Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 2, No. 2, (October 2011), pp. 131- 139
ISSN 2093–9310
http://www.afmi.or.kr

On direct product of t-norm (λ, μ)-fuzzy subrings and t-norm (λ, μ)-fuzzy subsemi-rings

X. Arul Selvaraj, D. Sivakumar

Received 6 January 2011; Revised 5 March 2011; Accepted 6 March 2011

Abstract. In this paper the direct product of t-norm $(\lambda,\ \mu)$-fuzzy subrings (ideals) has been discussed. We have proved that if the direct product of two t-norm $(\lambda,\ \mu)$-fuzzy subsets is a t-norm $(\lambda,\ \mu)$-fuzzy subring (ideal) then at least one of the t-norm $(\lambda,\ \mu)$-fuzzy subsets must be a t-norm $(\lambda,\ \mu)$-fuzzy subring (ideal) and also we introduce the concept of t-norm $(\lambda,\ \mu)$-fuzzy subsemi-ring and t-norm $(\lambda,\ \mu)$-fuzzy ideals of a semi-ring which can be regarded as a generalization of fuzzy subsemi-ring and fuzzy ideals and discuss some related properties.

2010 AMS Classification: 68A72, 16Y30, 16Y60

Keywords: Direct product of t-norm $(\lambda,\ \mu)$-fuzzy subsets, t-norm $(\lambda,\ \mu)$-fuzzy subring, t-norm $(\lambda,\ \mu)$-fuzzy ideal, t-norm $(\lambda,\ \mu)$-fuzzy subsemi-ring, homomorphism, epimorphism.

Corresponding Author: X. Arul Selvaraj (xaSelvarajmaths@gmail.com)

1. Introduction

The notions of fuzzy ideals were introduced by S-Abou-Zaid in 1991 [1, 2]. Using the notation of a fuzzy subset introduced by Zadeh [19] , W. Liu [10] defined fuzzy set and fuzzy ideals of a ring. The notion of fuzzy subgroup was introduced by A. Rosenfeld [14] in his pioneering paper. Subsequently the definition of fuzzy subgroup was generalized by Negoita and Ralescu [11] and by Anthony and Sherwood [3] product of fuzzy subgroups were first defined by C. V. Negoita and D. A. Ralescu [11]. H. Sherwood [3] also studied product of fuzzy subgroups in a generalized form. Ray [13] also obtained some results on product of fuzzy subgroups. Fuzzy ideals of a ring were first introduced by Liu [17]. In this paper Liu [18] introduced the concept of operations of fuzzy ideals of a ring. Here we have obtained a result relating to direct product of fuzzy subrings (ideals). Bhakat and Das [8, 9] introduced the concepts of $(\in, \in \vee q)$-fuzzy groups and $(\in, \in \vee q)$-fuzzy subring. Bingxue Yao [7] introduced the concepts of $(\lambda,\ \mu)$-fuzzy subring. We introduced the notion of

t-norm (λ, μ)-fuzzy subsemi-rings. This is an extension of the result of Bingxue Yao [7]. Tazid Ali [16] introduced the concepts of direct product of fuzzy subring. This is an extension of the result of X. Arul Selvaraj, D. Sivakumar and B. Anitha [6]. We always assume that $0 \le \lambda < \mu \le 1$.

2. Preliminaries

In this section we recall some useful definitions and examples.

Definition 2.1 ([4, 5, 13])**.** A triangular norm or t-norm is a function $t : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions for each $a, b, c, d, \in [0,1]$:

(i) $t(0,0) = 0, t(a,1) = a$;
(ii) $t(a,b) \le t(c,d)$,whenever $a \le c, b \le d$;
(iii) $t(a,b) = t(b,a)$;
(iv) $t(t(a,b),c) = t(a,t(b,c))$.

Example 2.2 ([4, 5])**.** A funcation $t : [0,1] \times [0,1] \to [0,1]$ defined as $t(a,b) = ab$ is a t-norm.

Example 2.3 ([4, 5])**.** A funcation $t : [0,1] \times [0,1] \to [0,1]$ defined as $t(a,b) = a \wedge b$ is a t-norm.

Example 2.4 ([4, 5])**.** A function $t : [0,1] \times [0,1] \to [0,1]$ defined as

$$t(a,b) = \begin{cases} a & \text{if b=1,} \\ b & \text{if a=1,} \\ 0 & \text{otherwise} \end{cases}$$

is a t-norm.

Definition 2.5 ([16])**.** By a fuzzy subset of a set X, we mean a function from X into [0,1]. The set of all fuzzy subsets of X is called fuzzy power set of X and is denoted by $I^X = [0,1]^X$.

Definition 2.6 ([16])**.** A fuzzy subset A of a ring R is said to be a fuzzy subring of R if $\forall\ a, b \in R$,

$$A(a-b) \ge A(a) \wedge A(b)$$
$$A(ab) \ge A(a) \wedge A(b)$$

The set of all fuzzy subrings of R is denoted by $I(R)$.

Definition 2.7 ([16])**.** A fuzzy subset A of a ring R is called a fuzzy ideal of R if $\forall\ a, b \in R$,

$$A(a-b) \ge A(a) \wedge A(b)$$
$$A(ab) \ge A(a) \vee A(b).$$

Definition 2.8 ([16])**.** Let A,B be fuzzy subsets of the sets X and Y respectively. The product of A and B, denoted by $A \times B$, is a fuzzy subset of $X \times Y$ defined as follows

$$(A \times B)(x,y) = A(x) \wedge B(y), \forall (x,y) \in X \times Y.$$

Definition 2.9. A non empty set R together with two binary operations "+" and "·" is said to be a semiring if

(i) $(R, +)$ is a monoid.
(ii) $(R, \cdot)$ is a semigroup.
(iii) $a(b+c) = ab + ac \quad \forall\ a, b, c \in R$.
(iv) $(a+b)c = ac + bc, \quad \forall\ a, b, c \in R$.

Definition 2.10. Let A be a fuzzy set on a set X. A level set A_α on X for $\alpha \in [0,1]$ is defined as $A_\alpha = \{x \in X \mid A(x) \geq \alpha\}$.

Definition 2.11. Let A be a fuzzy set of a semiring R. A is said to fuzzy subsemi-ring (fuzzy ideal) if for all $x, y \in R$

(i) $A(x+y) \geq A(x) \wedge A(y)$,
(ii) $A(xy) \geq A(x) \wedge A(y), \quad (A(xy) \geq A(x) \vee A(y))$.

Definition 2.12. Let A be a fuzzy subset of R. Then A is called a $(\lambda,\ \mu)$-fuzzy subsemi-ring of R if $\forall\ x, y \in R$

(i) $A(x+y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$,
(ii) $A(xy) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$.

3. On direct product of t-norm $(\lambda,\ \mu)$-fuzzy subrings

Based on the concepts of On direct product of $(\lambda,\ \mu)$-fuzzy subrings introduced by X. Arul Selvaraj, D. Sivakumar and B. Anitha [6]. We introduced t-norm $(\lambda,\ \mu)$ subrings. Let R be a ring with the zero element 0 and S be a ring with the zero element $0'$. The product of R and S of t-norm $(\lambda,\ \mu)$, denoted by $(R \times S) \vee \lambda$, is the set $(R \times S) \vee \lambda = \{t((r, s), \mu) \mid r \in R, s \in S\}$. Then $(R \times S) \vee \lambda$ is a ring where addition, multiplication and inverse are defined as

$$((r_1, s_1) + (r_2, s_2)) \vee \lambda = t((r_1 + r_2, s_1 + s_2), \mu)$$
$$(r_1, s_1)(r_2, s_2) \vee \lambda = t((r_1 r_2, s_1 s_2), \mu) \text{ and}$$
$$-(r, s) \vee \lambda = t((-r, -s), \mu).$$

Then zero element of $(R \times s) \vee \lambda$ is $(0, 0') \wedge \mu$.

Definition 3.1. By a t-norm $(\lambda,\ \mu)$-fuzzy subset of a set X, we mean a function from X into [0,1]. The set of all t-norm $(\lambda,\ \mu)$-fuzzy subsets of X is called t-norm $(\lambda,\ \mu)$-fuzzy power set of X and is denoted by $I_{t-norm\ (\lambda,\ \mu)}\ ^{(X)} = [0,1]^X$.

Definition 3.2. A t-norm $(\lambda,\ \mu)$-fuzzy subset A of a ring R is said to be a t-norm $(\lambda,\ \mu)$-fuzzy subring of R if $\forall a, b \in R$,

$$A(a-b) \vee \lambda \geq t(t(A(a), A(b)), \mu)$$
$$A(ab) \vee \lambda \geq t(t(A(a), A(b)), \mu)$$

The set of all t-norm $(\lambda,\ \mu)$-fuzzy subrings of R is denoted by $I_{t-norm\ (\lambda,\ \mu)}(R)$.

Lemma 3.3. *If A is a t-norm $(\lambda,\ \mu)$-fuzzy subring of a ring R, then*

$$A(0) \vee \lambda \geq t(A(r), \mu),\ \forall\ r \in R.$$

Proof. The proof is straightforward and omitted. □

Lemma 3.4. *Let $A \in I_{t-norm\ (\lambda,\ \mu)}(R)$. Then A is a t-norm $(\lambda,\ \mu)$ fuzzy subrings of R if and only if the level subset A_t is a subring of R $\forall\ t \in Im(\mu)$.*

Proof. The proof is straightforward and omitted. □

Definition 3.5. A t-norm $(\lambda,\ \mu)$-fuzzy subset A of a ring R is called a t-norm $(\lambda,\ \mu)$-fuzzy ideal of R if $\forall a, b \in R$,

$$A(a-b) \vee \lambda \geq t(t(A(a), A(b)), \mu)$$
$$A(ab) \vee \lambda \geq t((A(a) \vee A(b)), \mu)$$

Theorem 3.6. *A t-norm $(\lambda,\ \mu)$-fuzzy subset A of a ring R is a t-norm $(\lambda,\ \mu)$-fuzzy ideal of R if and only if the level subset A_t is an ideal of R $\forall\ t \in Im(A)$.*

Proof. The proof is straightforward and omitted. □

Definition 3.7. Let A,B be t-norm $(\lambda,\ \mu)$-fuzzy subsets of the sets X and Y respectively. The product of A and B , denoted by $(A \times B) \vee \lambda$, is a t-norm $(\lambda,\ \mu)$-fuzzy subset of $(X \times Y) \vee \lambda$ defined as follows

$$((A \times B)(x, y)) \vee \lambda = t(t((A(x), B(y))), \mu),\ \forall (x, y) \in X \times Y.$$

Theorem 3.8. *Let A be a t-norm $(\lambda,\ \mu)$-fuzzy subring of the ring R and B be a t-norm $(\lambda,\ \mu)$-fuzzy subring of the ring S. Then $A \times B$ is a t-norm $(\lambda,\ \mu)$-fuzzy subring of the ring $R \times S$.*

Proof. Let $((r_1, s_1), (r_2, s_2)) \vee \lambda \in (R \times S) \vee \lambda$. Then

$$((r_1, s_1)(r_2, s_2)) \vee \lambda = t(((r_1 r_2, s_1 s_2)), \mu).$$

Now

$$\begin{aligned}
&[(A \times B)((r_1, s_1) - (r_2, s_2))] \vee \lambda = [(A \times B)(r_1 - r_2, s_1 - s_2)] \vee \lambda \\
&= t(t(A(r_1 - r_2), B(s_1 - s_2)), \mu) = t(t(A(r_1 - r_2), \mu), t(B(s_1 - s_2), \mu)) \\
&\geq t(\{t(t(A(r_1), A(r_2), \mu), t(B(s_1), B(s_2), \mu))\}, \mu) \\
&= t(t(A(r_1), B(s_1), A(r_2), B(s_2)), \mu) \\
&= t(t((A \times B)(r_1, s_1), (A \times B)(r_2, s_2)), \mu).
\end{aligned}$$

Also

$$\begin{aligned}
&[(A \times B)((r_1, s_1)(r_2, s_2))] \vee \lambda = [(A \times B)(r_1 r_2, s_1 s_2)] \vee \lambda \\
&= t(t(A(r_1 r_2), B(s_1 s_2)), \mu) = t(t(A(r_1 r_2), \mu), t(B(s_1 s_2), \mu)) \\
&\geq t(t(A(r_1), A(r_2), \mu), t(B(s_1), B(s_2), \mu)) \\
&= t(t(A(r_1), B(s_1), A(r_2), B(s_2)), \mu) \\
&= t(t((A \times B)(r_1, s_1), (A \times B)(r_2, s_2)), \mu).
\end{aligned}$$

Hence $A \times B$ is a t-norm $(\lambda,\ \mu)$-fuzzy subring of $R \times S$. □

Theorem 3.9. *Let A be a t-norm $(\lambda,\ \mu)$-fuzzy ideal of the ring R and B be the t-norm $(\lambda,\ \mu)$-fuzzy ideal of the ring S. Then $(A \times B)$ is a t-norm $(\lambda,\ \mu)$-fuzzy ideal of the ring $(R \times S)$.*

Proof. It is similar to the proof of Theorem 3.8. □

However if A and B are t-norm $(\lambda,\ \mu)$-fuzzy subsets of R and S respectively, such that $A \times B$ is a t-norm $(\lambda,\ \mu)$-fuzzy subring (ideal) of $R \times S$ it is not necessarily true that both A and B are t-norm $(\lambda,\ \mu)$-fuzzy subrings (ideals) of R and S respectively as is evident from the following example.

Example 3.10. Consider $X_1 = \{.2, .3\}$ and $X_2 = \{.2, .3, .4\}$ be a any set and $\lambda = .1$ and $\mu = .9$. Let A and B be t-norm $(\lambda,\ \mu)$-fuzzy subsets of X_1 and X_2 respectively given by $A = \{(.2, .7), (.3, .6)\}$ and $B = \{(.2, .8), (.3, .3), (.4, .7)\}$. The t-norm $(\lambda,\ \mu)$-fuzzy subset $A \times B$ of $R \times S$ is given by

$$A \times B = \{((.2, .2), .7), ((.2, .3), .7), ((.2, .4), .7), ((.3, .2), .6), ((.3, .3), .6), ((.3, .4), .6)\}.$$

Then $A \times B$ is a t-norm $(\lambda,\ \mu)$-fuzzy ideal of $R \times S$ but B is not a t-norm $(\lambda,\ \mu)$-fuzzy ideal of S as $B_1 = \{.3\}$ is not an ideal of S. We will show that if $A \times B$ is a t-norm $(\lambda,\ \mu)$-fuzzy subring (ideal) of $R \times S$ then at least one of A and B is a t-norm $(\lambda,\ \mu)$-fuzzy subring (ideal) of R and S.

Theorem 3.11. *Let A and B be the t-norm $(\lambda,\ \mu)$-fuzzy subsets of R and S, respectively. If $(A \times B)$ is a t-norm $(\lambda,\ \mu)$-fuzzy subring of $R \times S$, then at least one of the following statements must hold.*

(1) $A(0) \vee \lambda \geq t(B(s), \mu), \forall\ s \in S.$
(2) $B(0') \vee \lambda \geq t(A(r), \mu),\ \forall\ r \in R.$

Proof. Since R and S are rings, $R \times S$ is also a ring. Let $A \times B$ be a t-norm $(\lambda,\ \mu)$-fuzzy subring of $R \times S$. By contraposition, suppose that none of the statements (1) and (2) hold. Then we can find $r \in R$ and $s \in S$ such that $A(r) \vee \lambda > t(B(0'), \mu)$ and $B(s) \vee \lambda > t(A(0), \mu)$. Now

$$\begin{aligned}
[(A \times B)(r, s)] \vee \lambda &= t(t(A(r), B(s)), \mu) = t(t(A(r), \mu), t(B(s), \mu)) \\
&> t(t(B(0'), \mu), t(A(0), \mu)) = t(t(B(0'), A(0)), \mu) \\
&= t((A \times B)(0, 0'), \mu).
\end{aligned}$$

Thus $(r, s) \in (R \times S), (0, 0')$ is the zero of the ring $R \times S$ and $A \times B$ is a t-norm $(\lambda,\ \mu)$-fuzzy subring of $R \times S$ satisfying $[(A \times B)(r, s)] \vee \lambda \geq t(((A \times B)(0, 0')), \mu)$, which contradicts the Lemma 3.3. Hence either $A(0) \vee \lambda \geq t(B(s), \mu)$, $\forall s \in S$ or $B(0') \vee \lambda \geq t(A(r), \mu), \forall\ r \in R$. □

Theorem 3.12. *Let A and B be t-norm $(\lambda,\ \mu)$-fuzzy subsets of R and S, respectively. If $A \times B$ is a t-norm $(\lambda,\ \mu)$-fuzzy subring of $R \times S$, then either A is a t-norm $(\lambda,\ \mu)$-fuzzy subring of R or B is a t-norm $(\lambda,\ \mu)$-fuzzy subring of S.*

Proof. Suppose $A \times B$ is a t-norm $(\lambda,\ \mu)$-fuzzy subring of $R \times S$. Then by Theorem 3.11 one of the following statements hold:

(i) $A(0) \vee \lambda \geq t(B(s), \mu), \forall\ s \in S.$
(ii) $B(0') \vee \lambda \geq t(A(r), \mu), \forall\ r \in R.$

Suppose (ii) holds. Since R and S are rings, $R \times S$ is also a ring with zero element $(0, 0')$. Let $x, y \in R$. Then $(x, 0'), (y, 0') \in R \times S$. Now using the property $B(0') \vee \lambda \geq t(A(r), \mu), \forall r \in R$, we have, $\forall x, y \in R$,

$$\begin{aligned}
A(x - y) \vee \lambda &= t(t(A(x - y), B(0')), \mu) = t(t(A(x - y), B(0' - 0')), \mu) \\
&= t(((A \times B)(x - y, 0' - 0')), \mu) = t(((A \times B)((x, 0') - (y, 0'))), \mu) \\
&\geq t(((A \times B)(x, 0'), (A \times B)(y, 0')), \mu) \\
&= t(((A(x), B(0')), (A(y), B(0'))), \mu) \\
&= t(t(A(x), A(y)), \mu).
\end{aligned}$$

Also,

$$\begin{aligned} A(xy) \vee \lambda &= t(t(A(xy), B(0')), \mu) = t(((A \times B)(xy, 0'0')), \mu) \\ &= t(((A \times B)((x, 0')(y, 0')), \mu) \\ &\geq t(t((A \times B)(x, 0'), (A \times B)(y, 0')), \mu) \\ &= t(t(t(A(x), B(0')), t(A(y), B(0'))), \mu) \\ &= t(t(A(x), A(y)), \mu). \end{aligned}$$

Hence A is a t-norm $(\lambda,\ \mu)$-fuzzy subring of R. Similarly we can show that, using the property $A(0) \vee \lambda \geq t(B(s), \mu), \forall\ s \in S$, B is a t-norm $(\lambda,\ \mu)$-fuzzy subring of S. □

Theorem 3.13. *Let A and B be fuzzy subsets of R and S, respectively. If $A \times B$ is a t-norm $(\lambda,\ \mu)$-fuzzy ideal of $R \times S$, then either A is a t-norm $(\lambda,\ \mu)$-fuzzy ideal of R or B is a t-norm $(\lambda,\ \mu)$-fuzzy ideal of S.*

Proof. We have already shown that under the stated condition either A is a t-norm $(\lambda,\ \mu)$-fuzzy subring of R or B is a t-norm $(\lambda,\ \mu)$-fuzzy subring of S. Suppose A is a t-norm $(\lambda,\ \mu)$-fuzzy subring of R. We will show that A is a t-norm $(\lambda,\ \mu)$-fuzzy ideal of R. Now using the property $B(0') \vee \lambda \geq t(A(r), \mu), r \in R$, we have, $\forall x, y \in R$

$$\begin{aligned} A(xy) \vee \lambda &= t(t(A(xy), B(0')), \mu) = t(((A \times B)(xy, 0'0')), \mu) \\ &= t(((A \times B)((x, 0')(y, 0'))), \mu) \\ &\geq t(((A \times B)((x, 0') \vee (A \times B)(y, 0'))), \mu) \\ &\qquad\qquad (\text{since } A \times B \text{ is a fuzzy ideal of } R \times S) \\ &= t(((A(x) \wedge B(0')) \vee (A(y) \wedge B(0'))), \mu) \\ &= t(((A(x) \vee A(y)), \mu). \end{aligned}$$

Hence A is a t-norm $(\lambda,\ \mu)$-fuzzy ideal of R. □

Corollary 3.14. *Let $A_1, A_2, ..., A_n$ be t-norm $(\lambda,\ \mu)$-fuzzy subsets of the rings $R_1, R_2, ..., R_n$ respectively. If $A_1 \times A_2 \times ... \times A_n$ is a t-norm $(\lambda,\ \mu)$-fuzzy subring(ideal) of $R_1 \times R_2 \times ... \times R_n$, then at least for one i, $A_i(0_i) \vee \lambda \geq A_k(x) \wedge \mu$, $\forall x \in R_k$, $k = 1, 2, ..., n$ where 0_i denotes the zero element of R_i.*

Corollary 3.15. *Let $A_1, A_2, ..., A_n$ be t-norm $(\lambda,\ \mu)$-fuzzy subsets of the rings $R_1, R_2, ..., R_n$ respectively. If $A_1 \times A_2 \times ... \times A_n$ is a t-norm $(\lambda,\ \mu)$-fuzzy subring(ideal) of $R_1 \times R_2 \times ... \times R_n$, then at least for one i, A_i is a t-norm $(\lambda,\ \mu)$-fuzzy subring of R_i*

4. t-norm $(\lambda,\ \mu)$-fuzzy subsemi-rings

Based on the concepts of $(\lambda,\ \mu)$-fuzzy group introduced by Bingxue Yao [7]. We introduced t-norm $(\lambda,\ \mu)$- subsemi-rings.

Definition 4.1. Let A be a fuzzy subset of R. Then A is called a t-norm $(\lambda,\ \mu)$-fuzzy subsemi-ring of R if $\forall\ x, y \in R$

(i) $A(x + y) \vee \lambda \geq t(t(A(x), A(y)), \mu)$,

(ii) $A(xy) \vee \lambda \geq t(t(A(x), A(y)), \mu)$.

Remark 4.2. A fuzzy subsemi-ring (ideal) is t-norm $(\lambda,\ \mu)$-fuzzy subsemi-ring (ideal)with $\lambda = 0$ and $\mu = 0.9$ but t-norm $(\lambda,\ \mu)$-fuzzy ideal need not be a fuzzy ideal.

Proposition 4.3. *Let A be fuzzy set of R. Then A is a t-norm $(\lambda,\ \mu)$-fuzzy subsemi-ring if and only if all nonempty A_α is a subsemi-ring of R for all $\alpha \in (\lambda,\ \mu]$.*

Proof. Let A be a t-norm $(\lambda,\ \mu)$-fuzzy subsemi-ring of R. Let $\alpha \in (\lambda, \mu]$ and $x, y \in A_\alpha$. Then $A(x) \geq \alpha$ and $A(y) \geq \alpha$. Thus

$$A(x+y) \vee \lambda \geq t(A(x), A(y), \mu) \geq t(\alpha, \alpha, \mu) = \alpha,$$

and so $x + y \in A_\alpha$. Now we have

$$A(xy) \vee \lambda \geq t(A(x), A(y), \mu) \geq t(\alpha, \alpha, \mu) = \alpha,$$

which implies that $xy \in A_\alpha$. Therefore A_α is a subsemi-ring R.

Conversely, let A_α be subsemi-ring of R for all $\alpha \in (\lambda,\ \mu]$, If there exist $x, y \in R$ such that $A(x+y) \vee \lambda < t(A(x), A(y), \mu) = \alpha$, then $A(x+y) < \alpha$ (since $\alpha > \lambda$). Hence $x + y \notin A_\alpha$ for $x, y \in A_\alpha$, which contradicts the fact that A_α is subsemi-ring of R. Hence $A(x+y) \vee \lambda \geq t(A(x), A(y), \mu)$ for all $x, y \in R$. Also

$$A(xy) \vee \lambda \geq t(A(x), A(y), \mu) \geq t(\alpha, \alpha, \mu) = \alpha.$$

In fact, suppose $A(xy) \vee \lambda < t(A(x), A(y), \mu) = \alpha$. Then $A(xy) < \alpha$ since $x > \alpha$. Thus $xy \notin A_\alpha$ for all $x, y \in A_\alpha$. This is a contradiction. So $A(xy) \vee \lambda \geq t(A(x), A(y), \mu)$. Therefore A is a t-norm $(\lambda,\ \mu)$-fuzzy subsemi-ring. □

Example 4.4. (Example of a t-norm $(\lambda,\ \mu)$-fuzzy subsemi-ring which is not a fuzzy subsemi-ring)

Let R be a semiring of positive rational numbers, and let $\lambda = 0.2$ and $\mu = 0.7$

$$A(x) = \begin{cases} 0.8 & \text{if } x\text{=7} \\ 0.7 & \text{if } x \text{ is an integer but } x \neq 7 \\ 0.3 & \text{if } x \text{ is rational} \end{cases}$$

Clearly A is a $(.2, .7)$-fuzzy subsemi-ring. But A is not fuzzy subsemi-ring since $A(14) < t(A(7), A(7))$.

Definition 4.5. Let A be a fuzzy subset of R. Then A is called a t-norm $(\lambda,\ \mu)$-fuzzy ideal of R if, $\forall\ x, y \in R$

(i) $A(x+y) \vee \lambda \geq t(t(A(x), A(y)), \mu)$,
(ii) $A(xy) \vee \lambda \geq t((A(x) \vee A(y)), \mu)$.

Proposition 4.6. *Let A be fuzzy set of R. Then A is a t-norm $(\lambda,\ \mu)$-fuzzy ideal of R if and only if non empty A_α is an ideal of R for all $\alpha \in (\lambda, \mu]$.*

Proof. Let A be a t-norm $(\lambda,\ \mu)$-fuzzy ideal of R. By Proposition 4.3 A_α is a subsemi-ring. Let $x \in A_\alpha$, $\ y \in R$. $A(xy) \vee \lambda \geq t((A(x) \vee A(r)), \mu) \geq \alpha$. Thus $A(xy) \geq \alpha$. Hence $xy \in A_\alpha$. Similarly $yx \in A_\alpha$. Hence A_α is an ideal of R.

Conversely, let A_α be an ideal of R for all $\alpha \in (\lambda, \mu]$. By proposition 4.3, $A(x + y) \vee \lambda \geq t(A(x), A(y), \mu.)$ If there exist $x, y \in R$ such that $A(xy) \vee \lambda < \alpha =$

$t((A(x) \vee A(y)), \mu) < \alpha$, then $A(x) \vee A(y) \geq \alpha$ and $A(xy) < \alpha$. It follows that $x \in A_\alpha$ or $y \in A_\alpha$ and $xy \notin A_\alpha$. This is a contradiction since A_α is an ideal. Therefore A is a t-norm $(\lambda,\ \mu)$-fuzzy ideal. $\square$

Theorem 4.7. *Let $f : R_1 \to R_2$ be a epimorphism of semirings. Let B be a fuzzy subset of R_2. Then B is a t-norm $(\lambda,\ \mu)$-fuzzy ideal (fuzzy subsemi-ring) of R_2 if and only if $f^{-1}(B)$ is t-norm $(\lambda,\ \mu)$-fuzzy ideal (fuzzy subsemi-ring) of R_1 where $[f^{-1}(B)](x) = B(f(x))\forall x \in R_1$.*

Proof. Let B be a t-norm $(\lambda,\ \mu)$-fuzzy ideal of R_2. Then

$$\begin{aligned} f^{-1}(B)(x_1 + x_2) \vee \lambda = B(f(x_1 + x_2) \vee \lambda) &= B(f(x_1) + f(x_2)) \vee \lambda \\ &\geq t(t(B(f(x_1)), B(f(x_2))), \mu) \\ &= t(t(f^{-1}(B)(x_1), f^{-1}(B)(x_2)), \mu) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(B)(x_1x_2) \vee \lambda = B(f(x_1.x_2)) \vee \lambda &= B(f(x_1).f(x_2)) \vee \lambda \\ &\geq t(B(f(x_1) \vee B(x_2)), \mu) \\ &= t((f^{-1}(B)(x_1) \vee f^{-1}(B)(x_2)), \mu) \end{aligned}$$

for all $x_1, x_2 \in R$. Therefore $f^{-1}(B)$ is a t-norm $(\lambda,\ \mu)$-fuzzy ideal of R_1.

Conversely suppose that $f^{-1}(B)$ is a t-norm $(\lambda,\ \mu)$-fuzzy ideal of R_1. Let $y_1, y_2 \in R_2$. Since f is onto, there exist $x_1, x_2 \in R$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then

$$\begin{aligned} B(y_1 + y_2) \vee \lambda = B(f(x_1 + x_2)) \vee \lambda &= f^{-1}(B)(x_1 + x_2) \vee \lambda \\ &\geq t(t(f^{-1}(B)(x_1), f^{-1}(B)(x_2)), \mu) \\ &= t(B(f(x_1) \vee B(f(x_2)), \mu) \\ &= t(t(B(y_1), B(y_2)), \mu) \end{aligned}$$

and

$$\begin{aligned} B(y_1y_2) \vee \lambda = B(f(x_1.x_2)) \vee \lambda &= f^{-1}(B)(x_1.x_2) \vee \lambda \\ &\geq t((f^{-1}(B)(x_1) \vee f^{-1}(B)(x_2)), \mu) \\ &= t((B(f(x_1)) \vee B(f(x_2))), \mu) \\ &= t((B(y_1) \vee B(y_2)), \mu). \end{aligned}$$

Therefore B is a t-norm $(\lambda,\ \mu)$-fuzzy ideal of R_2. $\square$

Acknowledgements. The authors are thankful to the reviewers for their valuable comments and suggestion towards the quality improvement of the paper in my future.

References

[1] S. Abou-Zaid, on Fuzzy subnear-ring and ideals, Fuzzy Sets and Systems 44 (1991) 139–146.
[2] S. Abou-Zaid, on Fuzzy ideals and Fuzzy quotient ring, Fuzzy Sets and Systems 59 (1993) 205–210.

[3] J. M. Anthony and H. Sherwood, Fuzzy groups redefined, J. Math. Anal. Appl. 69 (1979) 124–130.
[4] X. Arul Selvaraj and D. Sivakumar, t-norm (λ, μ)-fuzzy sub-near-rings and t-norm (λ, μ)-fuzzy ideals of near-rings, Int. J. Algebra, 5(3) (2011) 123–128.
[5] X. Arul Selvaraj and D. Sivakumar, t-norm (λ, μ)-fuzzy quotient near-rings and t-norm (λ, μ)-fuzzy quasi-ideals, Int. Math. Forum 6(5) (2011) 203–209.
[6] X. Arul Selvaraj, D. Sivakumar and B. Anitha, On direct product of (λ, μ)-fuzzy subrings, Int. Math. Forum (under review).
[7] Bingxue Yao, (λ, μ)-fuzzy subrings and (λ, μ)-fuzzy ideals, J. Fuzzy Math. 15(4) (2007) 981–987.
[8] S. K. Bhakat and P. Das, $(\in, \in \vee q)$-fuzzy group, Fuzzy Sets and Systems 80 (1996) 359–368.
[9] S. K. Bhakat and P. Das, $(\in, \in \vee q)$-fuzzy normal, quasinormal and maximal subgroups, Fuzzy Sets and Systems 112 (2000) 299–312.
[10] W. Liu, Fuzzy invariant subgroups and fuzzy ideals, Fuzzy Sets and Systems 59 (1993) 205–210.
[11] C. V. Negoita and D. A. Ralescu, Applications of fuzzy sets to system analysis, John Wiley and Sons, New York, Toronto, 1975.
[12] V. Novak, Fuzzy sets and their applications, Adam Hilger, Bristol, 1989.
[13] A. K. Ray, On product to fuzzy subgroups, Fuzzy Sets and Systems 105 (1999) 181–183.
[14] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512–517.
[15] H. Sherwood, Product of fuzzy subgroups, Fuzzy Sets and Systems 11 (1993) 79–89.
[16] Tazid Ali, On direct product of fuzzy subrings, J. Fuzzy Math. 17(2) (2009) 481–485.
[17] Wang-jin-Liu, Fuzzy invariant subgroups and fuzzy ideals, Fuzzy Sets and Systems 8 (1982) 133–139.
[18] Wang-jin-Liu, Operations on fuzzy ideals, Fuzzy Sets and Systems 11 (1983) 31–41.
[19] L. A. Zadeh, Fuzzy sets, Inform. Control 8 (1965) 338–353.

X. Arul Selvaraj (`xaSelvarajmaths@gmail.com`) – Mathematics Wing, D.D.E., Annamalai University, Annamalainagar - 608 002, Tamilnadu, India.

D. Sivakumar (`sivakumardmaths@yahoo.com`) – Mathematics Wing, D.D.E., Annamalai University, Annamalainagar - 608 002, Tamilnadu, India.