Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 2, No. 2, (October 2011), pp. 141- 150
ISSN 2093–9310
http://www.afmi.or.kr

On rough (m,n)-bi-ideals and generalized rough (m,n)-bi-ideals in semigroups

Muhammad Aslam, Muhammad Shabir, Naveed Yaqoob, Asim Shabir

Received 17 February 2011; Accepted 11 April 2011

Abstract. The purpose of this paper is to give some properties of rough (m,n)-bi-ideals and generalized rough (m,n)-bi-ideals in semigroups. In this paper, we have shown that the lower and upper approximation of an (m,n)-bi-ideal is an (m,n)-bi-ideal and generalized lower and upper approximation of an (m,n)-bi-ideal is an (m,n)-bi-ideal in semigroups. We also proved that the lower and upper approximation of an (m,n)-bi-ideal is an (m,n)-bi-ideal in the quotient semigroups.

2010 AMS Classification: 20M12, 20M17

Keywords: (m,n)-bi-ideals, Rough (m,n)-bi-ideals, Generalized rough (m,n)-bi-ideals.

Corresponding Author: Naveed Yaqoob (nayaqoob@ymail.com)

1. Introduction

The notion of (m,n)-ideal of a semigroup was introduced by Lajos [12, 13, 14]. Later (m,n)-quasi-ideals and (m,n)-bi-ideals and generalized (m,n)-bi-ideals were studied in various algebraic structures (see [1] and [2]).

The notion of a rough set was originally proposed by Pawlak [15, 16] as a formal tool for modeling and processing incomplete information in information systems. Some authors studied the algebraic properties of rough sets. Biswas and Nanda [4], introduced the notion of rough subgroups. Kuroki, in [10], introduced the notion of a rough ideal in a semigroup. Jun applied the rough set theory to BCK-algebras [7]. Aslam et al. [3] introduced the concept of roughness in left almost semigroups. And Davvaz applied the rough set theory to rings [6]. Kuroki and Mordeson [11], Xiao and Zhang [17], Zhang and Wu [18] and Chinram [5] presented many results on roughness in different structures. Michiro Kondo [9], studied the structure of generalized rough sets. Kondo consider some fundamental properties of generalized rough sets induced by binary relations and do not restrict the universe to be finite. Later Kim and Kim [8] studied the generalized rough sets and relations. In this paper, we have introduced the notion of rough $(m,0)$-ideals (rough $(0,n)$-ideals), generalized rough $(m,0)$-ideals (generalized rough $(0,n)$-ideals), rough (m,n)-bi-ideals (generalized rough (m,n)-bi-ideals) and rough m-left ideals (generalized rough m-left ideals), which are extended notions of $(m,0)$-ideals, $(0,n)$-ideals, (m,n)-bi-ideals, m-left ideals and n-right ideals in semigroups.

2. Rough subsets in semigroups

Let S be a semigroup and ρ be a congruence relation on S, that is, ρ is an equivalence relation on S such that

$$(a,b)\in\rho \text{ implies } (ax,bx)\in\rho \text{ and } (xa,xb)\in\rho$$

for all $a,b,x\in S$. If ρ is a congruence relation on S, then for every $x\in S$, $[x]_\rho$ denotes the congruence class of x with respect to the relation ρ.

A congruence ρ on S is called complete if $[a]_\rho[b]_\rho=[ab]_\rho$ for all $a,b\in S$.

Let A be a nonempty subset of a semigroup S. Then the sets

$$\rho_-(A)=\left\{x\in S:[x]_\rho\subseteq A\right\} \quad\text{and}\quad \rho^-(A)=\left\{x\in S:[x]_\rho\cap A\neq\phi\right\}$$

are called ρ-lower and ρ-upper approximations of A, respectively. For a nonempty subset A of S, $\rho(A)=(\rho_-(A),\rho^-(A))$ is called a rough set with respect to ρ if $\rho_-(A)\neq\rho^-(A)$.

Theorem 2.1 ([10])**.** *Let ρ be a congruence relation on a semigroup S. If A and B are nonempty subsets of S, then*

$$\rho_-(A\cap B)=\rho_-(A)\cap\rho_-(B).$$

Theorem 2.2 ([10])**.** *Let ρ be a congruence relation on a semigroup S. If A and B are nonempty subsets of S, then*

$$\rho^-(A)\rho^-(B)\subseteq\rho^-(AB).$$

Theorem 2.3 ([10])**.** *Let ρ be a complete congruence relation on a semigroup S. If A and B are nonempty subsets of S, then*

$$\rho_-(A)\rho_-(B)\subseteq\rho_-(AB).$$

Let S be a semigroup and ρ be a congruence relation on S. Then a subset A of a semigroup S is called a ρ-upper rough subsemigroup of S if $\rho^-(A)$ is a subsemigroup of S. Similarly a subset A of a semigroup S is called a ρ-lower rough subsemigroup of S if $\rho_-(A)$ is a subsemigroup of S. A subset A of a semigroup S is called a ρ-upper rough left [right, two-sided] ideal of S if $\rho^-(A)$ is a left [right, two-sided] ideal of S. Similarly a subset A of a semigroup S is called a ρ-lower rough left [right, two-sided] ideal of S if $\rho_-(A)$ is a left [right, two-sided] ideal of S.

Theorem 2.4 ([10])**.** *Let ρ be a congruence relation on a semigroup S. Then*

(1) *If A is a subsemigroup of S, then $\rho^-(A)$ is a subsemigroup of S.*
(2) *If A is a left [right, two-sided] ideal of S, then $\rho^-(A)$ is a left [right, two-sided] ideal of S.*

Theorem 2.5 ([10])**.** *Let ρ be a complete congruence relation on a semigroup S. Then*

(1) *If A is a subsemigroup of S, then $\rho_-(A)$ is, if it is nonempty, a subsemigroup of S.*
(2) *If A be a left [right, two-sided] ideal of S, then $\rho_-(A)$ is, if it is nonempty, a left [right, two-sided] ideal of S.*

Lemma 2.6. *Let ρ be a congruence relation on a semigroup S. Then for a nonempty subset A of S*
(1) $(\rho^-(A))^n \subseteq \rho^-(A^n)$ *for all* $n \in \mathbb{N}$.
(2) *If ρ is complete, then* $(\rho_-(A))^n \subseteq \rho_-(A^n)$ *for all* $n \in \mathbb{N}$.

3. Generalized rough subsets in semigroups

Let X be a nonempty set and θ be a binary relation on X. By $\mathcal{P}(X)$ we mean the power set of X. For all $A \subseteq X$, we define θ_- and $\theta_+ : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ by

$$\theta_-(A) = \{x \in X : \forall\ y,\ x\theta y \Rightarrow y \in A\} = \{x \in X : \theta N(x) \subseteq A\}$$

and

$$\theta_+(A) = \{x \in X : \exists\ y \in A,\ \text{such that } x\theta y\} = \{x \in X : \theta N(x) \cap A \neq \phi\}.$$

Where $\theta N(x) = \{y \in X : x\theta y\}$. $\theta_-(A)$ and $\theta_+(A)$ are called the θ-lower approximation and the θ-upper approximation operations, respectively (see [9]).

Example 3.1. Let $X = \{a, b, c\}$ and $\theta = \{(a, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$. Then $\theta N(a) = \{a\}$; $\theta N(b) = \{b, c\}$; $\theta N(c) = \{a, b, c\}$; $\theta_-(\{a\}) = \{a\}$; $\theta_-(\{b\}) = \phi$; $\theta_-(\{c\}) = \phi$; $\theta_-(\{a, b\}) = \{a\}$; $\theta_-(\{a, c\}) = \{a\}$; $\theta_-(\{b, c\}) = \{b\}$; $\theta_-(\{a, b, c\}) = \{a, b, c\}$; $\theta_+(\{a\}) = \{a, c\}$; $\theta_+(\{b\}) = \{b, c\}$; $\theta_+(\{c\}) = \{b, c\}$; $\theta_+(\{a, b\}) = \{a, b, c\}$; $\theta_+(\{a, c\}) = \{a, b, c\}$; $\theta_+(\{b, c\}) = \{b, c\}$; $\theta_+(\{a, b, c\}) = \{a, b, c\}$.

Theorem 3.2. *Let θ be a reflexive, transitive and compatible relation on a semigroup S. If A and B are nonempty subsets of S. Then*

$$\theta_-(A \cap B) = \theta_-(A) \cap \theta_-(B).$$

Proof. Let $a \in \theta_-(A \cap B)$. Then $\theta N(a) \subseteq A \cap B$. Thus

$$\theta N(a) \subseteq A \text{ and } \theta N(a) \subseteq B$$
$$\Longleftrightarrow a \in \theta_-(A) \text{ and } a \in \theta_-(B)$$
$$\Longleftrightarrow a \in \theta_-(A) \cap \theta_-(B).$$

Thus $\theta_-(A \cap B) = \theta_-(A) \cap \theta_-(B)$. □

Theorem 3.3. *Let θ be a reflexive, transitive and compatible relation on a semigroup S. If A and B are nonempty subsets of S. Then*

$$\theta_+(A)\theta_+(B) \subseteq \theta_+(AB).$$

Proof. Let c be any element of $\theta_+(A)\theta_+(B)$. Then $c = ab$ where $a \in \theta_+(A)$ and $b \in \theta_+(B)$. Thus there exist elements $x, y \in S$ such that $x \in A$ and $a\theta x$ and $y \in B$ and $b\theta y$. Since θ is compatible relation on S, so $ab\theta xy$. As $xy \in AB$, so we have $c = ab \in \theta_+(AB)$. Thus $\theta_+(A)\theta_+(B) \subseteq \theta_+(AB)$. □

Definition 3.4. Let θ be a transitive and compatible relation on a semigroup S. Then for each $a, b \in S$, $\theta N(a)\theta N(b) \subseteq \theta N(ab)$. If $\theta N(a)\theta N(b) = \theta N(ab)$, then θ is called complete compatible relation.

Theorem 3.5. *Let θ be a reflexive, transitive and complete compatible relation on a semigroup S and A, B are nonempty subsets of S. Then*

$$\theta_-(A)\theta_-(B) \subseteq \theta_-(AB).$$

Proof. Let c be any element of $\theta_-(A)\theta_-(B)$. Then $c = ab$ where $a \in \theta_-(A)$ and $b \in \theta_-(B)$. Thus we have $\theta N(a) \subseteq A$ and $\theta N(b) \subseteq B$. Since θ is complete compatible relation on S, so we have $\theta N(ab) = \theta N(a)\theta N(b) \subseteq AB$, which implies that $ab \in \theta_-(AB)$. Thus $\theta_-(A)\theta_-(B) \subseteq \theta_-(AB)$. □

Let θ be a reflexive, transitive and compatible relation on a semigroup S. Then a nonempty subset A of S is called a generalized θ-upper rough subsemigroup of S if $\theta_+(A)$ is a subsemigroup of S and A is called a generalized θ-lower rough subsemigroup of S if $\theta_-(A)$ is a subsemigroup of S

Theorem 3.6. *Let θ be a reflexive, transitive and compatible relation on a semigroup S. Then*

(1) *If A is a subsemigroup of S, then A is a generalized θ-upper rough subsemigroup of S.*
(2) *If θ is complete, then for a subsemigroup A of S, $\theta_-(A)$ is, if it is nonempty, a subsemigroup of S.*

Proof. (1) Let A be a subsemigroup of S. Then by Theorem 3.3, we have

$$\theta_+(A)\theta_+(A) \subseteq \theta_+(AA) \subseteq \theta_+(A).$$

Thus $\theta_+(A)$ is a subsemigroup of S, that is A is a generalized θ-upper rough subsemigroup of S.

(2) Let A be a subsemigroup of S. Then by Theorem 3.5, we have

$$\theta_-(A)\theta_-(A) \subseteq \theta_-(AA) \subseteq \theta_-(A).$$

Thus $\theta_-(A)$, if it is nonempty, is a subsemigroup of S. □

Lemma 3.7. *Let θ be a reflexive, transitive and compatible relation on a semigroup S, then for a nonempty subset A of S*

(1) $(\theta_+(A))^n \subseteq \theta_+(A^n)$ *for all* $n \in \mathbb{N}$.
(2) *If θ is complete, then* $(\theta_-(A))^n \subseteq \theta_-(A^n)$ *for all* $n \in \mathbb{N}$.

4. Rough (m, n)-bi-ideals in semigroups

Definition 4.1 ([1])**.** A subset A of a semigroup S is called an $(m, 0)$-ideal ($(0, n)$-ideal) if $A^m S \subseteq A$ ($SA^n \subseteq A$).

A subset A of a semigroup S is called a ρ-upper rough $(m, 0)$-ideal ($(0, n)$-ideal) of S if $\rho^-(A)$ is an $(m, 0)$-ideal ($(0, n)$-ideal) of S. Similarly a subset A of a semigroup S is called a ρ-lower rough $(m, 0)$-ideal ($(0, n)$-ideal) of S if $\rho_-(A)$ is an $(m, 0)$-ideal ($(0, n)$-ideal) of S.

Theorem 4.2. *Let ρ be a congruence relation on a semigroup S. If A is an $(m,0)$-ideal ($(0,n)$-ideal) of S. Then*

(1) *$\rho^-(A)$ is an $(m,0)$-ideal ($(0,n)$-ideal) of S.*
(2) *If ρ is complete, then $\rho_-(A)$ is, if it is nonempty, an $(m,0)$-ideal ($(0,n)$-ideal) of S.*

Proof. (1) Let A be an $(m,0)$-ideal of S, that is, $A^mS \subseteq A$. Note that $\rho^-(S) = S$. Then by Theorem 2.2 and Lemma 2.6(1), we have

$$\left(\rho^-(A)\right)^m S \subseteq \rho^-(A^m)\rho^-(S) \subseteq \rho^-(A^mS) \subseteq \rho^-(A).$$

This shows that $\rho^-(A)$ is an $(m,0)$-ideal of S, that is, A is an ρ-upper rough $(m,0)$-ideal of S. Similarly we can show that ρ-upper approximation of a $(0,n)$-ideal is a $(0,n)$-ideal of S.

(2) Let A be an $(m,0)$-ideal of S, that is, $A^mS \subseteq A$. Note that $\rho_-(S) = S$. Then by Theorem 2.3 and Lemma 2.6(2), we have

$$\left(\rho_-(A)\right)^m S \subseteq \rho_-(A^m)\rho_-(S) \subseteq \rho_-(A^mS) \subseteq \rho_-(A).$$

This shows that $\rho_-(A)$ is an $(m,0)$-ideal of S, that is, A is a ρ-lower rough $(m,0)$-ideal of S. Similarly we can show that ρ-lower approximation of a $(0,n)$-ideal is a $(0,n)$-ideal of S. □

Definition 4.3 ([1])**.** A subsemigroup A of a semigroup S is called an (m,n)-bi-ideal of S if $A^mSA^n \subseteq A$.

A subset A of a semigroup S is called a ρ-upper [ρ-lower] rough (m,n)-bi-ideal of S if $\rho^-(A)$ [$\rho_-(A)$] is an (m,n)-bi-ideal of S.

Theorem 4.4. *Let ρ be a congruence relation on a semigroup S. If A is an (m,n)-bi-ideal of S, then it is a ρ-upper rough (m,n)-bi-ideal of S.*

Proof. Let A be an (m,n)-bi-ideal of S. Then by Theorem 2.2 and Lemma 2.6(1), we have

$$\begin{aligned}(\rho^-(A))^m S(\rho^-(A))^n &\subseteq \rho^-(A^m)\rho^-(S)\ \rho^-(A^n)\\ &\subseteq \rho^-(A^mS)\ \rho^-(A^n)\\ &\subseteq \rho^-(A^mSA^n)\\ &\subseteq \rho^-(A).\end{aligned}$$

From this and Theorem 2.4(1), we obtain that $\rho^-(A)$ is an (m,n) bi-ideal of S, that is, A is a ρ-upper rough (m,n)-bi-ideal of S. □

Theorem 4.5. *Let ρ be a complete congruence relation on a semigroup S. If A is an (m,n)-bi-ideal of S, then $\rho_-(A)$ is, if it is nonempty, an (m,n)-bi-ideal of S.*

Proof. Let A be an (m,n)-bi-ideal of S. Then by Theorem 2.3 and Lemma 2.6(2), we have

$$\begin{aligned}(\rho_-(A))^m S(\rho_-(A))^n &\subseteq \rho_-(A^m)\rho_-(S)\rho_-(A^n)\\ &\subseteq \rho_-(A^mS)\ \rho_-(A^n)\\ &\subseteq \rho_-(A^mSA^n)\\ &\subseteq \rho_-(A).\end{aligned}$$

From this and Theorem 2.5(1), we obtain that $\rho_-(A)$ is, if it is nonempty, an (m,n)-bi-ideal of S. □

Definition 4.6 ([1])**.** A subset A of a semigroup S is called a generalized (m,n)-bi-ideal of S if $A^mSA^n \subseteq A$.

A subset A of a semigroup S is called a ρ-upper [ρ-lower] rough generalized (m,n)-bi-ideal of S if $\rho^-(A)$ [$\rho_-(A)$] is a generalized (m,n)-bi-ideal of S.

Theorem 4.7. *Let ρ be a congruence relation on a semigroup S. If A is a generalized (m,n)-bi-ideal of S, then it is a ρ-upper rough generalized (m,n)-bi-ideal of S.*

Proof. The proof of this theorem is similar to the proof of Theorem 4.4. □

Theorem 4.8. *Let ρ be a complete congruence relation on a semigroup S. If A is a generalized (m,n)-bi-ideal of S, then $\rho_-(A)$ is, if it is nonempty, a generalized (m,n)-bi-ideal of S.*

Proof. The proof of this theorem is similar to the proof of Theorem 4.5. □

Definition 4.9 ([2])**.** A subset A of a semigroup S is called an m-left ideal (n-right ideal) of S if $S^mA \subseteq A$ ($AS^n \subseteq A$).

A subset A of a semigroup S is called a ρ-upper rough m-left ideal (ρ-upper rough n-right ideal) of S if $\rho^-(A)$ is an m-left ideal (n-right ideal) of S. Similarly a subset A of a semigroup S is called a ρ-lower rough m-left ideal (ρ-lower rough n-right ideal) of S if $\rho_-(A)$ is an m-left ideal (n-right ideal) of S.

Theorem 4.10. *Let ρ be a congruence relation on a semigroup S. If A is an m-left ideal (n-right ideal) of S, then*

(1) *$\rho^-(A)$ is an m-left ideal (n-right ideal) of S.*
(2) *If ρ is complete, then $\rho_-(A)$ is, if it is nonempty, an m-left ideal (n-right ideal) of S.*

Proof. (1) Let A be an m-left ideal of S, that is, $S^mA \subseteq A$. Then by Theorem 2.2 and Lemma 2.6(1), we have

$$S^m\rho^-(A) = (\rho^-(S))^m\rho^-(A) \subseteq \rho^-(S^m)\rho^-(A) \subseteq \rho^-(S^mA) \subseteq \rho^-(A).$$

This shows that $\rho^-(A)$ is an m-left ideal of S, that is, A is an ρ-upper rough m-left ideal of S. Similarly we can show that ρ-upper approximation of a n-right ideal is a n-right ideal of S.

(2) Let A be an m-left ideal of S, that is, $S^mA \subseteq A$. Then by Theorem 2.3 and Lemma 2.6(2), we have

$$S^m\rho_-(A) = (\rho_-(S))^m\rho_-(A) \subseteq \rho_-(S^m)\rho_-(A) \subseteq \rho_-(S^mA) \subseteq \rho_-(A).$$

This shows that $\rho_-(A)$ is an m-left ideal of S, that is, A is a ρ-lower rough m-left ideal of S. Similarly we can show that ρ-lower approximation of an n-right ideal is an n-right ideal of S. □

5. Generalized rough (m, n)-bi-ideals in semigroups

Let θ be a reflexive, transitive and complete compatible relation on a semigroup S. A subset A of a semigroup S is called a generalized θ-upper rough $(m, 0)$-ideal ($(0, n)$-ideal) of S if $\theta_+(A)$ is an $(m, 0)$-ideal ($(0, n)$-ideal) of S. Similarly a subset A of a semigroup S is called a generalized θ-lower rough $(m, 0)$-ideal ($(0, n)$-ideal) of S if $\theta_-(A)$ is an $(m, 0)$-ideal ($(0, n)$-ideal) of S.

Theorem 5.1. *Let θ be a reflexive, transitive and compatible relation on a semigroup S. If A is an $(m, 0)$-ideal ($(0, n)$-ideal) of S. Then*

(1) *$\theta_+(A)$ is an $(m, 0)$-ideal ($(0, n)$-ideal) of S.*
(2) *If θ is complete, then $\theta_-(A)$ is, if it is nonempty, an $(m, 0)$-ideal ($(0, n)$-ideal) of S.*

Proof. (1) Let A be an $(m, 0)$-ideal of S, that is, $A^m S \subseteq A$. Then by Theorem 3.3 and Lemma 3.7(1), we have

$$(\theta_+(A))^m S \subseteq \theta_+(A^m)\theta_+(S) \subseteq \theta_+(A^m S) \subseteq \theta_+(A).$$

This shows that $\theta_+(A)$ is an $(m, 0)$-ideal of S, that is, A is a generalized θ-upper rough $(m, 0)$-ideal of S. Similarly we can show that generalized θ-upper approximation of a $(0, n)$-ideal is a $(0, n)$-ideal of S.

(2) Let A be an $(m, 0)$-ideal of S, that is, $A^m S \subseteq A$. Then by Theorem 3.5 and Lemma 3.7(2), we have

$$(\theta_-(A))^m S \subseteq \theta_-(A^m)\theta_-(S) \subseteq \theta_-(A^m S) \subseteq \theta_-(A).$$

This shows that $\theta_-(A)$ is an $(m, 0)$-ideal of S, that is, A is a generalized θ-lower rough $(m, 0)$-ideal of S. Similarly we can show that generalized θ-lower approximation of a $(0, n)$-ideal is a $(0, n)$-ideal of S. □

A subset A of a semigroup S is called a generalized θ-upper [generalized θ-lower] rough (m, n)-bi-ideal of S if $\theta_+(A)$ [$\theta_-(A)$] is an (m, n)-bi-ideal of S.

Theorem 5.2. *Let θ be a reflexive, transitive and compatible relation on a semigroup S. If A is an (m, n)-bi-ideal of S, then it is a generalized θ-upper rough (m, n)-bi-ideal of S.*

Proof. Let A be an (m, n)-bi-ideal of S. Then by Theorem 3.3 and Lemma 3.7(1), we have

$$\begin{aligned}(\theta_+(A))^m S(\theta_+(A))^n &\subseteq \theta_+(A^m)\theta_+(S)\ \theta_+(A^n)\\ &\subseteq \theta_+(A^m S)\ \theta_+(A^n)\\ &\subseteq \theta_+(A^m S A^n)\\ &\subseteq \theta_+(A).\end{aligned}$$

From this and Theorem 3.6(1), we obtain that $\theta_+(A)$ is an (m, n)-bi-ideal of S, that is, A is a generalized θ-upper rough (m, n)-bi-ideal of S. □

Theorem 5.3. *Let θ be a reflexive, transitive and complete compatible relation on a semigroup S. If A is an (m, n)-bi-ideal of S, then $\rho_-(A)$ is, if it is nonempty, an (m, n)-bi-ideal of S.*

Proof. Let A be an (m, n)-bi-ideal of S. Then by Theorem 3.5 and Lemma 3.7(2), we have

$$\begin{aligned}(\theta_-(A))^m S(\theta_-(A))^n &\subseteq \theta_-(A^m)\theta_-(S)\theta_-(A^n)\\ &\subseteq \theta_-(A^m S)\ \theta_-(A^n)\\ &\subseteq \theta_-(A^m S A^n)\\ &\subseteq \theta_-(A).\end{aligned}$$

From this and Theorem 3.6(2), we obtain that $\theta_-(A)$ is, if it is nonempty, an (m, n)-bi-ideal of S. □

A subset A of a semigroup S is called a generalized θ-upper rough m-left ideal (generalized θ-upper rough n-right ideal) of S if $\theta_+(A)$ is an m-left ideal (n-right ideal) of S. Similarly a subset A of a semigroup S is called a generalized θ-lower rough m-left ideal (generalized lower rough n-right ideal) of S if $\theta_-(A)$ is an m-left ideal (n-right ideal) of S.

Theorem 5.4. *Let θ be a reflexive, transitive and compatible relation on a semigroup S. If A is an m-left ideal (n-right ideal) of S. Then*

(1) *$\theta_+(A)$ is an m-left ideal (n-right ideal) of S.*
(2) *If θ is complete, then $\theta_-(A)$ is, if it is nonempty, an m-left ideal (n-right ideal) of S.*

Proof. (1) Let A be an m-left ideal of S, that is, $S^m A \subseteq A$. Then by Theorem 3.3 and Lemma 3.7(1), we have

$$S^m\theta_+(A) = (\theta_+(S))^m\theta_+(A) \subseteq \theta_+(S^m)\theta_+(A) \subseteq \theta_+(S^m A) \subseteq \theta_+(A).$$

This shows that $\theta_+(A)$ is an m-left ideal of S, that is, A is a generalized θ-upper rough m-left ideal of S. Similarly we can show that generalized θ-upper approximation of a n-right ideal is a n-right ideal of S.

(2) Let A be an m-left ideal of S, that is, $S^m A \subseteq A$. Then by Theorem 3.5 and Lemma 3.7(2), we have

$$S^m\theta_-(A) = (\theta_-(S))^m\theta_-(A) \subseteq \theta_-(S^m)\theta_-(A) \subseteq \theta_-(S^m A) \subseteq \theta_-(A).$$

This shows that $\theta_-(A)$ is an m-left ideal of S, that is, A is a generalized θ-lower rough m-left ideal of S. Similarly we can show that generalized θ-lower approximation of a n-right ideal is a n-right ideal of S. □

6. Rough and generalized rough (m, n)-quasi-ideals in semigroups

A nonempty subset Q of a semigroup S is called a quasi-ideal of S if $SQ \cap QS \subseteq Q$. A subset Q of a semigroup S is called a ρ-lower [generalized θ-lower] rough quasi-ideal of S if $\rho_-(Q)$ [$\theta_-(Q)$] is a quasi-ideal of S.

Theorem 6.1. *Let θ be a reflexive, transitive and complete compatible relation on a semigroup S. If Q is a quasi-ideal of S, then Q is a generalized θ-lower rough quasi-ideal of S.*

Proof. Let Q be a quasi-ideal of S. Now by Theorems 3.2 and 3.5, we get

$$\begin{aligned} \theta_-(Q)S \cap S\theta_-(Q) &= \theta_-(Q)\theta_-(S) \cap \theta_-(S)\theta_-(Q) \\ &\subseteq \theta_-(QS) \cap \theta_-(SQ) \\ &= \theta_-(QS \cap SQ) \\ &\subseteq \theta_-(Q). \end{aligned}$$

Thus we obtain that $\theta_-(Q)$ is a quasi-ideal of S, that is, Q is a generalized θ-lower rough quasi-ideal of S. □

Corollary 6.2. *Let ρ be a complete congruence relation on a semigroup S. If Q is a quasi-ideal of S, then Q is a ρ-lower rough quasi-ideal of S.*

Proof. This follows from Theorems 2.1 and 2.3. □

Definition 6.3 ([2])**.** A nonempty subset Q of a semigroup S is called an (m, n)-quasi-ideal of S if $S^mQ \cap QS^n \subseteq Q$.

A subset Q of a semigroup S is called a ρ-lower [generalized θ-lower] rough (m, n)-quasi-ideal of S if $\rho_-(Q)$ [$\theta_-(Q)$] is an (m, n)-quasi-ideal of S.

Theorem 6.4. *Let θ be a reflexive, transitive and complete compatible relation on a semigroup S. If Q is an (m, n)-quasi-ideal of S, then Q is a generalized θ-lower rough (m, n)-quasi-ideal of S.*

Proof. Let Q be an (m, n)-quasi-ideal of S. Now by Theorems 3.2 and 3.5 and Lemma 3.7(2), we get

$$\begin{aligned} S^m\theta_-(Q) \cap \theta_-(Q)S^n &= (\theta_-(S))^m\theta_-(Q) \cap \theta_-(Q)(\theta_-(S))^n \\ &\subseteq \theta_-(S^m)\theta_-(Q) \cap \theta_-(Q)\theta_-(S^n) \\ &\subseteq \theta_-(S^mQ) \cap \theta_-(QS^n) \\ &= \theta_-(S^mQ \cap QS^n) \\ &\subseteq \theta_-(Q). \end{aligned}$$

Thus we obtain that $\theta_-(Q)$ is an (m, n)-quasi-ideal of S, that is, Q is a generalized θ-lower rough (m, n)-quasi-ideal of S. □

Corollary 6.5. *Let ρ be a complete congruence relation on a semigroup S. If Q is an (m, n)-quasi-ideal of S, then Q is a ρ-lower rough (m, n)-quasi-ideal of S.*

Proof. This follows from Theorems 2.1 and 2.3, and Lemma 2.6(2). □

References

[1] M. Ansari, M. R. Khan and J. P. Kaushik, Notes on generalized (m, n)-bi-ideals in semigroups with involution, Int. J. Algebra. 3 (2009) 945–952.

[2] M. Ansari, M. R. Khan and J. P. Kaushik, A Note on (m, n)-quasi-ideals in semigroups, Int. J. Math. Anal. 3 (2009) 1853–1858.

[3] M. Aslam, M. Shabir and N. Yaqoob, Roughness in left almost semigroups, Accepted in J. Adv. Res. Pure Math.

[4] R. Biswas and S. Nanda, Rough groups and rough subgroups, Bull. Pol. Acad. Sci. Math. 42 (1994) 251–254.

[5] R. Chinram, Rough prime ideals and rough fuzzy prime ideals in gamma-semigroups, Commun. Korean Math. Soc. 24 (2009) 341–351.

[6] B. Davvaz, Roughness in rings, Inform. Sci. 164 (2004) 147–163.
[7] Y. B. Jun, Roughness of ideals in BCK-algebras, Sci. Math. Jpn. 57(1) (2003) 165–169.
[8] Y. C. Kim and Y. S. Kim, Generalized rough sets and relations, Int. Math. Forum 4 (2009) 257–266.
[9] M. Kondo, On the structure of generalized rough sets, Inform. Sci. 176 (2006) 589–600.
[10] N. Kuroki, Rough ideals in semigroups, Inform. Sci. 100 (1997) 139–163.
[11] N. Kuroki and J. N. Mordeson, Structure of rough sets and rough groups, J. Fuzzy Math. 5(1) (1997) 183–191.
[12] S. Lajos, Generalized ideals in semigroups, Acta Sci. Math. 22 (1961) 217–222.
[13] S. Lajos, Notes on (m,n)-ideals I, Proc. Japan Acad. 39 (1963) 419–421 .
[14] S. Lajos, Notes on (m,n)-ideals II, Proc. Japan Acad. 40 (1964) 631–632.
[15] Z. Pawlak, Rough sets, Int. J. Comput. Inform. Sci. 11 (1982) 341–356.
[16] Z. Pawlak, Rough sets: theoretical aspects of reasoning about data, Kluwer Academic Publishers, Dordrecht, 1991.
[17] Q. M. Xiao and Z. L. Zhang, Rough prime ideals and rough fuzzy prime ideals in semi-groups, Inform. Sci. 176 (2006) 725–733.
[18] W. Zhang and W. Wu, Theory and Method of Roughness, Science Press, Beijing, 2001.

Muhammad Aslam (draslamqau@yahoo.com) – Department of Mathematics, Quaid-i-Azam University, Islamabad-45320, Pakistan.

Muhammad Shabir (mshabirbhatti@yahoo.co.uk) – Department of Mathematics, Quaid-i-Azam University, Islamabad-45320, Pakistan.

Naveed Yaqoob (nayaqoob@ymail.com) – Department of Mathematics, Quaid-i-Azam University, Islamabad-45320, Pakistan.

Asim Shabir (asimshabir83@yahoo.com) – Department of Mathematics, Quaid-i-Azam University, Islamabad-45320, Pakistan.