigwedge_{\mathcal{U}\in 2^{\tau_2}}\left\{\bigwedge_{y\in Y}\left(f_L^{\rightarrow}(G)'(y)\vee\bigvee_{A\in\mathcal{U}}A(y)\right)\mapsto\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{y\in Y}\left(f_L^{\rightarrow}(G)'(y)\vee\bigvee_{A\in\mathcal{V}}A(y)\right)\right\}$$

$$=\bigwedge_{\mathcal{U}\in 2^{\tau_2}}\left\{\bigwedge_{x\in X}\left(G'(x)\vee\bigvee_{A\in\mathcal{U}}f_L^{\leftarrow}(A)(x)\right)\mapsto\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\left(G'(x)\vee\bigvee_{A\in\mathcal{V}}f_L^{\leftarrow}(A)(x)\right)\right\}$$

$$\geq\mathrm{com}(G).$$

□

4. The Generalized Tychonoff Theorem

In this section, we suppose that L be completely distributive.

Lemma 4.1. *Let (X,τ) be an L-topological space, η be a subbase of τ, and $G \in L^X$. Then*

$$\mathrm{com}(G) = \bigvee \left\{ a \in L : \left[G \widetilde{\subset} \bigvee \mathcal{U}\right] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[G \widetilde{\subset} \bigvee \mathcal{V}\right], \forall \mathcal{U} \in 2^{\eta} \right\}.$$

Proof. It is obvious that

$$\mathrm{com}(G) \leq \bigvee \left\{ a \in L : \left[G \widetilde{\subset} \bigvee \mathcal{U}\right] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[G \widetilde{\subset} \bigvee \mathcal{V}\right], \forall \mathcal{U} \in 2^{\eta} \right\}.$$

Now we prove that

$$\mathrm{com}(G) \geq \bigvee \left\{ a \in L : \left[G \widetilde{\subset} \bigvee \mathcal{U}\right] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[G \widetilde{\subset} \bigvee \mathcal{V}\right], \forall \mathcal{U} \in 2^{\eta} \right\}.$$

Thus we need to prove that

$$\begin{aligned} & \forall \mathcal{U} \in 2^{\eta}, \left[G \widetilde{\subset} \bigvee \mathcal{U}\right] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[G \widetilde{\subset} \bigvee \mathcal{V}\right] \\ \Rightarrow \quad & \forall \mathcal{U} \in 2^{\tau}, \left[G \widetilde{\subset} \bigvee \mathcal{U}\right] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[G \widetilde{\subset} \bigvee \mathcal{V}\right]. \end{aligned}$$

Suppose that there exists $\mathcal{U} \in 2^{\tau}$ such that $\left[G \widetilde{\subset} \bigvee \mathcal{U}\right] \wedge a \not\leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[G \widetilde{\subset} \bigvee \mathcal{V}\right]$. Then there exists $b \leq \left[G \widetilde{\subset} \bigvee \mathcal{U}\right] \wedge a$ such that $b \not\leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[G \widetilde{\subset} \bigvee \mathcal{V}\right]$. Let

$$\Gamma = \left\{ \mathcal{P} : \mathcal{U} \subseteq \mathcal{P} \subseteq \tau,\ b \leq \left[G \widetilde{\subset} \bigvee \mathcal{P}\right],\ b \not\leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{P})}} \left[G \widetilde{\subset} \bigvee \mathcal{V}\right] \right\}.$$

Then $(\Gamma, \subseteq)$ is a nonempty partially ordered set and each chain has an upper bound, hence by Zorn's Lemma, Γ has a maximal element Ω. Now we prove that Ω satisfies the following conditions:

(i) for every $B \in \tau$, if $C \in \Omega$ and $C \geq B$, then $B \in \Omega$;

(ii) if for each $B, C \in \tau$, $B \wedge C \in \Omega$, then $B \in \Omega$ or $C \in \Omega$.

We only verify (ii). If $B \notin \Omega$ and $C \notin \Omega$, then $\{B\} \cup \Omega \notin \Gamma$ and $\{C\} \cup \Omega \notin \Gamma$. This implies that $b \leq \bigvee_{\mathcal{V} \in 2^{(\Omega \cup \{B\})}} \left[G \widetilde{\subset} \bigvee \mathcal{V}\right]$ and $b \leq \bigvee_{\mathcal{V} \in 2^{(\Omega \cup \{C\})}} \left[G \widetilde{\subset} \bigvee \mathcal{V}\right]$. Hence for any $r \in \beta(b)$, there exists $A_1, A_2, \cdots, A_{m+n} \in \Omega$ such that

$$r \leq \left[G \widetilde{\subset} A_1 \vee A_2 \vee \cdots \vee A_m \vee B\right] \quad \text{and} \quad r \leq \left[G \widetilde{\subset} A_{m+1} \vee A_{m+2} \vee \cdots \vee A_{m+n} \vee C\right].$$

Further we have that

$$r \leq \left[G \widetilde{\subset} A_1 \vee A_2 \vee \cdots \vee A_{m+n} \vee B\right] \quad \text{and} \quad r \leq \left[G \widetilde{\subset} A_1 \vee A_2 \vee \cdots \vee A_{m+n} \vee C\right].$$

This shows that

$$r \leq \left[G \widetilde{\subset} A_1 \vee A_2 \vee \cdots \vee A_{m+n} \vee (B \wedge C)\right].$$

Therefore we obtain that $b \leq \bigvee_{\mathcal{V}\in 2^{(\Omega\cup\{B\wedge C\})}} \left[G\widetilde{\subset}\bigvee\mathcal{V}\right]$. This implies that $B\wedge C\notin\Omega$, which contradicts $B\wedge C\in\Omega$. (ii) is proved.

From (i) and (ii), it is immediate that if $D\in\Omega$, $P_1,P_2,\cdots,P_n\in\tau$ and $D\geq P_1\wedge P_2\wedge\cdots\wedge P_n$, then there exists $i(1\leq i\leq n)$ such that $P_i\in\Omega$.

Now let us consider $\eta\cap\Omega$. If $b\leq\left[G\widetilde{\subset}\bigvee(\eta\cap\Omega)\right]$, then

$$b\leq\bigvee_{\mathcal{V}\in 2^{(\eta\wedge\Omega)}}\left[G\widetilde{\subset}\bigvee\mathcal{V}\right]\leq\bigvee_{\mathcal{V}\in 2^{(\Omega)}}\left[G\widetilde{\subset}\bigvee\mathcal{V}\right],$$

this contradicts the sense of Ω. Therefore we have $b\not\leq\left[G\widetilde{\subset}\bigvee(\eta\cap\Omega)\right]$. Hence there exists $r\in\beta(b)$ such that $r\not\leq\left[G\widetilde{\subset}\bigvee(\eta\cap\Omega)\right]$. This implies that for any $A\in\eta\cap\Omega$, there exists an $x\in X$ such that $r\not\leq G'(x)\vee A(x)$.

By $b\leq\left[G\widetilde{\subset}\bigvee\Omega\right]$, we know that $b\leq\left(G'\vee\bigvee\Omega\right)(x)$. Hence there exists $D\in\Omega$ such that $r\in\beta\left(G'(x)\vee D(x)\right)$. Let

$$D=\bigvee_{i\in I}\bigwedge_{j\in J_i}A_{ij},\ \text{where for each } i\in I,\ J_i \text{ is a finite set and } A_{ij}\in\eta.$$

Then there exists $i\in I$ such that

$$r\in\beta\left(G'(x)\vee\bigwedge_{j\in J_i}A_{ij}(x)\right)\subseteq\bigcap_{j\in J_i}\beta\left(G'(x)\vee A_{ij}(x)\right).$$

This implies that $r\in\beta\left(G'(x)\vee A_{ij}(x)\right)$ for each $j\in J_i$. By $D\geq\bigwedge_{j\in J_i}A_{ij}$ we know that there is $j\in J_i$ such that $A_{ij}\in\Omega$, this contradicts $r\not\leq G'(x)\vee A_{ij}(x)$. The proof is obtained. □

Theorem 4.2. *Let (X,τ) be the product of a family of L-topological spaces $\{(X_i,\tau_i)\}_{i\in I}$, and $G_i\in L^{X_i}$ for any $i\in I$. Then* $\mathrm{com}\left(\prod_{i\in I}G_i\right)\geq\bigwedge_{i\in I}\mathrm{com}\left(G_i\right)$.

Proof. In order to prove $\mathrm{com}\left(\prod_{i\in I}G_i\right)\geq\bigwedge_{i\in I}\mathrm{com}\left(G_i\right)$, let $\bigwedge_{i\in I}\mathrm{com}\left(G_i\right)=a$. Then for any $i\in I$, $\mathrm{com}\left(G_i\right)\geq a$. Let $\eta=\{P_i^{\leftarrow}(D_i)\mid i\in I, D_i\in\tau_i\}$ be a subbase of τ. By Lemma 4.1, we only need to prove that for any $\mathcal{U}\in 2^{\eta}$,

$$\left[\prod_{i\in I}G_i\widetilde{\subset}\bigvee\mathcal{U}\right]\wedge a\leq\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\left[\prod_{i\in I}G_i\widetilde{\subset}\bigvee\mathcal{V}\right]. \tag{4.1}$$

Suppose that $\mathcal{U}\in 2^{\eta}$ and $b\in\beta\left(\left[\prod_{i\in I}G_i\widetilde{\subset}\bigvee\mathcal{U}\right]\wedge a\right)$. Let

$$J\subseteq I,\ \mathcal{U}=\bigcup_{i\in J}\mathcal{U}_i,\ \ \mathcal{U}_i=\left\{P_i^{\leftarrow}(B_i):B_i\in\mathcal{B}_i\subseteq\tau_i\right\}.$$

Then for any $x\in X$, we have

$$b\in\beta\left(\left(\prod_{i\in I}G_i\right)'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\right)=\beta\left(\bigvee_{i\in I}G_i'(x_i)\vee\bigvee_{i\in J}\bigvee_{A\in\mathcal{U}_i}A(x)\right). \tag{4.2}$$

(1) If $b \in \beta\left(\bigvee_{i\in I} G_i'(x_i)\right)$ for any $x = \{x_i\}_{i\in I} \in X$, then obviously

$$b \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \left[\prod_{i\in I} G_i \widetilde{\subset} \bigvee \mathcal{V}\right].$$

This shows that inequality (1) is true.

(2) Suppose that $b \notin \beta\left(\bigvee_{i\in I} G_i'(x_i)\right)$ for some $x = \{x_i\}_{i\in I} \in X$. Then $b \notin \beta\left(G_i'(x_i)\right)$ for any $i \in I$. Now we prove that there exists $k \in J$ such that $b \in \beta\left(G_k'(y_k) \vee \bigvee \mathcal{B}_k(y_k)\right)$ for any $y_k \in X_k$. If $\forall i \in J$, there exists $y_i \in X_i$ such that $b \notin \beta\left(G_i'(y_i) \vee \bigvee_{B\in\mathcal{B}_i} B(y_i)\right)$. Let $z = \{z_i\}_{i\in I}$ such that $z_i = y_i$ when $i \in J$, $z_i = x_i$ otherwise. By the following equality

$$\begin{aligned}\left(\prod_{i\in I} G_i\right)'(z) &= \left(\bigvee_{i\in J} P_i^{\leftarrow}(G_i')(z)\right) \vee \left(\bigvee_{i\notin J} P_i^{\leftarrow}(G_i')(z)\right)\\ &= \left(\bigvee_{i\in J} G_i'(y_i)\right) \vee \left(\bigvee_{i\notin J} G_i'(x_i)\right),\end{aligned}$$

we obtain that $b \notin \beta\left(\left(\prod_{i\in I} G_i\right)'(z)\right)$. Moreover for any $i \in J$, by the following fact

$$b \notin \beta\left(\bigvee_{B\in\mathcal{B}_i} B(y_i)\right) = \beta\left(\bigvee_{B\in\mathcal{B}_i} P_i^{\leftarrow}(B)(z)\right) = \beta\left(\bigvee_{A\in\mathcal{U}_i} A(z)\right),$$

we have

$$b \notin \bigcup_{i\in J} \beta\left(\bigvee_{A\in\mathcal{U}_i} A(z)\right) = \beta\left(\bigvee_{i\in J}\bigvee_{A\in\mathcal{U}_i} A(z)\right).$$

This implies

$$b \notin \beta\left(\left(\prod_{i\in I} G_i\right)'(z) \vee \bigvee_{A\in\mathcal{U}} A(z)\right).$$

This yields a contradiction with the formula (2). Thus we obtain the proof that there exists $k \in J$ such that $b \in \beta\left(G_k'(y_k) \vee \bigvee \mathcal{B}_k(y_k)\right)$ for any $y_k \in X_k$. This shows

$b \leq \left[G_k \widetilde{\subset} \bigvee \mathcal{B}_k\right]$. By $c(G_k) \geq a \geq b$ we can obtain

$$\begin{aligned}
b &\leq \bigvee_{\mathcal{D}_k \in 2^{(\mathcal{B}_k)}} \left[G_k \widetilde{\subset} \bigvee \mathcal{D}_k\right] \\
&= \bigvee_{\mathcal{D}_k \in 2^{(\mathcal{B}_k)}} \bigwedge_{y_k \in X_k} \left(G_k' \vee \bigvee \mathcal{D}_k\right)(y_i) \\
&= \bigvee_{\mathcal{D}_k \in 2^{(\mathcal{B}_k)}} \bigwedge_{y \in X} \left(G_k' \vee \bigvee \mathcal{D}_k\right)(P_k(y)) \\
&= \bigvee_{\mathcal{D}_k \in 2^{(\mathcal{B}_k)}} \bigwedge_{y \in X} \left(P_k^{\leftarrow}(G_k') \vee \bigvee_{D \in \mathcal{D}_k} P_k^{\leftarrow}(D)\right)(y) \\
&\leq \bigvee_{\mathcal{D}_k \in 2^{(\mathcal{B}_k)}} \bigwedge_{y \in X} \left(\left(\prod_{i \in I} G_i\right)' \vee \bigvee_{D \in \mathcal{D}_k} P_k^{\leftarrow}(D)\right)(y) \\
&\leq \bigvee_{\mathcal{V}_k \in 2^{(\mathcal{U}_k)}} \bigwedge_{y \in X} \left(\left(\prod_{i \in I} G_i\right)' \vee \bigvee \mathcal{V}_k\right)(y).
\end{aligned}$$

Thus we complete the proof of (1). Therefore $\operatorname{com}\left(\prod_{i\in I} G_i\right) \geq a = \bigwedge_{i \in I} \operatorname{com}(G_i)$. $\square$

By Theorem 3.6 and Theorem 4.2 we can obtain the following corollary.

Corollary 4.3. *Let (X, τ) be the product of a family of L-topological spaces $\{(X_i, \tau_i)\}_{i \in I}$. Then* $\operatorname{com}\left(\prod_{i\in I} \underline{\top}_i\right) = \bigwedge_{i\in I} \operatorname{com}(\underline{\top}_i)$, *where $\underline{\top}_i$ is the largest element in L^{X_i}.*

5. A comparison of different compactness degrees

In this section, we shall compare different notions of fuzzy compactness degrees in $[0,1]$-topological spaces.

In [17, 18], Šostak defined the compactness degree of a fuzzy set as follows:

Definition 5.1. Let (X, τ) be a $[0,1]$-topological space and $G \in [0,1]^X$. The compactness spectrum $C(G)$ of G is defined as follows:

$$C(G) = \left\{ b \in [0,1] : b \leq \left[G \widetilde{\subset} \bigvee \mathcal{U}\right] \Rightarrow b \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[G \widetilde{\subset} \bigvee \mathcal{V}\right], \forall \mathcal{U} \in 2^{\tau} \right\}.$$

The number $\mathbf{c}(G) = \inf([0,1] \backslash C(G))$ $(\inf \varnothing = 1)$ is called the compactness degree of G.

By Theorem 3.3 and Definition 5.1 we know that $\mathbf{c}(G) \leq \operatorname{com}(G)$. But in general, $\mathbf{c}(G) \neq \operatorname{com}(G)$. This can be seen from the following example.

Example 5.2. Let $X = [a, b]$ be a closed interval and let

$$\tau = \{\underline{\top}\} \cup \{0.5 \wedge \chi_A : A \in \delta\},$$

where δ denotes the natural topology on X. It is easy to check that $\mathbf{c}(0.5 \wedge \chi_X) = 0.5 < 1 = \operatorname{com}(0.5 \wedge \chi_X)$. Moreover $\mathbf{c}(\underline{\top}) = 0.5 < 1 = \operatorname{com}(\underline{\top})$.

In [11], E. Lowen and R. Lowen also introduce the notion of compactness degree of $[0,1]$-topological spaces as follows:

Definition 5.3. For a $[0,1]$-topological space (X,τ), the compactness degree of X is defined as

$$c(\underline{\top}) = \bigvee \Bigg\{ 1-\varepsilon : \forall \mathcal{U} \in 2^{\tau}, \forall \alpha > \varepsilon, \bigwedge_{x\in X} \Big(\bigvee \mathcal{U}\Big)(x) \geq \alpha \Rightarrow$$
$$\exists \mathcal{V} \in 2^{\mathcal{U}} \text{ such that } \bigwedge_{x\in X} \Big(\bigvee \mathcal{V}\Big)(x) \geq \alpha - \varepsilon \Bigg\}.$$

Now we compare $c(\underline{\top})$ and $\mathrm{com}(\underline{\top})$.

Theorem 5.4. *For a $[0,1]$-topological space (X,τ), $c(\underline{\top}) \geq \mathrm{com}(\underline{\top})$.*

Proof. When $c(\underline{\top}) = 1$, we know that the theorem is true since $c(\underline{\top}) = 1$ if and only if (X,τ) is fuzzy compact if and only if $\mathrm{com}(\underline{\top}) = 1$. Now we suppose that $c(\underline{\top}) < 1$. Let

$$S = \Bigg\{ \varepsilon : \forall \mathcal{U} \in 2^{\tau}, \forall \alpha > \varepsilon, \bigwedge_{x\in X} \Big(\bigvee \mathcal{U}\Big)(x) \geq \alpha \Rightarrow \exists \mathcal{V} \in 2^{\mathcal{U}}, \bigwedge_{x\in X} \Big(\bigvee \mathcal{V}\Big)(x) \geq \alpha - \varepsilon \Bigg\}.$$

Then $c(\underline{\top}) = 1 - \inf S < 1$. Hence $\inf S > 0$. In order to prove that $c(\underline{\top}) \geq \mathrm{com}(\underline{\top})$, take any r with $1 > r > c(\underline{\top})$. Now we prove $r > \mathrm{com}(\underline{\top})$. Take any s such that $r > s > c(\underline{\top})$. It is obvious that $0 < 1-s < 1-c(\underline{\top}) = \inf S$. This implies $1-s \notin S$. Hence there exists $\mathcal{U} \in 2^{\tau}$ and $\alpha > 1-s$ such that $\bigwedge_{x\in X} (\bigvee \mathcal{U})(x) \geq \alpha$, but $\forall \mathcal{V} \in 2^{\mathcal{U}}$, $\bigwedge_{x\in X} (\bigvee \mathcal{V})(x) < \alpha - 1 + s$. This implies $\bigwedge_{x\in X} (\bigvee \mathcal{U})(x) \geq \alpha > \alpha - (1-r)$, but $\bigvee_{\mathcal{V}\in 2^{\mathcal{U}}} \bigwedge_{x\in X} (\bigvee \mathcal{V})(x) \leq \alpha - 1 + s$. Thus we obtain

$$\bigwedge_{x\in X} \Big(\bigvee \mathcal{U}\Big)(x) \wedge r \not\leq \bigvee_{\mathcal{V}\in 2^{\mathcal{U}}} \bigwedge_{x\in X} \Big(\bigvee \mathcal{V}\Big)(x).$$

This shows $r > \mathrm{com}(\underline{\top})$. By the arbitrariness of r, we shows $c(\underline{\top}) \geq \mathrm{com}(\underline{\top})$. $\square$

In general, $c(\underline{\top}) \neq \mathrm{com}(\underline{\top})$. This can be seen from the following example.

Example 5.5. Let $X = \mathbb{N}$ be the set of all natural numbers. For any $n \in \mathbb{N}$, define $A_n \in [0,1]^X$ by

$$A_n(k) = \begin{cases} 0.5, & \text{if } k \leq n; \\ 0, & \text{if } k > n. \end{cases}$$

Let $\tau = \{\underline{\perp}, \underline{\top}, \underline{0.5}\} \cup \{A_n : n \in \mathbb{N}\}$. It is easy to check that τ is a $[0,1]$-topology on X. It is easy to check that $\bigwedge_{k\in X} \bigvee_{n\in\mathbb{N}} A_n(k) = 0.5$, but for any finite subfamily $\mathcal{V}$ of $\eta = \{A_n : n \in \mathbb{N}\}$, $\bigvee_{\mathcal{V}\in 2^{\eta}} \bigwedge_{k\in X} \bigvee_{A\in\mathcal{V}} A(k) = 0$. Therefore $\mathrm{com}(\underline{\top}) = 0$. Let

$$S = \Bigg\{ \varepsilon : \forall \mathcal{U} \in 2^{\tau}, \forall \alpha > \varepsilon, \bigwedge_{x\in X} \Big(\bigvee \mathcal{U}\Big)(x) \geq \alpha \Rightarrow \exists \mathcal{V} \in 2^{\mathcal{U}}, \bigwedge_{x\in X} \Big(\bigvee \mathcal{V}\Big)(x) \geq \alpha - \varepsilon \Bigg\}.$$

$\forall\alpha > 0.5$, if $\bigwedge_{x\in X} \left(\bigvee\mathcal{U}\right)(x) \geq \alpha$, then we must have $\underline{\top} \in \mathcal{U}$, hence $0.5 \in S$. It is easy to see $r \notin S$ for any $r < 0.5$. This implies $c(\underline{\top}) = 0.5$. Therefore $c(\underline{\top}) > \mathrm{com}(\underline{\top})$.

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