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# A new view of fuzzy ideals in rings

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Abstract. By means of a kind of new idea, we redefine some kinds of fuzzy ideals in a ring and investigate some of their related properties. The concepts of strong prime (semiprime) generalized fuzzy (bi-, quasi-) ideals in rings are introduced. In particular, we discuss the relationships between strong prime (resp., semiprime) generalized fuzzy (bi-, quasi-) ideals and prime (resp. semiprime) generalized fuzzy (bi-, quasi-) ideals in rings. Finally, we show that the regular and intra-regular rings can be described by using these kinds of generalized fuzzy ideals.

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## 1. Introduction

As is well known, algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and so on. This provides sufficient motivations to researchers to review various concepts and results from the realm of abstract algebras in the broader framework of fuzzy setting (see [3, 4, 5, 6, 7, 9, 10, 11, 12]).

After the introduction of fuzzy sets by Zadeh [14], there have been a number of generalizations of this fundamental concept. A new type of fuzzy subgroup, that is, the $(\in, \in \vee q)$-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [1] by using the combined notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets. In fact, the $(\in, \in \vee q)$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems with other algebraic structures (see [2, 8, 13, 15]).

In [5], Kuroki introduced the concepts of a fuzzy quasi-ideal and a fuzzy bi-ideal in a ring, and gave some characterization of a regular ring. They also gave some characterization of an intra-regular ring, both a regular and an intra-regular ring, and a regular duo ring in terms of fuzzy ideals.

As a continuation of the above paper. This present paper is organized as follows. In Section 2, we recall some basic definitions and results of rings. By means of a kind of new idea, we redefine some kinds of fuzzy ideals in a ring and investigate some of their related properties. Some fundamental aspects of generalized fuzzy ideals, generalized fuzzy bi-ideals and generalized fuzzy quasi-ideals of rings will be discussed in Sections 3 and 4, respectively. In Section 5, we introduce the concepts of strong prime (semiprime) generalized fuzzy (bi-, quasi-) ideals in rings. In particular, we discuss the relationships between strong prime (semiprime) generalized fuzzy ideals and prime (semiprime) generalized fuzzy (bi-, quasi-) ideals in rings. By using generalized fuzzy ideals (bi-ideals, quasi-ideals), we proceed to characterize the regular rings in Section 6. Finally, in Section 7, we continue to describe the intra-regular rings by using these kinds of generalized fuzzy ideals.

## 2. Preliminaries

Throughout this paper, let $R$ denote a ring unless other specified.

A subset $A$ of $R$ is called a *left (resp., right) ideal* of $R$ if $A$ is closed under addition and $RA \subseteq A$ (resp., $AR \subseteq A$). $A$ is called an *ideal* of $R$ if it is both a left ideal and a right ideal.

A subset $A$ of $R$ is called a *bi-ideal* if $A$ is an addition subgroup closed with multiplication and $ARA \subseteq A$. A subset $A$ of $R$ is called a *quasi-ideal* if $A$ is an addition subgroup with $AR \cap RA \subseteq A$ (see [6, 11]).

If $I$ and $J$ are ideals of $R$. An ideal $P$ of $R$ is called *prime* if $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$ for all ideals $I$ and $J$ of $R$. An ideal $P$ of $R$ is called *semiprime* if $I^2 \subseteq P$ implies $I \subseteq P$ for all ideals $I$ of $R$.

We next state some fuzzy logic concepts. Recall that a *fuzzy set* is a function $\mu : R \to [0,1]$. We denote by $\mathcal{F}(R)$ the set of all fuzzy sets of $R$. For any $A \subseteq R$, the characteristic function of $A$ is denoted by $\chi_A$. We define $\mu^{-1}$ by $\mu^{-1}(x) = \mu(-x)$, for all $x \in R$.

For any $t \in (0,1]$, define a fuzzy set $t_A$ of $R$ by

$$t_A(x) = \begin{cases} t & \text{if} \quad x \in A, \\ 0 & \text{if} \quad x \notin A, \end{cases}$$

for all $x \in R$.

A fuzzy set $\mu$ of $R$ of the form

$$\mu(y) = \begin{cases} t(\neq 0) & \text{if} \quad y = x, \\ 0 & \text{if} \quad y \neq x, \end{cases}$$

is said to be a *fuzzy point with support $x$ and value $t$* and is denoted by $x_t$. A fuzzy point $x_t$ is said to belong to (resp., be quasi-coincident with ) a fuzzy set $\mu$, written as $x_t \in \mu$ (resp., $x_t q \mu$) if $\mu(x) \geq t$ (resp., $\mu(x) + t > 1$). If $x_t \in \mu$ or $x_t\, q\, \mu$, then we write $x_t \in\vee q\mu$. If $\mu(x) < t$ (resp., $\mu(x) + t \leq 1$), then we call $x_t \overline{\in} \mu$ (resp., $x_t\, \overline{q}\, \mu$). We note that the symbol $\overline{\in\vee q}$ means that $\in\vee q$ does not hold.

Now, we give the concept of the product of two fuzzy sets of $R$.

**Definition 2.1** ([3])**.** Let $\mu$ and $\nu$ be fuzzy sets of $R$. Then the *product* of $\mu$ and $\nu$ is defined by

$$(\mu \odot \nu)(x) = \bigvee_{x=\sum\limits_{i=1}^{m} a_i b_i} (\mu(a_i) \wedge \nu(b_i))$$

and $(\mu \odot \nu)(x) = 0$ if $x$ cannot be expressed as $x = \sum\limits_{i=1}^{m} a_i b_i$.

**Definition 2.2** ([5, 10])**.** (1) A fuzzy set $\mu$ of $R$ is called a *fuzzy left (resp., right) ideal* of $R$ if for all $x, y \in R$, the following conditions are satisfied:

(F1a) $\mu(x+y) \geq \mu(x) \wedge \mu(y)$,
(F1a') $\mu(-x) \geq \mu(x)$,
(F1b) $\mu(xy) \geq \mu(y)$ (resp., $\mu(xy) \geq \mu(x)$).

(2) A fuzzy set $\mu$ of $R$ is called a *fuzzy bi-ideal* of $R$ if for all $x, y, z \in R$, it satisfies (F1a), (F1a') and

(F1c) $\mu(xy) \geq \mu(x) \wedge \mu(y)$,
(F1d) $\mu(xyz) \geq \mu(x) \wedge \mu(z)$.

(3) A fuzzy set $\mu$ of $R$ is called a *fuzzy quasi-ideal* of $R$ if it satisfies (F1a), (F1a') and

(F1e) $(\mu \odot \chi_R) \cap (\chi_R \odot \mu) \subseteq \mu$.

A fuzzy set $\mu$ of $R$ is called a *fuzzy ideal* of $R$ if it is both a fuzzy left ideal and a fuzzy right ideal of $R$.

**Remark 2.3** ([5])**.** (1) Every fuzzy ideal of $R$ is a fuzzy quasi-ideal of $R$, but the converse is not true;

(2) Every fuzzy quasi-ideal of $R$ is a fuzzy bi-ideal of $R$, but the converse is not true.

**Definition 2.4** ([1])**.** A fuzzy set $\mu$ of $R$ is called an $(\in, \in\vee q)$*-fuzzy left (resp., right) ideal* of $R$ if for all $t, r \in (0,1]$ and $x, y \in R$,

(F2a) $x_t \in \mu$ and $y_r \in \mu$ imply $(x+y)_{t\wedge r} \in\vee q\mu$,
(F2a') $x_t \in \mu$ implies $(-x)_t \in\vee q\mu$,
(F2b) $y_t \in \mu$ (resp., $x_t \in \mu$) implies $(xy)_t \in\vee q\mu$.

A fuzzy set $\mu$ of $R$ is called an $(\in, \in\vee q)$*-fuzzy ideal* of $R$ if it is both an $(\in, \in\vee q)$-fuzzy left ideal and an $(\in, \in\vee q)$-fuzzy right ideal of $R$.

**Theorem 2.5.** *A fuzzy set $\mu$ of $R$ is an $(\in, \in\vee q)$-fuzzy left (resp., right) ideal of $R$ if and only if for any $x, y \in R$,*

(F3a) $\mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge 0.5$,
(F3a') $\mu(-x) \geq \mu(x) \wedge 0.5$,
(F3b) $\mu(xy) \geq \mu(y) \wedge 0.5$ *(resp.,* $\mu(xy) \geq \mu(x) \wedge 0.5$*).*

For a fuzzy set $\mu$ of $R$ and $t \in (0,1]$, the crisp set $\mu_t = \{x \in R \mid \mu(x) \geq t\}$ is called the *level subset* of $\mu$.

**Theorem 2.6** ([1]). *A fuzzy set $\mu$ of $R$ is an $(\in, \in\vee q)$-fuzzy ideal of $R$ if and only if $\mu_t(\neq \varnothing)$ is an ideal of $R$ for all $t \in (0, 0.5]$.*

Naturally, we consider the concept of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideals of $R$ by means of Bhakat's way.

**Definition 2.7.** A fuzzy set $\mu$ of $R$ is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-*fuzzy left (resp., right)* of $R$ if for all $t, r \in (0, 1]$ and for all $x, y \in R$,
(F4a) $(x + y)_{t\wedge r}\overline{\in}\mu$ implies $x_t\overline{\in} \vee \overline{q}\mu$ or $y_r\overline{\in} \vee \overline{q}\mu$,
(F4a') $(-x)_t\overline{\in}\mu$ implies $x_t\overline{\in} \vee \overline{q}\mu$,
(F4b) $(xy)_t\overline{\in}\mu$ implies $y_t\overline{\in} \vee \overline{q}\mu$ (resp., $x_t\overline{\in} \vee \overline{q}\mu$).

A fuzzy set $\mu$ of $R$ is called an $(\overline{\in}, \overline{\in}\vee\overline{q})$-*fuzzy ideal* of $R$ if it is both an $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy left ideal and an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal of $R$.

**Theorem 2.8.** *A fuzzy set $\mu$ of $R$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (resp., right) of $R$ if and only if for any $x, y \in R$,*
(F5a) $\mu(x + y) \vee 0.5 \geq \mu(x) \wedge \mu(y)$,
(F5a') $\mu(-x) \vee 0.5 \geq \mu(x)$,
(F5b) $\mu(xy) \vee 0.5 \geq \mu(y)$ *(resp., $\mu(xy) \vee 0.5 \geq \mu(x)$).*

*Proof.* We only prove (F4a) $\Leftrightarrow$ (F5a). The others are similar.

(F4a)$\Rightarrow$ (F5a) If there exist $x, y \in R$ such that $\mu(x + y) \vee 0.5 < t = \mu(x) \wedge \mu(y)$, then $0.5 < t \leq 1, (x + y)_t\overline{\in}\mu$, but $x_t \in \mu, y_t \in \mu$. By (F1), we have $x_t\overline{q}\mu$ or $y_t\overline{q}\mu$. Then, ($t \leq \mu(x)$ and $t + \mu(x) \leq 1$) or ($t \leq \mu(y)$ and $t + \mu(y) \leq 1$). Thus, $t \leq 0.5$, contradiction.

(F5a)$\Rightarrow$ (F4a) Let $(x + y)_{t\wedge r}\overline{\in}\mu$, then $\mu(x + y) < t \wedge r$.

(1) If $\mu(x+y) \geq \mu(x) \wedge \mu(y)$, then $\mu(x) \wedge \mu(y) < t \wedge r$, and consequently, $\mu(x) < t$ or $\mu(y) < r$. It follows that $x_t\overline{\in}\mu$ or $y_r\overline{\in}\mu$. Thus, $x_t\overline{\in} \vee \overline{q}\mu$ or $y_r\overline{\in} \vee \overline{q}\mu$.

(2) If $\mu(x + y) < \mu(x) \wedge \mu(y)$ then by (F4), we have $0.5 \geq \mu(x) \wedge \mu(y)$. Putting $x_t\overline{\in}\mu$ or, $y_r\overline{\in}\mu$, then $t \leq \mu(x) \leq 0.5$ or $r \leq \mu(y) \leq 0.5$. It follows that $x_t\overline{q}\mu$ or $y_r\overline{q}\mu$, and thus, $x_t\overline{\in} \vee \overline{q}\mu$ or $y_r\overline{\in} \vee \overline{q}\mu$. This completes the proof. □

**Theorem 2.9.** *A fuzzy set $\mu$ of $R$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of $R$ if and only if $\mu_t(\neq \varnothing)$ is an ideal of $R$ for all $t \in (0.5, 1]$.*

*Proof.* This proof is similar to the proof of Theorem 2.6. □

## 3. Generalized fuzzy ideals

In this Section, we introduce the concepts of generalized fuzzy ideals of rings by means of a new way, which is different with the related topic.

**Remark 3.1.** Let $\mu$ and $\nu$ be any two fuzzy sets of $R$. Then

(i) If $x_t \in \mu$ implies $x_t \in\vee q\ \nu$ for all $x \in R$ and $t \in (0, 1]$, then we can write $\mu \subseteq \vee q\ \nu$.

(ii) If $x_t\overline{\in}\ \mu$ implies $x_t\overline{\in} \vee \overline{q}\ \nu$ for all $x \in R$ and $t \in (0, 1]$, then we can write $\mu \supseteq \vee\ \overline{q}\ \nu$.

**Proposition 3.2.** *For any two fuzzy sets $\mu$ and $\nu$ of $R$,*

(i) $\mu \subseteq \vee\ q\ \nu$ *if and only if* $\nu(x) \geq \mu(x) \wedge 0.5, \forall x \in R$;

(ii) $\mu \supseteq \vee\ \overline{q}\ \nu$ *if and only if* $\mu(x) \vee 0.5 \geq \nu(x), \forall x \in R$.

*Proof.* (i) Let $\mu \subseteq \vee q\ \nu$. If there exists $x \in R$ such that $\nu(x) < t = \mu(x) \wedge 0.5$, then $x_t \in \mu$, but $x_t \overline{\in \vee q} \nu$, a contradiction.

Conversely, let $\nu(x) \geq \mu(x) \wedge 0.5, \forall x \in R$. If $\mu \overline{\subseteq \vee q} \nu$, then there exists $x_t \in \mu$, but $x_t \overline{\in \vee q} \nu$, and so $\mu(x) \geq t$, $\nu(x) < t$ and $\nu(x) < 0.5$, a contradiction.

(ii) Let $\mu \supseteq \vee\ \overline{q}\ \nu$. If there exists $x \in R$ such that $\mu(x) \vee 0.5 < t = \nu(x)$, then $x_t \in \nu$, but $x_t \overline{\in} \mu$ and $t > 0.5$. Hence $x_t \overline{\in} \vee\ \overline{q} \nu$, and so $x_t \overline{q} \nu$, that is, $\nu(x) + t \leq 1$, and so $t \leq 0.5$, a contradiction.

Conversely, let $\mu(x) \vee 0.5 \geq \nu(x), \forall x \in R$. If $\mu \overline{\supseteq \vee\ \overline{q}} \nu$, then there exists $x_t \overline{\in} \mu$, but $x_t \overline{\overline{\in} \vee\ \overline{q}} \nu$. Hence $\mu(x) < t, \nu(x) \geq t$ and $\nu(x) + t > 1$.

Case (1) $\mu(x) > 0.5$. Then $\mu(x) \geq \nu(x)$, contradiction.

Case (2) $\mu(x) \leq 0.5$. Then $0.5 \geq \nu(x)$. By $\nu(x) \geq t, 0.5 \geq \nu(x) \geq t$. But $2\nu(x) \geq \nu(x) + t > 1$, and so $\nu(x) > 0.5$, a contradiction. $\square$

Now, we give the concept of the sum of two fuzzy sets of $R$.

**Definition 3.3.** Let $\mu$ and $\nu$ be fuzzy sets of $R$. Then the *sum* of $\mu$ and $\nu$ is defined by

$$(\mu + \nu)(x) = \bigvee_{x=a+b} (\mu(a) \wedge \nu(b))$$

and $(\mu + \nu)(x) = 0$ if $x$ cannot be expressed as $x = a + b$.

The following is obvious and we omit the proof.

**Proposition 3.4.** *Let* $A, B \subseteq R$. *Then we have*

(1) $A \subseteq B \Leftrightarrow \chi_A \subseteq \vee\ q\ \ \chi_B \Leftrightarrow \chi_B \supseteq \vee\ \overline{q}\ \chi_A$;
(2) $\chi_A \cap \chi_B = \chi_{A \cap B}$;
(3) $\chi_A \odot \chi_B = \chi_{AB}$;
(4) $\chi_A + \chi_B = \chi_{A+B}$.

Now, by means of a new way, we consider another generalized fuzzy ideals of rings, which is called a new $(\in, \in \vee q)$-fuzzy ideal.

**Definition 3.5.** A fuzzy set $\mu$ of $R$ is called a *new* $(\in, \in \vee q)$*-fuzzy left (resp., right) ideal* of $R$ if it satisfies:

(F6a) $(\mu + \mu) \subseteq \vee q \mu$,
(F6a') $\mu^{-1} \subseteq \vee q \mu$,
(F6b) $(\chi_R \odot \mu) \subseteq \vee q \mu$ (resp., $(\mu \odot \chi_R) \subseteq \vee q \mu$).

**Theorem 3.6.** *A fuzzy set* $\mu$ *of* $R$ *is a new* $(\in, \in \vee q)$*-fuzzy left (resp., right) ideal of* $R$ *if and only if it satisfies* (F3a), (F3a') *and* (F3b).

*Proof.* Let $\mu$ be a new $(\in, \in \vee q)$-fuzzy left (resp., right) ideal of $R$. If there exist $x, y \in R$ such that $\mu(x + y) < t < \mu(x) \wedge \mu(y) \wedge 0.5$, then $t < 0.5, x_t \in \mu, y_t \in \mu$, but $(x + y)_t \overline{\in} \mu$, and so $(x + y)_t \overline{\in \vee q} \mu$. But, $(\mu + \mu)(x + y) = \bigvee_{x+y=a+b} (\mu(a) \wedge \mu(b)) \geq \mu(x) \wedge \mu(y) \geq t$, and so $(x+y)_t \in (\mu+\mu)$. Thus, $(x+y)_t \in \vee q \mu$, contradiction. This proves (F3a) holds.

Now, if there exists $x \in R$ such that $\mu(-x) < t < \mu(x) \wedge 0.5$, then $\mu(x) \geq t$ and $t < 0.5$, but $(-x) \overline{\in} \mu$. Thus, $(-x)_t \overline{\in \vee q} \mu$. But $\mu^{-1}(-x) = \mu(x) \geq t$, and so

$(-x)_t \in \mu^{-1}$, which implies, $(-x)_t \in\vee q\mu$, contradiction. This proves (F3a') holds. The proof of (F3b) is similar to the proof of (F3a).

Conversely, assume that $\mu$ satisfies (F3a), (F3a') and (F3b). Let $x_t \in (\mu+\mu)$, but $x_t\overline{\in\vee q}\mu$. Then $\mu(x) < t$ and $\mu(x) < 0.5$. By definition, $(\mu+\mu)(x+y) = \bigvee\limits_{x+y=a+b}(\mu(a)\wedge\mu(b))$. Since $0.5 > \mu(x) = \mu(a+b) \geq \mu(a)\wedge\mu(b)\wedge 0.5$, and so $\mu(x) \geq \mu(a)\wedge\mu(b)$. Thus, $t \leq (\mu+\mu)(x) \leq \bigvee\limits_{x=a+b}\mu(x) = \mu(x)$, that is, $\mu(x) \geq t$, a contradiction. This proves that (F6a) holds.

Now, let $x_t \in \mu^{-1}$, but $x_t\overline{\in\vee q}\mu$, then $\mu(x) < t$ and $\mu(x) < 0.5$. Thus $\mu(-x) \geq \mu(x)\wedge 0.5 = \mu(x)$, and so $\mu(-x) = \mu(x)$. Hence $t \leq \mu^{-1}(x) = \mu(-x) = \mu(x)$, a contradiction. The proof of (F6b) is similar to the proof of (F6a). Therefore, $\mu$ is a new $(\in, \in\vee q)$-fuzzy left(resp., right) ideal of $R$. □

The following is a consequence of Theorems 2.5 and 3.6.

**Corollary 3.7.** *The concepts of new $(\in, \in\vee q)$-fuzzy left (resp., right) ideals of $R$ and $(\in, \in\vee q)$-fuzzy left (resp., right) ideals are equivalent.*

Next, by means of a new way, we consider another generalized fuzzy left(resp., right) ideals of rings, which is called a new $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy left(resp., right) ideal.

**Definition 3.8.** A fuzzy set $\mu$ of $R$ is called a *new $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy left (resp., right) ideal* of $R$ if it satisfies:

(F7a) $\mu \supseteq\vee\,\overline{q}(\mu+\mu)$,
(F7a') $\mu \supseteq\vee\,\overline{q}\mu^{-1}$,
(F7b) $\mu \supseteq\vee\,\overline{q}(\chi_R\odot\mu)$ (resp., $\mu \supseteq\vee\,\overline{q}(\mu\odot\chi_R)$).

**Theorem 3.9.** *A fuzzy set $\mu$ of $R$ is a new $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy left(resp., right) ideal of $R$ if and only if it satisfies* (F5a), (F5a') *and* (F5b).

*Proof.* Let $\mu$ be a new $(\in, \in\vee q)$-fuzzy left(resp., right) ideal of $R$. If there exist $x, y \in R$ such that $\mu(x+y)\vee 0.5 < t < \mu(x)\wedge\mu(y)$, then $t > 0.5, x_t \in \mu, y_t \in \mu$, but $(x+y)_t\overline{\in}\mu$, and so $(x+y)_t\overline{\in}\vee\overline{q}(\mu+\mu)$. Thus, $(\mu+\mu)(x+y) < t$ (*) and $(\mu+\mu)(x+y)+t \leq 1$. (**) But, $(\mu+\mu)(x+y) = \bigvee\limits_{x+y=a+b}(\mu(a)\wedge\mu(b)) \geq \mu(x)\wedge\mu(y) \geq t$, which implies, $(\mu+\mu)(x+y) \geq t$. By(*) and (**), we have $(\mu+\mu)(x+y)+t \leq 1$, and so $t \leq 0.5$, contradiction. This proves (F5a) holds. Similarly, we can prove (F5a') and (F5b).

Conversely, assume that $\mu$ satisfies (F5a), (F5a') and (F5b). Let $x_t \in \mu$, but $x_t\overline{\overline{\in}\vee\overline{q}}(\mu+\mu)$. Then $\mu(x) < t$, but $(\mu+\mu)(x) \geq t$ and $(\mu+\mu)(x)+t > 1$, and so $(\mu+\mu)(x) > 0.5$. By definition, $(\mu+\mu)(x) = \bigvee\limits_{x=a+b}(\mu(a)\wedge\mu(b))$. Since $0.5\vee\mu(x) = \mu(a+b)\vee 0.5 \geq \mu(a)\wedge\mu(b)$, and so $0.5\vee\mu(x) \geq \mu(a)\wedge\mu(b)$. Thus, $t \leq (\mu+\mu)(x) \leq \bigvee\limits_{x=a+b}\mu(x)\vee 0.5$. Since $(\mu+\mu)(x) > 0.5$, then $\mu(x) \geq 0.5$, and so $\mu(x) \geq t$, contradiction. This proves that (F7a) holds. Similarly, we can prove (F7a') and (F7b) hold. Therefore, $\mu$ is a new $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy left(resp., right) ideal of $R$. □

The following is a consequence of Theorems 2.8 and 3.9.

**Corollary 3.10.** *The concepts of new* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy left(resp., right) ideals of* $R$ *and* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy left(resp., right) ideals are equivalent.*

## 4. Generalized fuzzy bi-(quasi-) ideals

In this Section, we introduce the concepts of generalized fuzzy bi-ideals and generalized fuzzy quasi-ideals in rings by means of a new way, which is different with the related topic.

**Definition 4.1.** A fuzzy set $\mu$ of $R$ is called an $(\in, \in \vee q)$*-fuzzy bi-ideal* of $R$ if it satisfies (F6a), (F6a') and

(F6c) $(\mu \odot \mu) \subseteq \vee q\mu$,

(F6d) $(\mu \odot \chi_R \odot \mu) \subseteq \vee q\mu$.

**Theorem 4.2.** *A fuzzy set* $\mu$ *of* $R$ *is an* $(\in, \in \vee q)$*-fuzzy bi-ideal of* $R$ *if and only if it satisfies* (F3a), (F3a') *and*

(F6c') $\mu(xy) \geq \mu(x) \wedge \mu(y) \wedge 0.5$,

(F6d') $\mu(xyz) \geq \mu(x) \wedge \mu(z) \wedge 0.5$.

*Proof.* We only need to consider (F6d) $\Leftrightarrow$ (F6d').

(F6d) $\Rightarrow$ (F6d'). If there exist $x, y, z \in R$ such that $\mu(xyz) < t < \mu(x) \wedge \mu(z) \wedge 0.5$, then $\mu(xyz) < t < 0.5, x_t \in \mu, y_t \in \mu$. Thus, $(xyz)_t \overline{\in \vee q} \mu$. But,

$$\begin{aligned}
(\mu \odot \chi_R \odot \mu)(xyz) &= \bigvee_{xyz=\sum\limits_{i=1}^{m} a_i b_i} ((\mu \odot \chi_R)(a_i) \wedge \mu(b_i)) \\
&\geq (\mu \odot \chi_R)(xy) \wedge \mu(z) \\
&= \bigvee_{xy=\sum\limits_{i=1}^{m} a_i b_i} (\mu(a_i) \wedge \chi_R(b_i) \wedge \mu(z)) \\
&\geq \mu(x) \wedge \mu(z) \geq t,
\end{aligned}$$

and so $(xyz)_t \in (\mu \odot \chi_R \odot \mu)$. Thus, $(xyz)_t \in \vee q\mu$, contradiction. This proves (F6d') holds.

(F6d') $\Rightarrow$ (F6d) Let $x_t \in (\mu \odot \chi_R \odot \mu)$, but $x_t \overline{\in \vee q} \mu$. Then $\mu(x) < t$ and $\mu(x) < 0.5$, but $(\mu \odot \chi_R \odot \mu)(x) \geq t$. By definition, we have

$$\begin{aligned}
(\mu \odot \chi_R \odot \mu)(x) &= \bigvee_{x=\sum\limits_{i=1}^{m} a_i b_i} ((\mu \odot \chi_R)(a_i) \wedge \mu(b_i)) \\
&= \bigvee_{x=\sum\limits_{i=1}^{m} a_i b_i} \left( \bigvee_{a_i=\sum\limits_{i=1}^{m_i} a_{ik} b_{ik}} (\mu(a_{ik}) \wedge \chi_R(b_{ik}) \wedge \mu(b_i)) \right) \\
&= \bigvee_{x=\sum\limits_{i=1}^{m'} a_i b_i c_i} (\mu(a_i) \wedge \mu(c_i)).
\end{aligned}$$

Since $0.5 > \mu(x) = \mu(\sum_{i=1}^{m'} a_i b_i c_i) \geq \mu(a_i) \wedge \mu(c_i) \wedge 0.5$, and so $\mu(x) \geq \mu(a_i) \wedge \mu(c_i)$. Thus, $t \leq (\mu \odot \chi_R \odot \mu)(x) \leq \bigvee_{x=\sum_{i=1}^{m'} a_i b_i c_i} (\mu(a_i) \wedge \mu(c_i)) \leq \bigvee_{x=\sum_{i=1}^{m'} a_i b_i c_i} \mu(x) = \mu(x)$, that is, $\mu(x) \geq t$, a contradiction. This proves that (F6d) holds. □

**Corollary 4.3.** *A fuzzy set $\mu$ of $R$ is an $(\in, \in \vee\, q)$-fuzzy bi-ideal of $R$ if and only if it satisfies* (F2a), (F2a') *and*

(F6c") $x_t, y_r \in \mu \Rightarrow (xy)_{t\wedge r} \in \vee\, q\mu$,

(F6d") $x_t, z_r \in \mu \Rightarrow (xyz)_{t\wedge r} \in \vee\, q\mu$.

Next, by means of a new way, we consider another generalized fuzzy bi-ideals of rings, which is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal.

**Definition 4.4.** A fuzzy set $\mu$ of $R$ is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-*fuzzy bi-ideal* of $R$ if it satisfies (F7a), (F7a') and

(F7c) $\mu \supseteq \vee\, \overline{q}\ (\mu \odot \mu)$,

(F7d) $\mu \supseteq \vee\, \overline{q}\ (\mu \odot \chi_R \odot \mu)$.

**Theorem 4.5.** *A fuzzy set $\mu$ of $R$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of $R$ if and only if for all $x, y, z \in R$, it satisfies* (F5a), (F5a') *and*

(F7c') $\mu(x) \vee \mu(y) \vee 0.5 \geq \mu(x) \wedge \mu(y)$,

(F7d') $\mu(xyz) \vee 0.5 \geq \mu(x) \wedge \mu(z)$.

*Proof.* This proof is similar to the proof of Theorem 4.2. □

**Corollary 4.6.** *A fuzzy set $\mu$ of $R$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of $R$ if and only if for all $x, y, z \in R$, it satisfies* (F4a), (4a') *and*

(F7c") $(xy)_{t\wedge r} \overline{\in} \mu \Rightarrow x_t \overline{\in} \vee \overline{q}\mu$ *or* $y_r \overline{\in} \vee \overline{q}\mu$,

(F7d") $(xyz)_{t\wedge r} \overline{\in} \mu \Rightarrow x_t \overline{\in} \vee \overline{q}\mu$ *or* $z_r \overline{\in} \vee \overline{q}\mu$.

Next, we introduce the concepts of generalized fuzzy quasi-ideals in rings by means of a new way, which is different with the related topic.

**Definition 4.7.** A fuzzy set $\mu$ of $R$ is called an $(\in, \in \vee\, q)$-*fuzzy quasi-ideal* of $R$ if it satisfies (F6a), (F6a') and

(F6e) $(\chi_R \odot \mu) \cap (\mu \odot \chi_R) \subseteq \vee q\mu$.

**Theorem 4.8.** *A fuzzy set $\mu$ of $R$ is an $(\in, \in \vee\, q)$-fuzzy quasi-ideal of $R$ if and only if it satisfies* (F3a), (F3a') *and*

(F6e') $\mu(x) \geq ((\chi_R \odot \mu) \cap (\mu \odot \chi_R))(x) \wedge 0.5$.

*Proof.* The proof is similar to the proof of Theorem 4.2. □

**Corollary 4.9.** *A fuzzy set $\mu$ of $R$ is an $(\in, \in \vee\, q)$-fuzzy quasi-ideal of $R$ if and only if it satisfies* (F2a), (F2a') *and*

(F6e") $x_t \in (\chi_R \odot \mu) \cap (\mu \odot \chi_R) \Rightarrow x_t \in \vee\, q\mu$.

Next, we consider another generalized fuzzy quasi-ideals of rings, which is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal.

**Definition 4.10.** A fuzzy set $\mu$ of $R$ is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-*fuzzy quasi-ideal* of $R$ if it satisfies (F7a), (F7a') and

(F7e) $\mu \supseteq \vee\, \overline{q}\ ((\chi_R \odot \mu) \cap (\mu \odot \chi_R))$.

**Theorem 4.11.** *A fuzzy set $\mu$ of $R$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of $R$ if and only if for all $x \in R$, it satisfies* (F5a), (F5a') *and*

(F7e') $\mu(x) \vee 0.5 \geq ((\chi_R \odot \mu) \cap (\mu \odot \chi_R))(x)$.

*Proof.* The proof is similar to the proof of Theorem 4.2. □

**Corollary 4.12.** *A fuzzy set $\mu$ of $R$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of $R$ if and only if for all $x \in R$, it satisfies* (F4a), (4a') *and*

(F7e") $x_t \overline{\in} \mu \Rightarrow x_t \overline{\in} \vee \overline{q}\ ((\chi_R \odot \mu) \cap (\mu \odot \chi_R))$.

Finally, we give the characterization of $(\in, \in \vee q)$-fuzzy bi-ideals (resp., quasi-ideals) and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideals (resp., quasi-ideals) of rings.

**Theorem 4.13.** (i) *A fuzzy set $\mu$ of $R$ is an $(\in, \in \vee q)$-fuzzy bi-ideal (quasi-ideal) of $R$ if and only if $\mu_t (\neq \varnothing)$ is a bi-ideal (quasi-ideal) of $R$ for all $t \in (0, 0.5]$, respectively.*

(ii) *A fuzzy set $\mu$ of $R$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal (quasi-ideal) of $R$ if and only if $\mu_t (\neq \varnothing)$ is a bi-ideal (quasi-ideal) of $R$ for all $t \in (0.5, 1]$, respectively.*

## 5. Strong prime (semiprime) generalized fuzzy (bi-,quasi-) ideals

In this Section, we introduce the concepts of strong prime (semiprime) generalized fuzzy (bi-, quasi-) ideals in rings. In particular, we discuss the relationships between strong prime (semiprime) generalized fuzzy ideals and prime (semiprime) generalized fuzzy (bi-, quasi-) ideals in rings.

**Definition 5.1.** (i) An $(\in, \in \vee q)$-fuzzy (resp., bi-, quasi-) ideal $\mu$ of $R$ is called prime if for all $x, y \in R$ and $t \in (0, 1]$, we have

(P1) $(xy)_t \in \mu \Rightarrow x_t \in \vee\, q\mu$ or $y_t \in \vee\, q\mu$.

(ii) An $(\in, \in \vee q)$-fuzzy (resp., bi-, quasi-) ideal $\mu$ of $R$ is called semiprime if for all $x \in R, t \in (0, 1]$, we have

(SP1) $(x^2)_t \in \mu \Rightarrow x_t \in \vee\, q\mu$.

**Theorem 5.2.** (i) *An $(\in, \in \vee q)$-fuzzy (resp., bi-, quasi-) ideal $\mu$ of $R$ is prime if for all $x, y \in R$, it satisfies:*

(P2) $\mu(x) \vee \mu(y) \geq \mu(xy) \wedge 0.5$.

(ii) *An $(\in, \in \vee q)$-fuzzy (resp., bi-, quasi-) ideal $\mu$ of $R$ is semiprime if for all $x \in R$, it satisfies:*

(SP2) $\mu(x) \geq \mu(x^2) \wedge 0.5$.

**Theorem 5.3.** *An $(\in, \in \vee q)$-fuzzy (resp., bi-, quasi-) ideal $\mu$ of $R$ is prime (semiprime) if and only if $\mu_t (\neq \varnothing)$ is a prime (semiprime) (resp., bi-, quasi-) ideal of $R$ for all $t \in (0, 0.5]$, respectively.*

First, we give the concept of strong prime (semiprime) $(\in, \in \vee q)$-fuzzy (resp., bi-, quasi-) ideals in rings.

**Definition 5.4.** (i) An $(\in, \in\vee q)$-fuzzy (resp., bi-, quasi-) ideal $\rho$ of $R$ is called strong prime if for every $(\in, \in\vee q)$-fuzzy (resp., bi-, quasi-) ideals $\mu$ and $\nu$ of $R$, it satisfies:

(P3) $\mu \odot \nu \subseteq \rho$ implies $\mu \subseteq \rho$ or $\nu \subseteq \rho$.

(ii) An $(\in, \in\vee q)$-fuzzy (resp., bi-, quasi-) ideal $\mu$ of $R$ is called strong semiprime if for every $(\in, \in\vee q)$-fuzzy (resp., bi-, quasi-) ideal $\mu$ of $R$, it satisfies:

(SP3) $\mu \odot \mu \subseteq \rho$ implies $\mu \subseteq \rho$.

**Theorem 5.5.** *Let $\mu$ be a strong prime (semiprime) $(\in, \in\vee q)$-fuzzy (resp., bi-, quasi-) ideal of $R$. Then the $\mu_t (\mu_t \neq \varnothing)$ is a prime (semiprime) (resp., bi-, quasi-) ideal of $R$ for all $t \in (0, 0.5]$, respectively.*

*Proof.* We only consider strong prime $(\in, \in\vee q)$-fuzzy (bi-, quasi-) ideals. The case for strong semiprimeness is similar.

Let $t \in (0, 0.5]$ be such that $\mu_t$ is non-empty. Then it follows from Theorems 2.6 and 4.13 that $\mu_t$ is a (resp., bi-, quasi-) ideal of $R$. Now we show that $\mu_t$ is prime. Let $I$ and $J$ be two (resp., bi-, quasi-) ideals of $R$ such that $IJ \subseteq \mu_t$. Then it is easy to see that $t_I$ and $t_J$ are two $(\in, \in\vee q)$-fuzzy (resp., bi-, quasi-) ideals of $R$ and that $t_I \odot t_J \subseteq \mu$. In fact, let $x \in R$. If $(t_I \odot t_J)(x) = 0$, then $(t_I \odot t_J)(x) = 0 \leq \mu(x)$. Otherwise, there exist $a_i, b_i \in R$ such that $x = \sum\limits_{i=1}^{m} a_i b_i$ and $t_I(a_i) \wedge t_J(b_i) \neq 0$. This implies $a_i \in I$ and $b_i \in J$, hence $x \in IJ \subseteq \mu_t$, that is, $\mu(x) \geq t$. Hence $(t_I \odot t_J)(x) = \bigvee\limits_{x=\sum_{i=1}^{m} y_i z_i} t_I(y_i) \wedge t_J(z_i) \leq t \leq \mu(x)$. Therefore, $t_I \odot t_J \subseteq \mu$. Since $\mu$ is a strong prime $(\in, \in\vee q)$-fuzzy (resp., bi-, quasi-) ideal of $R$, we have $t_I \subseteq \mu$ or $t_J \subseteq \mu$, this implies $I \subseteq \mu_t$ or $J \subseteq \mu_t$. This completes the proof. □

The following is a consequence of Theorems 5.3 and 5.5.

**Theorem 5.6.** *Every strong prime (semiprime) $(\in, \in\vee q)$-fuzzy (resp., bi- and quasi-) ideal of a ring is a prime (semiprime) $(\in, \in\vee q)$-fuzzy (resp., bi- and quasi-) ideal, respectively.*

**Remark 5.7.** The converse of Theorem 5.6 is not true in general as shown in the following example.

**Example 5.8.** Let $(\mathbb{Z}, +, \cdot)$ be the ring of all integers. Then $0.4_{(2)}$ is an $(\in, \in\vee q)$-fuzzy ideal of $\mathbb{Z}$ and non-empty subset $(0.4_{(2)})_t$ is a prime ideal of $R$ for all $t \in (0, 0.5]$. By Theorem 5.3, we know that $0.4_{(2)}$ is a prime $(\in, \in\vee q)$-fuzzy ideal of $\mathbb{Z}$, but it is not strong prime. In fact, $0.4_{(3)} \odot 0.5_{(4)} \subseteq 0.4_{(2)}$, but $0.4_{(3)} \nsubseteq 0.4_{(2)}$ and $0.5_{(4)} \nsubseteq 0.4_{(2)}$, where both $0.4_{(3)}$ and $0.5_{(4)}$ are $(\in, \in\vee q)$-fuzzy ideals of $\mathbb{Z}$.

Now, we give the concept of prime (semiprime) $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy (resp., bi-, quasi-) ideals in rings.

**Definition 5.9.** (i) An $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy (resp., bi-, quasi-) ideal $\mu$ of $R$ is called prime if for all $x, y \in R$ and $t \in (0, 1]$, we have

(P4) $(xy)_t \overline{\in} \mu \Rightarrow x_t \overline{\in}\vee \overline{q}\mu$ or $y_t \overline{\in}\vee \overline{q}\mu$.

(ii) An $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy (resp., bi-, quasi-) ideal $\mu$ of $R$ is called semiprime if for all $x \in R, t \in (0, 1]$, we have (SP4) $(x^2)_t \overline{\in} \mu \Rightarrow x_t \overline{\in}\vee \overline{q}\mu$.

**Theorem 5.10.** (i) *An* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy (resp., bi-, quasi-) ideal* $\mu$ *of* $R$ *is prime if for all* $x, y \in R$*, it satisfies:*

(P5) $\mu(xy) \vee 0.5 \geq \mu(x) \wedge \mu(y)$.

(ii) *An* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy (resp., bi-, quasi-) ideal* $\mu$ *of* $R$ *is semiprime if for all* $x \in R$*, it satisfies:*

(SP5) $\mu(x^2) \vee 0.5 \geq \mu(x)$.

*Proof.* We only prove (i). The case (ii) is similar. Let $\mu$ be a prime $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy (resp., bi-, quasi-) ideal of $R$. If there exist $x, y \in R$ such that $\mu(xy) \vee 0.5 < t = \mu(x) \wedge \mu(y)$, then $t > 0.5, x_t, y_t \in \mu$, but $(xy)_t \overline{\in} \mu$. Thus, $x_t \overline{\in} \vee \overline{q} \mu$ or $y_t \overline{\in} \vee \overline{q} \mu$. Since $x_t, y_t \in \mu$, we have $x_t \overline{q} \mu$ or $y_t \overline{q} \mu$, that is, $\mu(x) + t \leq 1$ or $\mu(x) + t \leq 1$. Thus, $t \leq 0.5$, contradiction.

Conversely, suppose that condition (P5) holds. Let $(xy)_t \overline{\in} \mu$. Then $\mu(xy) < t$ and so $\mu(x) \wedge \mu(y) \leq \mu(xy) \vee 0.5 < t \vee 0.5$. We consider the following two cases:

(i) If $t \leq 0.5$, then $\mu(x) \wedge \mu(y) < 0.5$. Thus, $\mu(x) < 0.5$ or $\mu(y) < 0.5$, that is, $\mu(x) + t < 1$ or $\mu(y) + t < 1$. Hence, $x_t \overline{q} \mu$ or $y_t \overline{q} \mu$, and so $x_t \overline{\in} \vee \overline{q} \mu$ or $y_t \overline{\in} \vee \overline{q} \mu$.

(ii) If $t > 0.5$, then $\mu(x) \wedge \mu(y) < t$. Thus, $\mu(x) < t$ or $\mu(y) < t$, and so, $x_t \overline{\in} \mu$ or $y_t \overline{\in} \mu$. Thus, $x_t \overline{\in} \vee \overline{q} \mu$ or $y_t \overline{\in} \vee \overline{q} \mu$. This proves that $\mu$ is prime. □

**Theorem 5.11.** *An* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy (resp., bi-, quasi-) ideal* $\mu$ *of* $R$ *is prime (semiprime) if and only if* $\mu_t (\neq \varnothing)$ *is a prime (semiprime) (resp., bi-, quasi-) ideal of* $R$ *for all* $t \in (0.5, 1]$*, respectively.*

*Proof.* It is a consequence of Theorems 2.9, 4.13 and 5.3. □

Finally, we give the concept of strong prime (semiprime) $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy (resp., bi-, quasi-) ideals in rings.

**Definition 5.12.** (i) An $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy (resp., bi-, quasi-) ideal $\rho$ of $R$ is called strong prime if for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy (resp., bi-, quasi-) ideals $\mu$ and $\nu$ of $R$, it satisfies (P3).

(ii) An $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy (resp., bi-, quasi-) ideal $\mu$ of $R$ is called strong semiprime if for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy (resp., bi-, quasi-) ideal $\mu$ of $R$, it satisfies (SP3).

**Theorem 5.13.** *Let* $\mu$ *be a strong prime (semiprime)* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy (resp., bi-, quasi-) ideal of* $R$*. Then* $\mu_t (\neq \varnothing)$ *is a prime (semiprime) (resp., bi-, quasi-) ideal of* $R$ *for all* $t \in (0.5, 1]$*, respectively.*

*Proof.* The proof is similar to the proof of Theorem 5.5. □

The following is a consequence of Theorems 5.3 and 5.13.

**Theorem 5.14.** *Every strong prime (semiprime)* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy (resp., bi- and quasi-) ideal of a ring is a prime (semiprime)* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy (resp., bi- and quasi-) ideal, respectively.*

## 6. Characterizations of regular rings

We characterize the regular rings in this Section by generalized fuzzy ideals, generalized fuzzy bi-ideals and generalized fuzzy quasi-ideals.

Define two relations "$\underset{>}{\leqq}$" and "$\underset{<}{\geqq}$" on $\mathcal{F}(R)$ by

$$\mu \lesseqqgtr \nu \text{ if and only if } \mu \subseteq \vee\, q\ \nu \text{ and } \nu \subseteq \vee\, q\ \mu \text{ for all } \mu, \nu \in \mathcal{F}(R);$$

and

$$\mu \gtreqqless \nu \text{ if and only if } \mu \supseteq \vee\, \overline{q}\ \nu \text{ and } \nu \supseteq \vee\, \overline{q}\ \mu \text{ for all } \mu, \nu \in \mathcal{F}(R).$$

Clearly, " $\lesseqqgtr$ " and " $\gtreqqless$ " are equivalence relations on $\mathcal{F}(R)$. A ring $R$ is said to be *regular* if for each $x \in R$, there exists $a \in R$ such that $x = xax$ ([11]).

**Lemma 6.1.** *A ring $R$ is regular if and only if for any right ideal $A$ and any left ideal $B$, we have $AB = A \cap B$.*

Now, we characterize the regular rings by $(\in, \in\vee q)$-fuzzy ideals.

**Theorem 6.2.** *A ring $R$ is regular if and only if for any $(\in, \in\vee q)$-fuzzy right ideal $\mu$ and $(\in, \in\vee q)$-fuzzy left ideal $\nu$, we have $\mu \cap \nu \lesseqqgtr \mu \odot \nu$.*

*Proof.* Let $R$ be a regular ring, $\mu$ an $(\in, \in\vee q)$-fuzzy right ideal and $\nu$ an $(\in, \in\vee q)$-fuzzy left ideal. Then, $\mu \odot \nu \subseteq \vee\, q\ \mu \odot \chi_R \subseteq \vee\, q\ \mu$ and $\mu \odot \nu \subseteq \vee\, q\ \chi_R \odot \nu \subseteq \vee\, q\ \nu$. Thus, $\mu \odot \nu \subseteq \vee\, q\ \mu \cap \nu$. For any $x \in R$, there exists $a \in R$ such that $x = xax$ since $R$ is regular. Thus

$$\begin{aligned}(\mu \odot \nu)(x) &= \bigvee_{x=a_1b_1} (\mu(a_1) \wedge \nu(b_1)) \\ &\geq \mu(xa) \wedge \nu(x) \\ &\geq \mu(x) \wedge \nu(x) \wedge 0.5 \\ &= (\mu \cap \nu)(x) \wedge 0.5,\end{aligned}$$

i.e., $\mu \cap \nu \subseteq \vee\, q\ \mu \odot \nu$, whence $\mu \cap \nu \lesseqqgtr \mu \odot \nu$.

Conversely, let $A$ and $B$ be, respectively right and left ideal of $R$. Then, it is easy to see that their characteristic functions $\chi_A$ and $\chi_B$ are, respectively, $(\in, \in\vee q)$-fuzzy right ideal and $(\in, \in\vee q)$-fuzzy left ideal. Thus, by Proposition 3.4, we have $\chi_{AB} = \chi_A \odot \chi_B \lesseqqgtr \chi_A \cap \chi_B = \chi_{A\cap B}$. Thus, by Proposition 3.4, $AB = A \cap B$. It follows from Lemma 6.1 that $R$ is regular. $\square$

Similarly, we also can characterize the regular rings by using $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideals.

**Theorem 6.3.** *A ring $R$ is regular if and only if for any $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy right ideal $\mu$ and $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy left ideal $\nu$, we have $\mu \cap \nu \gtreqqless \mu \odot \nu$.*

*Proof.* Let $R$ be a regular ring, $\mu$ an $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy right ideal and $\nu$ an $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy left ideal. Then, $\mu \supseteq \vee\, \overline{q}\ \mu \odot \chi_R \supseteq \vee\, \overline{q}\ \mu \odot \nu$ and $\mu \supseteq \vee\, \overline{q}\ \chi_R \odot \nu \supseteq \vee\, \overline{q}\ \mu \odot \nu$. Thus, $\mu \cap \nu \supseteq \vee\, \overline{q}\ \mu \odot \nu$. For any $x \in R$, there exist $a, a' \in R$ such that $x = xaa'x$ since $R$ is regular. Thus

$$\begin{aligned}(\mu \odot \nu)(x) \vee 0.5 &= \bigvee_{x=a_1b_1} (\mu(a_1) \wedge \nu(b_1)) \vee 0.5 \\ &\geq (\mu(xa) \vee 0.5) \wedge (\nu(a'x) \vee 0.5) \\ &\geq \mu(x) \wedge \nu(x) \\ &= (\mu \cap \nu)(x),\end{aligned}$$

i.e., $\mu \odot \nu \supseteq \vee\, \overline{q}\ \mu \cap \nu$, whence $\mu \cap \nu \gtreqqless \mu \odot \nu$.

Conversely, let $A$ and $B$ be, respectively right and left ideal of $R$. Then, it is easy to see that their characteristic functions $\chi_A$ and $\chi_B$ are, respectively, $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal. Thus, by Proposition 3.4, we have $\chi_{AB} = \chi_A \odot \chi_B \underset{<}{\geqq} \chi_A \cap \chi_B = \chi_{A \cap B}$. Thus, by Proposition 3.4, $AB = A \cap B$. It follows from Lemma 6.1 that $R$ is regular. □

**Lemma 6.4.** *Let $R$ be a ring. Then the following are equivalent:*
(1) *$R$ is regular;*
(2) *$B = BSB$ for every bi-ideal $B$ of $R$;*
(3) *$Q = QSQ$ for every quasi-ideal $Q$ of $R$.*

Now, we characterize the regular rings by $(\in, \in \vee q)$-fuzzy bi-ideals (resp., quasi-ideals).

**Theorem 6.5.** *Let $R$ be a ring. Then the following are equivalent:*
(1) *$R$ is regular;*
(2) *$\mu \underset{>}{\leqq} \mu \odot \chi_R \odot \mu$ for every $(\in, \in \vee q)$-fuzzy bi-ideal $\mu$ of $R$;*
(3) *$\mu \underset{>}{\leqq} \mu \odot \chi_R \odot \mu$ for every $(\in, \in \vee q)$-fuzzy quasi-ideal $\mu$ of $R$.*

**Theorem 6.6.** *Let $R$ be a ring. Then the following are equivalent:*
(1) *$R$ is regular;*
(2) *$\mu \underset{<}{\geqq} \mu \odot \chi_R \odot \mu$ for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal $\mu$ of $R$;*
(3) *$\mu \underset{<}{\geqq} \mu \odot \chi_R \odot \mu$ for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal $\mu$ of $R$.*

Finally, we characterize the regular rings by $(\in, \in \vee q)$-fuzzy ideals, $(\in, \in \vee q)$-fuzzy bi-ideals and $(\in, \in \vee q)$-fuzzy quasi-ideals.

**Theorem 6.7.** *Let $R$ be a ring. Then the following are equivalent:*
(1) *$R$ is regular;*
(2) *$\mu \cap \nu \underset{>}{\leqq} \mu \odot \nu \odot \mu$ for every $(\in, \in \vee q)$-fuzzy bi-ideal $\mu$ and every $(\in, \in \vee q)$-fuzzy ideal $\nu$ of $R$;*
(3) *$\mu \cap \nu \underset{>}{\leqq} \mu \odot \nu \odot \mu$ for every $(\in, \in \vee q)$-fuzzy quasi-ideal $\mu$ and every $(\in, \in \vee q)$-fuzzy ideal $\nu$ of $R$.*

*Proof.* (1) ⇒ (2) Let $\mu$ and $\nu$ be an $(\in, \in \vee q)$-fuzzy bi-ideal and an $(\in, \in \vee q)$-fuzzy ideal of $R$, respectively. For any $x \in R$, there exists $a \in R$ such that $x = xax$ since $R$ is regular. Thus,

$$\begin{aligned}(\mu \odot \nu \odot \mu)(x) &= \bigvee_{x=\sum_{i=1}^{m} a_i b_i} ((\mu \odot \nu)(a_i) \wedge \mu(b_i)) \\ &\geq (\mu \odot \nu)(xa) \wedge \mu(x) \\ &= \bigvee_{xa=\sum_{i=1}^{m} a_i b_i} (\mu(a_i) \wedge \nu(b_i) \wedge \mu(x)) \\ &\geq \mu(x) \wedge \nu(xax) \geq \mu(x) \wedge \nu(x) \wedge 0.5\end{aligned}$$

which implies $\mu \cap \nu \subseteq \vee q(\mu \odot \nu \odot \mu)$. On the other hand,

$(\mu \odot \nu \odot \mu) \subseteq \vee q(\mu \odot \chi_R \odot \mu) \subseteq \vee q\mu$ and $(\mu \odot \nu \odot \mu) \subseteq \vee q(\chi_R \odot \nu \odot \chi_R) \subseteq \vee q\nu$,

and so $(\mu \odot \nu \odot \mu) \subseteq \vee q(\mu \cap \nu)$. Thus, $\mu \cap \nu \lesseqqgtr \mu \odot \chi_R \odot \mu$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Let $\mu$ be any $(\in, \in\vee q)$-fuzzy quasi-ideal of $R$. Then we have

$$\mu = \mu \cap \chi_R \lesseqqgtr \mu \odot \chi_R \odot \mu.$$

It follows from Theorem 6.5 that $R$ is regular. □

Similarly, we also can characterize the regular rings by $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideals, $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideals and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideals.

**Theorem 6.8.** *Let $R$ be a ring. Then the following are equivalent:*

(1) *$R$ is regular;*

(2) *$\mu \cap \nu \gtreqqless \mu \odot \nu \odot \mu$ for every $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy bi-ideal $\mu$ and every $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy ideal $\nu$ of $R$;*

(3) *$\mu \cap \nu \gtreqqless \mu \odot \nu \odot \mu$ for every $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy quasi-ideal $\mu$ and every $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy ideal $\nu$ of $R$.*

## 7. Characterizations of intra-regular rings

In this Section, we characterize the intra-regular rings by generalized fuzzy ideals, generalized fuzzy bi-ideals and generalized fuzzy quasi-ideals.

A ring $R$ is said to be *intra-regular* if for each $x \in R$, there exist $x_i, y_i \in R$ such that $x = \sum_{i=1}^{m} x_i x y_i$ ([11]).

**Lemma 7.1.** *A ring $R$ is intra-regular if and only if for any left ideal $A$ and any right ideal $B$, we have $A \cap B \subseteq AB$.*

Now, we characterize the intra-regular rings by $(\in, \in\vee q)$-fuzzy ideals.

**Theorem 7.2.** *A ring $R$ is intra-regular if and only if for any $(\in, \in\vee q)$-fuzzy left ideal $\mu$ and any $(\in, \in\vee q)$-fuzzy right ideal $\nu$, we have $\mu \cap \nu \subseteq \vee q\ \mu \odot \nu$.*

*Proof.* Let $R$ be an intra-regular ring, $\mu$ an $(\in, \in\vee q)$-fuzzy left ideal and $\nu$ an $(\in, \in\vee q)$-fuzzy right ideal. For any $x \in R$, there exist $x_i, y_i \in R$ such that $x = \sum_{i=1}^{m} x_i x y_i$ since $R$ is intra-regular. Thus

$$\begin{aligned}(\mu \odot \nu)(x) &= \bigvee_{x=\sum_{i=1}^{m} a_i b_i} (\mu(a_i) \wedge \nu(b_i)) \\ &\geq \mu(x_i x) \wedge \nu(x y_i) \\ &\geq \mu(x) \wedge \nu(x) \wedge 0.5 \\ &= (\mu \cap \nu)(x) \wedge 0.5,\end{aligned}$$

i.e., $\mu \cap \nu \subseteq \vee\ q\ \mu \odot \nu$.

Conversely, let $A$ and $B$ be, respectively left and right ideal of $R$. Then, it is easy to see that their characteristic functions $\chi_A$ and $\chi_B$ are, respectively, $(\in, \in\vee q)$-fuzzy left ideal and $(\in, \in\vee q)$-fuzzy right ideal. Thus, by Proposition 3.4, we have

$\chi_{A\cap B} = \chi_A \cap \chi_B \subseteq \vee q\ \chi_A \odot \chi_B = \chi_{AB}$. Hence, by Proposition 3.4, $A \cap B \subseteq AB$. It follows from Lemma 7.1 that $R$ is intra-regular. $\square$

Similarly, we also can characterize the intra-regular rings by using $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideals.

**Theorem 7.3.** *ring $R$ is intra-regular if and only if for any $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal $\mu$ and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal $\nu$, we have $\mu \cap \nu \supseteq \vee\overline{q}\ \mu \odot \nu$.*

**Lemma 7.4.** *Let $R$ be a ring. Then the following are equivalent:*

(1) *$R$ is both regular and intra-regular;*
(2) *$B = B^2$ for every bi-ideal $B$ of $R$;*
(3) *$Q = Q^2$ for every quasi-ideal $Q$ of $R$.*

Next, we characterize the both regular and intra-regular rings by $(\in, \in \vee q)$-fuzzy bi-ideals (resp., quasi-ideals).

**Theorem 7.5.** *Let $R$ be a ring. Then the following are equivalent:*

(1) *$R$ is both regular and intra-regular;*
(2) *$\mu \lesseqqgtr \mu \odot \mu$ for every $(\in, \in \vee q)$-fuzzy bi-ideal $\mu$ of $R$;*
(3) *$\mu \lesseqqgtr \mu \odot \mu$ for every $(\in, \in \vee q)$-fuzzy quasi-ideal $\mu$ of $R$.*

*Proof.* (1) $\Rightarrow$ (2) Let $\mu$ be an $(\in, \in \vee q)$-fuzzy bi-ideal of $R$. It is clear that $\mu \odot \mu \subseteq \vee q\mu$. For any $x \in R$, there exist $x_i, y_i \in R$ such that $x = \sum\limits_{i=1}^{m} x x_i x^2 y_i x$ since $R$ is both regular and intra-regular. Thus

$$\begin{aligned}(\mu \odot \mu)(x) &= \bigvee_{x=\sum\limits_{i=1}^{m} a_i b_i} (\mu(a_i) \wedge \mu(b_i)) \\ &\geq \mu(x x_i x) \wedge \mu(x y_i x) \\ &\geq \mu(x) \wedge \mu(x) \wedge 0.5 \\ &= (\mu \cap \mu)(x) \wedge 0.5,\end{aligned}$$

i.e., $\mu \subseteq \vee\ q\ \mu \odot \mu$. Thus, $\mu \lesseqqgtr \mu \odot \mu$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Let $Q$ be any quasi-ideal of $R$. Then $\chi_Q$ is an $(\in, \in \vee q)$-fuzzy quasi-ideal of $R$. Thus, we have $\chi_Q \lesseqqgtr \chi_Q \odot \chi_Q = \chi_{Q^2}$, which implies, $Q = Q^2$. Thus, $R$ is both regular and intra-regular by Lemma 7.4. $\square$

Similarly, we also can characterize the rings that are both regular and intra-regular by $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideals (resp., quasi-ideals).

**Theorem 7.6.** *Let $R$ be a ring. Then the following are equivalent:*

(1) *$R$ is both regular and intra-regular;*
(2) *$\mu \gtreqqless \mu \odot \mu$ for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal $\mu$ of $R$;*
(3) *$\mu \gtreqqless \mu \odot \mu$ for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal $\mu$ of $R$.*

## 8. Conclusions

In study the structure of a fuzzy algebraic system, we notice that the (fuzzy) ideals with special properties always play an important role. In this paper, by means of a kind of new idea, we redefine some kinds of fuzzy ideals in a ring and investigate some of their related properties.

We hope that the research along this direction can be continued, and in fact, some results in this paper have already constituted a platform for further discussion concerning the future development of rings. In our future study of fuzzy structure of rings, may be the following topics should be considered:

(1) To describe soft rings and its applications;
(2) To give some types of fuzzy ideals in pseudo rings;
(3) To investigate rough ideals in pseudo rings.

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