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# Direct product of intuitionistic fuzzy sets in LA-semigroups-II

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Abstract. In this paper, we have introduced the notion of direct product of intuitionistic fuzzy bi-ideals and direct product of intuitionistic fuzzy $P$-systems of LA-semigroups and discussed some of their fundamental properties. We have also proved that the IFS $A \times B = \langle \mu_{A\times B}, \lambda_{A\times B} \rangle$ is an IF bi-ideal of an LA-semigroup $S_1 \times S_2$ if and only if the upper and lower level sets are bi-ideals of $S_1 \times S_2$. Moreovere, we have proved that if $A = \langle \mu_A, \lambda_A \rangle$ and $B = \langle \mu_B, \lambda_B \rangle$ are IF $P$-systems of LA-semigroups $S_1$ and $S_2$ respectively, then $A \times B$ is an IF $P$-system of an LA-semigroup $S_1 \times S_2$.

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## 1. Introduction

L. A. Zadeh initiated the concept of fuzzy set in his pioneer paper (see [14]) and since then this concept has been applied to various algebraic structures, which provides a natural framework for generalizing some basic notions of algebra e.g. set theory, group theory, ring theory, groupoids, real analysis, measure theory, topology, and differential equations etc. The idea of "Intuitionistic fuzzy set" was first introduced by K.T. Atanassov (see [4, 5]) as generalization of the notion of fuzzy set. Kazim and Naseerudin introduced the concept of LA-semigroup in their definitive paper (see [7]). Let $S$ be a non empty set. Then $(S, *)$ is called an LA-semigroup, if $S$ is closed and satisfies the identity $(x*y)*z = (z*y)*x$ for all $x, y, z \in S$, which is called left invertive law. Later, Q. Mushtaq and others have investigated the structure further and added many useful results to the theory of LA-semigroups (see [11]). It is a useful non associative algebraic structure, midway between a groupoids and a commutative semigroup. The direct product of LA-semigroups was first introduced by Q. Mushtaq and M. Khan (see [10]). In the following manner: If $S_1$ and $S_2$ are LA-semigroups, then $S_1 \times S_2 = \{(s_1, s_2) : s_1 \in S_1 \text{ and } s_2 \in S_2\}$ is an LA-semigroup under point-wise multiplication of order pairs. Q. Mushtaq and M. Khan defined the direct product of left (resp, right) ideals, prime ideals, maximal ideals and investigate the properties of such ideals. In 1983, A. K. Ray introduced the concept of product of fuzzy subgroups in his paper [12]. Recently, in [1], H. Aktas and N. Cagman introduced the concept of generalized product of fuzzy subgroups and some fundamental properties. Recently, M. Khan et al. introduced the concept of fuzzy ideals and anti fuzzy ideals of LA-semigroups in his papers [8, 9]. In [2], M. Aslam, S. Abdullah and T. Khan introduced the concept of generalized direct product of fuzzy ideals in LA-semigroup. In [3], M. Aslam, S. Abdullah and N. Tabbasum used the idea of direct(Cartesian) product of intuitionistic fuzzy set in LA-semigroup and obtained some usful results.

In this paper, we introduce the concept of direct product of intuitionistic fuzzy bi-ideals and direct product of intuitionistic fuzzy $P$-systems of an LA-semigroup $S_1 \times S_2$, and some related properties are investigated. We have also proved if $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B}\rangle$ is an IFS of an LA-semigroup $S_1 \times S_2$, then $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B}\rangle$ is an IF bi-ideal of $S_1 \times S_2$ if and only if the upper and lower level sets are bi-ideals of $S_1 \times S_2$. Moreover, we also prove that if $A = \langle \mu_A, \lambda_A\rangle$ and $B = \langle \mu_B, \gamma_B\rangle$ are IF $P$-systems of LA-semigroups $S_1$ and $S_2$ respectively, then $A \times B$ is an IF $P$-system of an LA-semigroup $S_1 \times S_2$.

## 2. Preliminaries

In this part we introduce some concepts and results that are needed in the sequel.

Let $S$ be non-empty set. Then $(S, *)$ is called an LA-semigroup if $x * y \in S$ and $(x * y) * z = (z * y) * x$ for all $x, y, z \in S$. A non-empty subset $U$ of an LA-semigroup $S$ is said to be a subLA-semigroup if $UU \subseteq U$, where $UU = \{u_1 * u_2 : u_1, u_2 \in U\}$. A left (right) ideal $U$ of an LA-semigroup $S$ is a non-empty subset $I$ of $S$ such that $SI \subseteq I$ ($IS \subseteq I$). If $I$ is both left and right ideal of an LA-semigroup $S$, then we say that $I$ is an ideal of $S$. A non-empty subset $B$ of an LA-semigroup $S$ is called bi-ideal of $S$ if $BB \subseteq B$ and $(BS)\,B \subseteq B$. A non-empty subset $M$ of an LA-semigroup $S$ is called M-system if for all $a, b \in M$ and $x \in S$ such that $(ax)\,b \in M$ and a non-empty $P$ of $S$ is called P-system of $S$ if for all $a \in M$ and $x \in S$ such that $(ax)\,a \in M$.

**Definition 2.1** ([10])**.** Let $I_{S_1}$ and $I_{S_2}$ be subsets of LA-semigroups $S_1$ and $S_2$ respectively. The direct product $I_{S_1} \times I_{S_2}$ is called left (resp. right) ideal of LA-semigroup $S_1 \times S_2$ if

$$(S_1 \times S_2)(I_{S_1} \times I_{S_2}) \subseteq (I_{S_1} \times I_{S_2}) \text{ (resp. } (I_{S_1} \times I_{S_2})(S_1 \times S_2) \subseteq (I_{S_1} \times I_{S_2})).$$

**Lemma 2.2** ([10])**.** *If $I_{S_1}$ and $I_{S_2}$ are ideals of LA-semigroups $S_1$ and $S_2$, respectively, then $I_{S_1} \times I_{S_2}$ is an ideal of LA-semigroup $S_1 \times S_2$.*

**Lemma 2.3** ([10])**.** *If $I_{S_1} \times I_{S_2}$ and $J_{S_1} \times J_{S_2}$ are ideals of LA-semigroups $S_1 \times S_2$, respectively, then $(I_{S_1} \times I_{S_2}) \cap (J_{S_1} \times J_{S_2})$ is an ideal.*

**Definition 2.4** ([8])**.** A fuzzy subset $f$ of LA-semigroup is called fuzzy subLA-semigroup of LA-semigroup $S$ if for all $x, y \in S$, $f(xy) \geq \max\{f(x), f(y)\}$

**Definition 2.5** ([8]). A fuzzy subset $f$ of LA-semigroup $S$ is called fuzzy a left (resp. right) ideal of LA-semigroup $S$ if $f(xy) \geq f(y)$ (resp. $f(xy) \geq f(x)$ for all $x, y \in S$) and a fuzzy subset $f$ is called a fuzzy ideal of an LA-semigroup $S$ if it is both fuzzy left and fuzzy right ideal of $S$.

**Definition 2.6** ([12]). Let $f : S_1 \longrightarrow [0,1]$ and $g : S_2 \longrightarrow [0,1]$ be two fuzzy subsets of LA-semigroups $S_1$ and $S_2$ respectively. Then the product of fuzzy subsets is denoted by $f \times g$ and defined as $f \times g : S_1 \times S_2 \longrightarrow [0,1]$, where $(f \times g)(s_1, s_2) = \min\{f(s_1), g(s_2)\}$.

**Definition 2.7** ([4, 5]). Let $X$ be a nonempty fixed set. An intuitionistic fuzzy set (briefly, IFS) $A$ is an object having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$$

where the functions $\mu_A : X \longrightarrow [0,1]$ and $\gamma_A : X \longrightarrow [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x \in X$ to the set $A$ respectively and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in S$. For the sake of simplicity, we use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$.

**Definition 2.8** ([5]). If $A = \langle \mu_A, \gamma_A \rangle$ and $B = \langle \mu_B, \gamma_B \rangle$ are two IFSs of the set $X$, then

$A \subseteq B$ iff $\forall x \in X$, $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$,
$A = B$ iff $\forall x \in X$, $\mu_A(x) = \mu_B(x)$ and $\gamma_A(x) = \gamma_B(x)$,
$\Box A = \{x, \mu_A(x), 1 - \mu_A(x) | x \in X\}$,
$\Diamond\ A = \{x, 1 - \gamma_A(x), \gamma_A(x) | x \in X\}$.

**Definition 2.9** ([6]). Let $A = \langle \mu_A, \gamma_A \rangle$ and $B = \langle \mu_B, \gamma_B \rangle$ be intuitionistic fuzzy sets of non-empty sets $X_1$ and $X_2$ respectively. The direct product of intuitionistic fuzzy sets denoted by $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B} \rangle$ and defined as $\mu_{A\times B}(x,y) = \min\{\mu_A(x), \mu_B(y)\}$ and $\gamma_{A\times B}(x,y) = \max\{\mu_A(x), \mu_B(y)\}$ for all $(x,y) \in S_1 \times S_2$.

**Definition 2.10** ([3]). Let $A = \langle \mu_A, \gamma_A \rangle$ and $B = \langle \mu_B, \gamma_B \rangle$ be intuitionistic fuzzy set of LA-semigroups $S_1$ and $S_2$ respectively. The direct product of intuitionistic fuzzy set $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B} \rangle$ is called an intuitionistic fuzzy subLA-semigroup of LA-semigroup $S_1 \times S_2$ If

$$\begin{aligned} \mu_{A\times B}((x_1,y_1)(x_2,y_2)) &\geq \min\{\mu_{A\times B}((x_1,y_1), \mu_{A\times B}(x_2,y_2))\} \\ \text{and } \gamma_{A\times B}((x_1,y_1),(x_2,y_2)) &\leq \max\{\gamma_{A\times B}((x_1,y_1), \gamma_{A\times B}(x_2,y_2))\} \end{aligned}$$

for all $(x_1,y_1)$, $(x_2,y_2) \in S_1 \times S_2$.

**Definition 2.11** ([3]). Let $A = \langle \mu_A, \gamma_A \rangle$ and $B = \langle \mu_B, \gamma_B \rangle$ be intuitionistic fuzzy sets of LA-semigroups $S_1$ and $S_2$ respectively. The direct product of intuitionistic fuzzy set $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B} \rangle$ is called an intuitionistic fuzzy right (resp. left) ideal of an LA-semigroup $S_1 \times S_2$ if

$$\begin{aligned} &\mu_{A\times B}((x_1,y_1)(x_2,y_2)) \geq \mu_{A\times B}((x_1,y_1) \\ &\qquad\qquad (\text{resp. } \mu_{A\times B}((x_1,y_1)(x_2,y_2)) \geq \mu_{A\times B}(x_2,y_2) \\ &\gamma_{A\times B}((x_1,y_1),(x_2,y_2)) \leq \gamma_{A\times B}(x_1,y_1) \\ &\qquad\qquad (\text{resp. } \gamma_{A\times B}((x_1,y_1)(x_2,y_2)) \leq \gamma_{A\times B}(x_2,y_2) \end{aligned}$$

for all $(x_1, y_1)$, $(x_2, y_2) \in S_1 \times S_2$.

$A \times B = \langle \mu_{A\times B}, \gamma_{A\times B} \rangle$ is called an intuitionistic fuzzy ideal of an LA-semigroup $S_1 \times S_2$ if it is both intuitionistic fuzzy right and intuitionistic fuzzy left ideal of an LA-semigroup $S_1 \times S_2$.

**Example 2.12** ([3])**.** Let $S_1 = \{0, 1, 2\}$ and $S_2 = \{a, b, c\}$ be two LA-semigroups with the following tables:

| $\cdot$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 1 | 2 | 1 |
| 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 |

| $\cdot$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $c$ | $b$ | $c$ |
| $b$ | $b$ | $b$ | $b$ |
| $c$ | $b$ | $b$ | $b$ |

Then $S_1 \times S_2 = \{(0, a), (0, b), (0, c), (1, a), (1, b), (1, c), (2, a), (2, b)(2, , c)\}$ is an LA-semigroup with point wise multiplication. We define direct product of intuitionistic fuzzy set $A\times B = \langle \mu_{A\times B}, \gamma_{A\times B} \rangle$ on LA-semigroup $S_1\times S_2$ as $\mu_{A\times B} : S_1\times S_2 \longrightarrow [0, 1]$ by $\mu_{A\times B}(0, a) = \mu_{A\times B}(1, a) = \mu_{A\times B}(2, a) = 0.2$, $\mu_{A\times B}(0, b) = \mu_{A\times B}(0, c) = \mu_{A\times B}(1, b) = \mu_{A\times B}(1, c) = \mu_{A\times B}(2, b) = \mu_{A\times B}(2, c) = 0.3$ and $\gamma_{A\times B} : S_1 \times S_2 \longrightarrow [0, 1]$ by $\gamma_{A\times B}(0, a) = \gamma_{A\times B}(0, b) = \gamma_{A\times B}(0, c) = 0.6$, $\gamma_{A\times B}(1, a) = \gamma_{A\times B}(1, b) = \gamma_{A\times B}(1, c) = \mu\gamma_{A\times B}(2, a) = \gamma_{A\times B}(2, b) = \gamma_{A\times B}(2, c) = 0.5$. Then by routine calculations $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B} \rangle$ is an intuitionistic fuzzy ideal of LA-semigroup $S_1 \times S_2$.

## 3. Major Section

In [6], there are five different forms of direct (or Cartesian ) product, while here only one of them is used.

**Definition 3.1.** Let $A = \langle \mu_A, \gamma_A \rangle$ and $B = \langle \mu_B, \gamma_B \rangle$ be intuitionistic fuzzy sets of LA-semigroups $S_1$ and $S_2$ respectively. The direct product of intuitionistic fuzzy set $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B} \rangle$ is called an intuitionistic fuzzy bi-ideal of LA-semigroup $S_1 \times S$ if

$(IFB1)$ $\mu_{A\times B}((x_1, y_1)(x_2, y_2)) \geq \min\{\mu_{A\times B}((x_1, y_1), \mu_{A\times B}(x_2, y_2))\}$,

$(IFB2)$ $\mu_{A\times B}(((x_1, y_1)(x_2, y_2))(x_3, y_3)) \geq \min\{\mu_{A\times B}((x_1, y_1), \mu_{A\times B}(x_3, y_3))\}$,

$(IFB3)$ $\gamma_{A\times B}((x_1, y_1)(x_2, y_2)) \leq \max\{\gamma_{A\times B}((x_1, y_1), \gamma_{A\times B}(x_2, y_2))\}$,

$(IFB3)$ $\gamma_{A\times B}(((x_1, y_1)(x_2, y_2))(x_3, y_3)) \leq \max\{\gamma_{A\times B}((x_1, y_1), \gamma_{A\times B}(x_3, y_3))\}$,

for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in S_1 \times S_2$.

**Theorem 3.2.** *If $A = \langle \mu_A, \gamma_A \rangle$ and $B = \langle \mu_B, \gamma_B \rangle$ are any two IF bi-ideals of LA-semigroups $S_1$ and $S_2$ respectively, then $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B} \rangle$ is an IF bi-ideal of an LA-semigroup $S_1 \times S_2$.*

*Proof.* Let $A = \langle \mu_A, \gamma_A \rangle$ and $B = \langle \mu_B, \gamma_B \rangle$ be any two IF bi-ideals of LA-semigroups $S_1$ and $S_2$ respectively. Then for any $(x_1, y_1)$, $(x_2, y_2) \in S_1 \times S_2$, we have

$$\begin{aligned}
\mu_{A\times B}((x_1, y_1)(x_2, y_2)) &= \mu_{A\times B}(x_1x_2, y_1y_2) = \min\{\mu_A(x_1, y_1), \mu_B(x_2, y_2)\} \\
&\geq \min\{\min\{\mu_A(x_1), \mu_A(x_2)\}, \min\{\mu_B(y_1), \mu_B(y_2)\}\} \\
&= \min\{\min\{\mu_A(x_1), \mu_B(y_1)\}, \min\{\mu_A(x_2), \mu_B(y_2)\}\} \\
&= \min\{\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)\}
\end{aligned}$$

and

$$\begin{aligned}\gamma_{A\times B}((x_1,y_1)(x_2,y_2)) &= \gamma_{A\times B}(x_1x_2,y_1y_2)=\max\{\gamma_A(x_1x_2),\gamma_B(y_1y_2)\}\\ &\leq \max\{\max\{\gamma_A(x_1),\gamma_A(x_2)\},\max\{\gamma_B(y_1),\gamma_B(y_2)\}\}\\ &= \max\{\max\{\gamma_A(x_1),\gamma_B(y_1)\},\max\{\gamma_A(x_2),\gamma_B(y_2)\}\}\\ &= \max\{\gamma_{A\times B}(x_1,y_1),\gamma_{A\times B}(x_2,y_2)\}.\end{aligned}$$

Now, let $(x_1,y_1)$, $(x_2,y_2)$, $(x_3,y_3)\in S_1\times S_2$. Then

$$\begin{aligned}&\mu_{A\times B}(((x_1,y_1)(x_2,y_2))(x_3,y_3))=\mu_{A\times B}((x_1x_2)x_3,(y_1y_2)y_3)\\ &=\min\{\mu_A((x_1x_2)x_3),\mu_B((y_1y_2)y_3)\}\\ &\geq \min\{\min\{\mu_A(x_1),\mu_A(x_3)\},\min\{\mu_B(y_1),\mu_B y_3)\}\}\\ &=\min\{\min\{\mu_A(x_1),\mu_B(y_1)\},\min\{\mu_A(x_3),\mu_B(y_3)\}\}\\ &=\min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_3,y_3)\},\end{aligned}$$

$$\begin{aligned}&\gamma_{A\times B}(((x_1,y_1)(x_2,y_2))(x_3,y_3)),=\gamma_{A\times B}((x_1x_2)x_3,(y_1y_2)y_3)\\ &=\max\{\gamma_A((x_1x_2)x_3),\gamma_B((y_1y_2)y_3)\}\\ &\leq \max\{\max\{\gamma_A(x_1),\gamma_A(x_3)\},\max\{\gamma_B(y_1),\gamma_B y_3)\}\}\\ &=\max\{\max\{\gamma_A(x_1),\gamma_B(y_1)\},\max\{\gamma_A(x_3),\gamma_B(y_3)\}\}\\ &=\max\{\gamma_{A\times B}(x_1,y_1),\gamma_{A\times B}(x_3,y_3)\}.\end{aligned}$$

This completes the proof. □

**Theorem 3.3.** *If $A=\langle\mu_A,\gamma_A\rangle$ and $B=\langle\mu_B,\gamma_B\rangle$ are any two IF left(right) ideals of LA-semigroups $S_1$ and $S_2$ respectively, then $A\times B=\langle\mu_{A\times B},\gamma_{A\times B}\rangle$ is an IF bi-ideal of an LA-semigroup $S_1\times S_2$.*

*Proof.* Let $A=\langle\mu_A,\gamma_A\rangle$ and $B=\langle\mu_B,\gamma_B\rangle$ be any two IF left(right) ideals of LA-semigroups $S_1$ and $S_2$ respectively. Then for any $(x_1,y_1)$, $(x_2,y_2)\in S_1\times S_2$, we have

$$\begin{aligned}\mu_{A\times B}((x_1,y_1)(x_2,y_2)) &\geq \min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}\text{ and}\\ \gamma_{A\times B}((x_1,y_1)(x_2,y_2)) &\leq \max\{\gamma_{A\times B}(x_1,y_1),\gamma_{A\times B}(x_2,y_2)\}\end{aligned}$$

Now let $(x_1,y_1)$, $(x_2,y_2)$, $(x_3,y_3)\in S_1\times S_2$. Then

$$\begin{aligned}\mu_{A\times B}(((x_1,y_1)(x_2,y_2))(x_3,y_3)) &\geq \mu_{A\times B}(x_3,y_3)\\ &\geq \min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_3,y_3)\},\\ \gamma_{A\times B}(((x_1,y_1)(x_2,y_2))(x_3,y_3)) &\leq \gamma_{A\times B}(x_3,y_3)\\ &\leq \max\{\gamma_{A\times B}(x_1,y_1),\gamma_{A\times B}(x_3,y_3)\}.\end{aligned}$$

This completes the proof. □

**Example 3.4.** Let $S_1=\{0,1,2,3\}$ and $S_2=\{a,b,c,d\}$. Consider the following tables:

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 2 | 2 | 2 | 3 |
| 1 | 3 | 3 | 2 | 2 |
| 2 | 3 | 3 | 3 | 3 |
| 3 | 3 | 3 | 3 | 3 |

| $\cdot$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $a$ | $b$ | $c$ | $d$ |
| $b$ | $d$ | $c$ | $c$ | $c$ |
| $c$ | $c$ | $c$ | $c$ | $c$ |
| $d$ | $b$ | $c$ | $c$ | $c$ |

$$\begin{aligned} \mu_A(b) &= 0.7 = \mu_A(c),\ \mu_A(d) = 0.6,\ \mu_A(a) = 0.5. \\ \gamma_A(b) &= 0.2 = \gamma_A(c),\ \gamma_A(d) = 0.3,\ \gamma_A(a) = 0.5. \end{aligned}$$

$$\begin{aligned} \mu_B(1) &= 0.7 = \mu_B(3),\ \mu_B(2) = 0.6,\ \mu_B(0) = 0.5. \\ \gamma_B(1) &= 0.1 = \gamma_B(3),\ \gamma_B(2) = 0.3,\ \gamma_B(0) = 0.5. \end{aligned}$$

Then $A = \langle \mu_A, \mu_A \rangle$ and $B = \langle \mu_B, \mu_B \rangle$ are IFS on $S_1$ and $S_2$ respectively and $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B} \rangle$ is an IF bi-ideal but not an IF ideal.

**Proposition 3.5.** *If $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B} \rangle$ and $C \times D = \langle \mu_{C\times D}, \gamma_{C\times D} \rangle$ are any two left(right) ideals of LA-semigroups $S_1 \times S_2$, then $A \times B \cap C \times D$ is an IF bi-ideal of an LA-semigroup $S_1 \times S_2$.*

*Proof.* The proof is straightforward. □

**Proposition 3.6.** *If $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B} \rangle$ is an IF bi-ideal of an LA-semigroup $S_1 \times S_2$, then $\Box(A \times B) = \langle \mu_{A\times B}, \bar{\mu}_{A\times B} \rangle$ is an IF bi-ideal of $S_1 \times S_2$.*

*Proof.* Let $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B} \rangle$ be an IF bi-ideal of an LA-semigroups $S_1 \times S_2$. Then for any $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$, we have

$$\mu_{A\times B}((x_1, y_1)(x_2, y_2)) \geq \min\{\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)\} \text{ and}$$

$$\begin{aligned} \bar{\mu}_{A\times B}((x_1, y_1)(x_2, y_2)) &= 1 - \mu_{A\times B}((x_1, y_1)(x_2, y_2)) \\ &\leq 1 - \min\{\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)\} \\ &= \max\{1 - \mu_{A\times B}(x_1, y_1), 1 - \mu_{A\times B}(x_2, y_2)\} \\ &= \max\{\bar{\mu}_{A\times B}(x_1, y_1), \bar{\mu}_{A\times B}(x_2, y_2)\} \end{aligned}$$

and

$$\begin{aligned} \bar{\mu}_{A\times B}(((x_1, y_1)(x_2, y_2))(x_3, y_3)) &= 1 - \mu_{A\times B}(((x_1, y_1)(x_2, y_2))(x_3, y_3)) \\ &\leq 1 - \min\{\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_3, y_3)\} \\ &= \max\{1 - \mu_{A\times B}(x_1, y_1), 1 - \mu_{A\times B}(x_3, y_3)\} \\ &= \max\{\bar{\mu}_{A\times B}(x_1, y_1), \bar{\mu}_{A\times B}(x_3, y_3)\}. \end{aligned}$$

This completes the proof. □

**Proposition 3.7.** *If $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B} \rangle$ is an IF bi-ideal of an LA-semigroup $S_1 \times S_2$, then $\Diamond(A \times B) = \langle \bar{\gamma}_{A\times B}, \gamma_{A\times B} \rangle$ is an IF bi-ideal of $S_1 \times S_2$.*

*Proof.* Let $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B} \rangle$ be an IF bi-ideal of an LA-semigroup $S_1 \times S_2$. Then for any $(x_1, y_1)$, $(x_2, y_2) \in S_1 \times S_2$, we have

$$\begin{aligned}
\gamma_{A\times B}((x_1,y_1)(x_2,y_2)) &\leq \max\{\gamma_{A\times B}(x_1,y_1), \gamma_{A\times B}(x_2,y_2)\} \text{ and} \\
\bar{\gamma}_{A\times B}((x_1,y_1)(x_2,y_2)) &= 1-\gamma_{A\times B}((x_1,y_1)(x_2,y_2)) \\
&\geq 1-\max\{\gamma_{A\times B}(x_1,y_1), \gamma_{A\times B}(x_2,y_2)\} \\
&= \min\{1-\gamma_{A\times B}(x_1,y_1), 1-\gamma_{A\times B}(x_2,y_2)\} \\
&= \min\{\bar{\gamma}_{A\times B}(x_1,y_1), \bar{\gamma}_{A\times B}(x_2,y_2)\}.
\end{aligned}$$

Also

$$\begin{aligned}
\bar{\gamma}_{A\times B}(((x_1,y_1)(x_2,y_2))(x_3,y_3)) &= 1-\gamma_{A\times B}(((x_1,y_1)(x_2,y_2))(x_3,y_3)) \\
&\geq 1-\max\{\gamma_{A\times B}(x_1,y_1), \gamma_{A\times B}(x_3,y_3)\} \\
&= \min\{1-\gamma_{A\times B}(x_1,y_1), 1-\gamma_{A\times B}(x_3,y_3)\} \\
&= \min\{\bar{\gamma}_{A\times B}(x_1,y_1), \bar{\gamma}_{A\times B}(x_3,y_3)\}
\end{aligned}$$

This completes the proof. □

**Theorem 3.8.** *If $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B} \rangle$ is an IF bi-ideal of an LA-semigroup $S_1 \times S_2$, then $\mu_{A\times B}$ and $\bar{\gamma}_{A\times B}$ are fuzzy bi-ideals of $S_1 \times S_2$.*

*Proof.* Let $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B} \rangle$ be an IF bi-ideal of an LA-semigroup $S_1 \times S_2$. Then we have for any $(x_1, y_1)$, $(x_2, y_2) \in S_1 \times S_2$

$$\begin{aligned}
\mu_{A\times B}((x_1,y_1)(x_2,y_2)) &\geq \min\{\mu_{A\times B}(x_1,y_1), \mu_{A\times B}(x_2,y_2)\} \text{ and} \\
\gamma_{A\times B}((x_1,y_1)(x_2,y_2)) &\leq \max\{\gamma_{A\times B}(x_1,y_1), \gamma_{A\times B}(x_2,y_2)\} \\
1-\gamma_{A\times B}((x_1,y_1)(x_2,y_2)) &\geq 1-\max\{\gamma_{A\times B}(x_1,y_1), \gamma_{A\times B}(x_2,y_2)\} \\
\bar{\gamma}_{A\times B}((x_1,y_1)(x_2,y_2)) &\leq \min\{1-\gamma_{A\times B}(x_1,y_1), 1-\gamma_{A\times B}(x_2,y_2)\} \\
\bar{\gamma}_{A\times B}((x_1,y_1)(x_2,y_2)) &\leq \max\{\bar{\gamma}_{A\times B}(x_1,y_1), \bar{\gamma}_{A\times B}(x_2,y_2)\}
\end{aligned}$$

Now, for any $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3) \in S_1 \times S_2$, we have

$$\begin{aligned}
\mu_{A\times B}(((x_1,y_1)(x_2,y_2))(x_3,y_3)) &\geq \min\{\mu_{A\times B}(x_1,y_1), \mu_{A\times B}(x_3,y_3)\} \\
\gamma_{A\times B}(((x_1,y_1)(x_2,y_2))(x_3,y_3)) &\leq \max\{\gamma_{A\times B}(x_1,y_1), \gamma_{A\times B}(x_3,y_3)\} \\
1-\gamma_{A\times B}(((x_1,y_1)(x_2,y_2))(x_3,y_3)) &\leq 1-\max\{\gamma_{A\times B}(x_1,y_1), \gamma_{A\times B}(x_3,y_3)\} \\
\bar{\gamma}_{A\times B}(((x_1,y_1)(x_2,y_2))(x_3,y_3)) &\leq \min\{1-\gamma_{A\times B}(x_1,y_1), 1-\gamma_{A\times B}(x_3,y_3)\} \\
\bar{\gamma}_{A\times B}(((x_1,y_1)(x_2,y_2))(x_3,y_3)) &\leq \min\left\{\bar{\gamma}_{A\times B}(x_1,y_1), \bar{\gamma}_{A\times B}(x_3,y_3)\right\}
\end{aligned}$$

Therefore $\mu_{A\times B}$ and $\bar{\gamma}_{A\times B}$ are fuzzy bi-ideals of $S_1 \times S_2$. □

**Theorem 3.9.** *If $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B} \rangle$ is an IF bi-ideal of an LA-semigroup $S_1 \times S_2$, then $\bar{\mu}_{A\times B}$ and $\gamma_{A\times B}$ are anti fuzzy bi-ideals of $S_1 \times S_2$.*

*Proof.* Let $A\times B=\langle\mu_{A\times B},\gamma_{A\times B}\rangle$ be an IF bi-ideal of an LA-semigroup $S_1\times S_2$. Then we have for any $(x_1,y_1)$, $(x_2,y_2)\in S_1\times S_2$

$$\begin{aligned}
\mu_{A\times B}((x_1,y_1)(x_2,y_2)) &\geq \min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}\\
1-\mu_{A\times B}((x_1,y_1)(x_2,y_2)) &\leq 1-\min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}\\
\bar{\mu}_{A\times B}((x_1,y_1)(x_2,y_2) &\leq \max\{1-\mu_{A\times B}(x_1,y_1),1-\mu_{A\times B}(x_2,y_2)\}\\
\bar{\mu}_{A\times B}((x_1,y_1)(x_2,y_2)) &\leq \max\left\{\bar{\mu}_{A\times B}(x_1,y_1),\bar{\mu}_{A\times B}(x_2,y_2)\right\}, \text{ and}\\
\gamma_{A\times B}((x_1,y_1)(x_2,y_2)) &\leq \max\{\gamma_{A\times B}(x_1,y_1),\gamma_{A\times B}(x_2,y_2)\}
\end{aligned}$$

Now, for any $(x_1,y_1)$, $(x_2,y_2)$, $(x_3,y_3)\in S_1\times S_2$, we have

$$\begin{aligned}
\mu_{A\times B}(((x_1,y_1)(x_2,y_2))(x_3,y_3)) &\geq \min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_3,y_3)\}\\
1-\mu_{A\times B}(((x_1,y_1)(x_2,y_2))(x_3,y_3)) &\leq 1-\min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_3,y_3)\}\\
\bar{\mu}_{A\times B}(((x_1,y_1)(x_2,y_2))(x_3,y_3)) &\leq \max\{1-\mu_{A\times B}(x_1,y_1),1-\mu_{A\times B}(x_3,y_3)\}\\
\bar{\mu}_{A\times B}(((x_1,y_1)(x_2,y_2))(x_3,y_3)) &\leq \max\{\bar{\mu}_{A\times B}(x_1,y_1),\bar{\mu}_{A\times B}(x_3,y_3)\}, \text{ and}\\
\gamma_{A\times B}(((x_1,y_1)(x_2,y_2))(x_3,y_3)) &\leq \max\{\gamma_{A\times B}(x_1,y_1),\gamma_{A\times B}(x_3,y_3)\}
\end{aligned}$$

This completes the proof. □

**Definition 3.10.** If $A=\langle\mu_A,\gamma_A\rangle$ and $B=\langle\mu_B,\gamma_B\rangle$ are intuitionistic fuzzy sets of LA-semigroups $S_1$ and $S_2$ respectively, then for any $s,t\in[0,1]$, the set

$$U(\mu_{A\times B},s)=\{(x,y)\in S_1\times S_2:\mu_{A\times B}(x,y)\geq s\}$$

is called the upper level set of $\mu_{A\times B}(x,y)$ and the set

$$L(\gamma_{A\times B},t)=\{(x,y)\in S_1\times S_2:\gamma_{A\times B}(x,y)\leq t\}$$

is called the lower level set of $\gamma_{A\times B}(x,y)$.

**Theorem 3.11.** *If $A\times B=\langle\mu_{A\times B},\gamma_{A\times B}\rangle$ is an IFS of an LA-semigroup $S_1\times S_2$, then $A\times B=\langle\mu_{A\times B},\gamma_{A\times B}\rangle$ is an IF bi-ideal of $S_1\times S_2$ if and only if the upper and lower level sets are bi-ideals of $S_1\times S_2$.*

*Proof.* Let $A\times B=\langle\mu_{A\times B},\gamma_{A\times B}\rangle$ be an IF bi-ideal of an LA-semigroup $S_1\times S_2$ and $(x_1,y_1)$, $(x_2,y_2)\in U(\mu_{A\times B},t)$. Then $\mu_{A\times B}(x_1,y_1)\geq t$ and $\mu_{A\times B}(x_2,y_2)\geq t$ since

$$\begin{aligned}
\mu_{A\times B}((x_1,y_1)(x_2,y_2)) &\geq \min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}\\
&\geq \min\{t,t\}=t\\
\mu_{A\times B}((x_1,y_1)(x_2,y_2)) &\geq t.
\end{aligned}$$

Hence $(x_1,y_1)(x_2,y_2)\in U(\mu_{A\times B},t)$.

Let $(x_1,y_1)$, $(x_3,y_3)\in U(\mu_{A\times B},t)$ and $(x_2,y_2)\in S_1\times S_2$. Then $\mu_{A\times B}(x_1,y_1)\geq t$ and $\mu_{A\times B}(x_2,y_2)\geq t$, since

$$\begin{aligned}
\mu_{A\times B}((x_1,y_1)(x_2,y_2))(x_3,y_3)) &\geq \min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_3,y_3)\}\geq t,\\
\mu_{A\times B}((x_1,y_1)(x_2,y_2))(x_3,y_3)) &\geq t.
\end{aligned}$$

Hence $((x_1,y_1)(x_2,y_2))(x_3,y_3)\in U(\mu_{A\times B},t)$. Therefore $U(\mu_{A\times B},t)$ is bi-ideal of $S_1\times S_2$. Similarly $L(\gamma_{A\times B},t)$ is bi-ideal of $S_1\times S_2$.

Conversely, let the upper and lower level sets are bi-ideals of $S_1 \times S_2$ and $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B}\rangle$ is not an IF bi-ideal of an LA-semigroup $S_1 \times S_2$. Then for some $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$, we have

$$\mu_{A\times B}((x_1, y_1)(x_2, y_2)) < \min\{\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)\}.$$

Let $t_0 = \frac{1}{2}\{\mu_{A\times B}((x_1, y_1)(x_2, y_2)) + \min\{\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)\}\}$. Then

$$\begin{aligned} \mu_{A\times B}((x_1, y_1)(x_2, y_2)) &< t_0 < \min\{\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)\} \\ \text{implies } \mu_{A\times B}((x_1, y_1)(x_2, y_2)) &< t_0 \text{ and } t_0 < \min\{\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)\} \\ \text{implies } t_0 &< \mu_{A\times B}(x_1, y_1) \text{ and } t_0 < \mu_{A\times B}(x_2, y_2) \\ (x_1, y_1) &\in U(\mu_{A\times B}, t_0) \text{ and } (x_2, y_2) \in U(\mu_{A\times B}, t_0) \text{ but} \\ \mu_{A\times B}((x_1, y_1)(x_2, y_2)) &< t_0 \implies (x_1, y_1)(x_2, y_2) \notin U(\mu_{A\times B}, t_0), \end{aligned}$$

which is a contradiction. Hence $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B}\rangle$ is an IF bi-ideal of an LA-semigroup $S_1 \times S_2$. □

**Definition 3.12.** An IFS $A = \langle \mu_A, \gamma_A\rangle$ of an LA-semigroup $S$ is called an intuitionistic fuzzy $P$-system if for all $a, x \in S$

(1) $\mu_A(x(ax)) \geq \mu_A(x)$,
(2) $\gamma_A(x(ax)) \leq \gamma_A(x)$.

**Theorem 3.13.** *If $A = \langle \mu_A, \gamma_A\rangle$ and $B = \langle \mu_B, \gamma_B\rangle$ are IF $P$-systems of LA-semigroups $S_1$ and $S_2$ respectively, then $A\times B$ is an IF $P$-system of an LA-semigroup $S_1 \times S_2$.*

*Proof.* Let $A = \langle \mu_A, \gamma_A\rangle$ and $B = \langle \mu_B, \gamma_B\rangle$ be IF $P$-systems of LA-semigroups $S_1$ and $S_2$ respectively. Let $(x, y), (a, b) \in S_1 \times S_2$. Then

$$\begin{aligned} \mu_{A\times B}((x, y)((a, b)(x, y))) &= \mu_{A\times B}(x_1(ax_2), (y_1(by_2)) \\ &\geq \min\{\mu_A(x(ax)), \mu_B(y(by))\} \\ &\geq \min\{\mu_A(x) \wedge \mu_A(x), \mu_B(y) \wedge \mu_B(y)\} \\ &= \min\{\mu_A(x) \wedge \mu_B(y_1), \mu_A(x) \wedge \mu_B(y)\} \\ &= \min\{\mu_{A\times B}(x, y), \mu_{A\times B}(x, y)\} \\ \mu_{A\times B}((x, y)((a, b)(x, y))) &\geq \mu_{A\times B}(x, y) \wedge \mu_{A\times B}(x, y) \text{ and} \\ \gamma_{A\times B}((x, y)((a, b)(x, y))) &= \gamma_{A\times B}(x(ax), (y(by)) \\ &\leq \max\{\gamma_A(x(ax)), \gamma_B(y(by))\} \\ &\leq \max\{\gamma_A(x) \vee \gamma_A(x), \gamma_B(y) \vee \gamma_B(y)\} \\ &= \max\{\gamma_A(x) \vee \gamma_B(y_1), \gamma_A(x) \vee \gamma_B(y)\} \\ &= \max\{\gamma_{A\times B}(x, y), \gamma_{A\times B}(x, y)\} \\ \gamma_{A\times B}((x, y)((a, b)(x, y))) &\leq \gamma_{A\times B}(x, y) \vee \gamma_{A\times B}(x, y). \end{aligned}$$

Hence $A \times B$ is an IF $P$-system of an LA-semigroup $S_1 \times S_2$. □

**Theorem 3.14.** *If $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B}\rangle$ is an IF $P$-system of an LA-semigroup $S_1 \times S_2$, then $\Box(A \times B) = \langle \mu_{A\times B}, \bar{\mu}_{A\times B}\rangle$ and $\Diamond(A \times B) = \langle \bar{\gamma}_{A\times B}, \gamma_{A\times B}\rangle$ are IF $P$-systems of $S_1 \times S_2$.*

*Proof.* Straightforward. □

**Theorem 3.15.** *If $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B}\rangle$ is an IFS of an LA-semigroup $S_1 \times S_2$, then $A \times B = \langle \mu_{A\times B}, \gamma_{A\times B}\rangle$ is an IF $P$-system of $S_1 \times S_2$ if and only if the upper and lower level sets are $P$-systems of $S_1 \times S_2$.*

*Proof.* It follows from Theorem 3.11. □

## 4. Conclusions

M. Aslam and S. Abdullah have introduced the concept of direct product of intuitionistic fuzzy ideals of LA-semigroups. In this paper we have introduced the concept of direct product of intuitionistic fuzzy bi-ideals of LA-semigroups and discussed some of their fundamental properties.

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