

Semicompactness degree in L -topological spaces

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ABSTRACT. In this paper, we introduce the notion of the degree to which an L -fuzzy set is semicompact in an L -topological space by means of the implication operator of L . An L -fuzzy set G is semicompact if and only if its semicompactness degree $\text{scom}(G) = \top$. Some properties of semicompactness degree are investigated.

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1. INTRODUCTION

The notion of semicompactness [2] was introduced in L -topological spaces by Kudri in [4]. However Kudri's semicompactness relies on the structure of L and L is required to be completely distributive. Subsequently Shi introduced a new definition of semicompactness in L -topological spaces by means of semiopen L -sets and their inequality (see [5]). This definition does not rely on the structure of basis lattice L and no distributivity in L is required.

For the above notions of semicompactness, an L -fuzzy set is either semicompact or not. Considering fuzziness degree of semicompactness, we shall introduce the concept of the fuzzy semicompactness degree of an L -fuzzy set in L -topological spaces by means of the implication operator of L .

2. PRELIMINARIES

Throughout this paper, $(L, \vee, \wedge, ')$ is a complete De Morgan frame [3]. The smallest element and the largest element in L are denoted by \perp and \top , respectively.

In a complete De Morgan frame L , there exists a binary operation \mapsto . Explicitly the implication is given by

$$a \mapsto b = \bigvee \{c \in L \mid a \wedge c \leq b\}.$$

It is easy to check the following properties of \mapsto .

- (1) $(a \mapsto b) \geq c \Leftrightarrow a \wedge c \leq b$;
- (2) $a \mapsto b = \top \Leftrightarrow a \leq b$;
- (3) $a \mapsto (\bigwedge_i b_i) = \bigwedge_i (a \mapsto b_i)$;
- (4) $(\bigvee_i a_i) \mapsto b = \bigwedge_i (a_i \mapsto b)$;
- (5) $(a \mapsto c) \wedge (c \mapsto b) \leq a \mapsto b$;
- (6) $a \leq b \Rightarrow c \mapsto a \leq c \mapsto b$.
- (7) $a \leq b \Rightarrow b \mapsto c \leq a \mapsto c$.
- (8) $(a \mapsto b) \wedge (c \mapsto d) \leq a \wedge c \mapsto b \wedge d$.

For a nonempty set X , L^X denotes the set of all L -fuzzy sets on X . \underline{a} denotes the constant L -fuzzy sets on X taking the value a .

An L -topological space is a pair (X, τ) , where τ is a subfamily of L^X which contains $\underline{0}$, $\underline{1}$ and is closed for any suprema and finite infima. Each member of τ is called an open L -set and its quasi-complement is called a closed L -set.

For a subfamily $\Phi \subseteq L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamilies of Φ . $2^{[\Phi]}$ denotes the set of all countable subfamilies of Φ .

Definition 2.1 ([1]). An L -set G in an L -space (X, τ) is called semiopen if there exists $A \in \tau$ such that $A \leq G \leq cl(A)$. G is called semiclosed if G' is semiopen.

Definition 2.2. Let (X, τ_1) and (Y, τ_2) be two L -spaces. A map $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ is said to be

- (1) semicontinuous [1] if $f_L^-(G)$ is semiopen in (X, τ_1) for every open L -set G in (Y, τ_2) .
- (2) irresolute [8] if $f_L^-(G)$ is semiopen in (X, τ_1) for every semiopen L -set G in (Y, τ_2) .
- (3) strong irresolute [5] if $f_L^-(G)$ is open in (X, τ_1) for every semiopen L -set G in (Y, τ_2) .

Definition 2.3 ([7]). An L -fuzzy inclusion on X is a mapping $\tilde{\subset} : L^X \times L^X \rightarrow L$ defined by the equality $\tilde{\subset}(A, B) = \bigwedge_{x \in X} (A'(x) \vee B(x))$.

In the sequel, we shall write $[A \tilde{\subset} B]$ instead of $\tilde{\subset}(A, B)$. It is evident that for ordinary subsets A and B of a set X , if $A \subseteq B$, then $[A \tilde{\subset} B] = 1$, and $[A \tilde{\subset} B] = 0$ otherwise. Moreover we have the following properties:

- (1) $[A \tilde{\subset} B] \wedge [A \tilde{\subset} C] = [A \tilde{\subset} B \wedge C]$;
- (2) $[A \vee B \tilde{\subset} C] = [A \tilde{\subset} C] \wedge [B \tilde{\subset} C]$.

Lemma 2.4 ([6, 7]). Let $f : X \rightarrow Y$ be a set map and $f_L^{\rightarrow} : L^X \rightarrow L^Y$ be the extension of f (see [3]). Then for any $\mathcal{P} \subseteq L^Y$, we have that

$$\bigwedge_{y \in Y} \left(f_L^{\rightarrow}(G)'(y) \vee \bigvee_{B \in \mathcal{P}} B(y) \right) = \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{B \in \mathcal{P}} f_L^-(B)(x) \right),$$

i.e.,

$$[f_L^{\rightarrow}(G) \tilde{\subset} \bigvee \mathcal{P}] = [G \tilde{\subset} f_L^-(\bigvee \mathcal{P})].$$

Definition 2.5 ([5]). Let (X, τ) be an L -topological space. An L -set $G \in L^X$ is called (countably) semicompact if for every (countable) family \mathcal{U} of semiopen L -sets, it follows that

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right),$$

i.e.,

$$[G \widetilde{\vee} \mathcal{U}] \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \widetilde{\vee} \mathcal{V}].$$

Definition 2.6 ([5]). Let (X, τ) be an L -topological space. An L -set $G \in L^X$ is said to have the semi-Lindelöf property or be a semi-Lindelöf L -set if for every family \mathcal{U} of semiopen L -sets, it follows that

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{[\mathcal{U}]}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right).$$

Definition 2.7 ([7]). Let (X, τ) be an L -topological space and $G \in L^X$. The compactness degree $\text{com}(G)$ of G is defined as

$$\begin{aligned} \text{com}(G) &= \bigwedge_{\mathcal{U} \in 2^\tau} \left(\bigwedge_{x \in X} \left(G' \vee \bigvee_{A \in \mathcal{U}} A \right)(x) \mapsto \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G' \vee \bigvee_{A \in \mathcal{V}} A \right)(x) \right) \\ &= \bigwedge_{\mathcal{U} \in 2^\tau} \left([G \widetilde{\vee} \mathcal{U}] \mapsto \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \widetilde{\vee} \mathcal{V}] \right). \end{aligned}$$

Definition 2.8 ([7]). Let (X, τ) be an L -topological space and $G \in L^X$. The countable compactness degree $\text{ccom}(G)$ of G is defined as

$$\begin{aligned} \text{ccom}(G) &= \bigwedge_{\mathcal{U} \in 2^{[\tau]}} \left(\bigwedge_{x \in X} \left(G' \vee \bigvee_{A \in \mathcal{U}} A \right)(x) \mapsto \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G' \vee \bigvee_{A \in \mathcal{V}} A \right)(x) \right) \\ &= \bigwedge_{\mathcal{U} \in 2^{[\tau]}} \left([G \widetilde{\vee} \mathcal{U}] \mapsto \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \widetilde{\vee} \mathcal{V}] \right). \end{aligned}$$

Definition 2.9 ([7]). Let (X, τ) be an L -topological space and $G \in L^X$. The degree $\text{Lp}(G)$ to which G has the Lindelöf property is defined as

$$\begin{aligned} \text{Lp}(G) &= \bigwedge_{\mathcal{U} \in 2^\tau} \left(\bigwedge_{x \in X} \left(G' \vee \bigvee_{A \in \mathcal{U}} A \right)(x) \mapsto \bigvee_{\mathcal{V} \in 2^{[\mathcal{U}]}} \bigwedge_{x \in X} \left(G' \vee \bigvee_{A \in \mathcal{V}} A \right)(x) \right) \\ &= \bigwedge_{\mathcal{U} \in 2^\tau} \left([G \widetilde{\vee} \mathcal{U}] \mapsto \bigvee_{\mathcal{V} \in 2^{[\mathcal{U}]}} [G \widetilde{\vee} \mathcal{V}] \right). \end{aligned}$$

3. THE SEMICOMPACTNESS DEGREE OF L -SETS

Let τ_s denote the set of all semiopen L -sets in τ . Based on Definitions 2.5 and 2.6, we can naturally introduce the notion of semicompactness degree as follows:

Definition 3.1. Let (X, τ) be an L -topological space and $G \in L^X$. The semicompactness degree $\text{scom}(G)$ of G is defined as

$$\begin{aligned} \text{scom}(G) &= \bigwedge_{\mathcal{U} \in 2^{\tau_s}} \left(\bigwedge_{x \in X} \left(G' \vee \bigvee_{A \in \mathcal{U}} A \right)(x) \mapsto \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G' \vee \bigvee_{A \in \mathcal{V}} A \right)(x) \right) \\ &= \bigwedge_{\mathcal{U} \in 2^{\tau_s}} \left(\left[G \widetilde{\vee} \bigvee \mathcal{U} \right] \mapsto \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[G \widetilde{\vee} \bigvee \mathcal{V} \right] \right). \end{aligned}$$

Obviously G is semicompact if and only if $\text{scom}(G) = \top$.

Since an open L -set is semiopen, we easily obtain the the following result.

Theorem 3.2. Let (X, τ) be an L -topological space and $G \in L^X$. Then $\text{scom}(G) \leq \text{com}(G)$.

The next lemma is obvious.

Lemma 3.3. Let (X, τ) be an L -topological space and $G \in L^X$. Then $\text{scom}(G) \geq a$ if and only if for any $\mathcal{U} \in 2^{\tau_s}$,

$$\left[G \widetilde{\vee} \bigvee \mathcal{U} \right] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[G \widetilde{\vee} \bigvee \mathcal{V} \right].$$

By Lemma 3.3 we can easily obtain the following characterization of the semicompactness degree.

Theorem 3.4. Let (X, τ) be an L -topological space and $G \in L^X$. Then

$$\text{scom}(G) = \bigvee \left\{ a \in L : \left[G \widetilde{\vee} \bigvee \mathcal{U} \right] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[G \widetilde{\vee} \bigvee \mathcal{V} \right] \text{ for any } \mathcal{U} \in 2^{\tau_s} \right\}.$$

Now we consider some properties of semicompactness degree.

Theorem 3.5. For any $G, H \in L^X$, $\text{scom}(G \vee H) \geq \text{scom}(G) \wedge \text{scom}(H)$.

Proof. By Theorem 3.4 we have

$$\begin{aligned}
 & \text{scom}(G \vee H) \\
 &= \bigvee \left\{ a \in L : \left[G \vee H \widetilde{\vee} \bigvee \mathcal{U} \right] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\tau_s)}} \left[G \vee H \widetilde{\vee} \bigvee \mathcal{V} \right], \forall \mathcal{U} \in 2^{(\tau_s)} \right\} \\
 &= \bigvee \left\{ a \in L : \left[G \widetilde{\vee} \bigvee \mathcal{U} \right] \wedge \left[H \widetilde{\vee} \bigvee \mathcal{U} \right] \wedge a \right. \\
 &\quad \left. \leq \bigvee_{\mathcal{V} \in 2^{(\tau_s)}} \left[G \widetilde{\vee} \bigvee \mathcal{V} \right] \wedge \left[H \widetilde{\vee} \bigvee \mathcal{V} \right], \forall \mathcal{U} \in 2^{(\tau_s)} \right\} \\
 &\geq \bigvee \left\{ a \in L : \left[G \widetilde{\vee} \bigvee \mathcal{U} \right] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\tau_s)}} \left[G \widetilde{\vee} \bigvee \mathcal{V} \right] \right\} \wedge \\
 &\quad \bigvee \left\{ a \in L : \left[H \widetilde{\vee} \bigvee \mathcal{U} \right] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\tau_s)}} \left[H \widetilde{\vee} \bigvee \mathcal{V} \right], \forall \mathcal{U} \in 2^{(\tau_s)} \right\} \\
 &= \text{scom}(G) \wedge \text{scom}(H).
 \end{aligned}$$

This completes the proof. \square

Theorem 3.6. For any $G \in L^X$ and any semiclosed L -set H , $\text{scom}(G \wedge H) \geq \text{scom}(G)$.

Proof. By Theorem 3.4 we have

$$\begin{aligned}
 & \text{scom}(G \wedge H) \\
 &= \bigvee \left\{ a \in L : \left[G \wedge H \widetilde{\vee} \bigvee \mathcal{U} \right] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\tau_s)}} \left[G \wedge H \widetilde{\vee} \bigvee \mathcal{V} \right], \forall \mathcal{U} \in 2^{(\tau_s)} \right\} \\
 &= \bigvee \left\{ a \in L : \left[G \widetilde{\vee} H' \vee \bigvee \mathcal{U} \right] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\tau_s)}} \left[G \widetilde{\vee} H' \vee \bigvee \mathcal{V} \right], \forall \mathcal{U} \in 2^{(\tau_s)} \right\} \\
 &\geq \text{scom}(G).
 \end{aligned}$$

This completes the proof. \square

Theorem 3.7. Let $f : X \rightarrow Y$ be a set map, τ_1 be an L -topology on X , τ_2 be an L -topology on Y , and $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ be irresolute. Then $\text{scom}(f_L^{\rightarrow}(G)) \geq \text{scom}(G)$.

Proof. Let $(\tau_1)_s$ and $(\tau_2)_s$ denote respectively the sets of all semiopen L -sets in τ_1 and τ_2 . By means of Lemma 2.4, we can obtain the following inequality.

$$\begin{aligned}
 & \text{scom}(f_L^{\rightarrow}(G)) \\
 &= \bigwedge_{\mathcal{U} \in 2^{((\tau_2)_s)}} \left\{ \bigwedge_{y \in Y} \left(f_L^{\rightarrow}(G)'(y) \vee \bigvee_{A \in \mathcal{U}} A(y) \right) \mapsto \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{y \in Y} \left(f_L^{\rightarrow}(G)'(y) \vee \bigvee_{A \in \mathcal{V}} A(y) \right) \right\} \\
 &= \bigwedge_{\mathcal{U} \in 2^{((\tau_2)_s)}} \left\{ \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} f_L^{\leftarrow}(A)(x) \right) \mapsto \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} f_L^{\leftarrow}(A)(x) \right) \right\} \\
 &\geq \text{scom}(G).
 \end{aligned}$$

This completes the proof. \square

Analogously we can prove the following two results.

Theorem 3.8. *Let $f : X \rightarrow Y$ be a set map, τ_1 be an L -topology on X , τ_2 be an L -topology on Y , and $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ be semicontinuous. Then $\text{com}(f_L^{\rightarrow}(G)) \geq \text{scom}(G)$.*

Theorem 3.9. *Let $f : X \rightarrow Y$ be a set map, τ_1 be an L -topology on X , τ_2 be an L -topology on Y , and $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ be strong irresolute. Then $\text{scom}(f_L^{\rightarrow}(G)) \geq \text{com}(G)$.*

4. THE COUNTABLE SEMICOMPACTNESS DEGREE OF L -SETS

Analogous to the last section, we can naturally introduce the notion of countable semicompactness degree and the degree to which an L -set has the semi-Lindelöf property as follows:

Definition 4.1. Let (X, τ) be an L -topological space and $G \in L^X$. The countable semicompactness degree $\text{scom}(G)$ of G is defined as

$$\begin{aligned}
 \text{scom}(G) &= \bigwedge_{\mathcal{U} \in 2^{[\tau_s]}} \left(\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \mapsto \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right) \right) \\
 &= \bigwedge_{\mathcal{U} \in 2^{[\tau_s]}} \left([G \tilde{\vee} \mathcal{U}] \mapsto \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \tilde{\vee} \mathcal{V}] \right).
 \end{aligned}$$

Definition 4.2. Let (X, τ) be an L -topological space and $G \in L^X$. The degree $\text{slp}(G)$ to which G has semi-Lindelöf property is defined as

$$\begin{aligned}
 \text{slp}(G) &= \bigwedge_{\mathcal{U} \in 2^{\tau_s}} \left(\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \mapsto \bigvee_{\mathcal{V} \in 2^{[\mathcal{U}]}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right) \right) \\
 &= \bigwedge_{\mathcal{U} \in 2^{\tau_s}} \left([G \tilde{\vee} \mathcal{U}] \mapsto \bigvee_{\mathcal{V} \in 2^{[\mathcal{U}]}} [G \tilde{\vee} \mathcal{V}] \right).
 \end{aligned}$$

Obviously G is countably semicompact if and only if $\text{scom}(G) = \top$, and G has the semi-Lindelöf property if and only if $\text{slp}(G) = \top$.

Analogous to Theorem 3.2 we easily obtain the the following result.

Theorem 4.3. *Let (X, τ) be an L -topological space and $G \in L^X$. Then $\text{sccom}(G) \leq \text{ccom}(G)$, $\text{slp}(G) \leq \text{Lp}(G)$.*

The next theorem is obvious.

Theorem 4.4. *Let (X, τ) be an L -topological space and $G \in L^X$. Then $\text{scom}(G) = \text{sccom}(G) \wedge \text{slp}(G)$.*

Analogous to the last section we can easily obtain the following some results.

Theorem 4.5. *Let (X, τ) be an L -topological space and $G \in L^X$. Then*

$$\text{sccom}(G) = \bigvee \left\{ a \in L : \left[G \widetilde{\subset} \bigvee \mathcal{U} \right] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[G \widetilde{\subset} \bigvee \mathcal{V} \right] \text{ for any } \mathcal{U} \in 2^{[\tau_s]} \right\}.$$

and

$$\text{slp}(G) = \bigvee \left\{ a \in L : \left[G \widetilde{\subset} \bigvee \mathcal{U} \right] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{[\mathcal{U}]}} \left[G \widetilde{\subset} \bigvee \mathcal{V} \right] \text{ for any } \mathcal{U} \in 2^{\tau_s} \right\}.$$

Theorem 4.6. *For any $G, H \in L^X$, we have*

$$\text{sccom}(G \vee H) \geq \text{sccom}(G) \wedge \text{sccom}(H) \quad \text{and} \quad \text{slp}(G \vee H) \geq \text{slp}(G) \wedge \text{slp}(H).$$

Theorem 4.7. *For any $G \in L^X$ and any semiclosed L -set H , we have*

$$\text{sccom}(G \wedge H) \geq \text{sccom}(G) \quad \text{and} \quad \text{slp}(G \wedge H) \geq \text{slp}(G).$$

Theorem 4.8. *Let $f : X \rightarrow Y$ be a set map, τ_1 be an L -topology on X , τ_2 be an L -topology on Y , and $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ be irresolute. Then $\text{sccom}(f_L^{\rightarrow}(G)) \geq \text{sccom}(G)$ and $\text{slp}(f_L^{\rightarrow}(G)) \geq \text{slp}(G)$.*

Theorem 4.9. *Let $f : X \rightarrow Y$ be a set map, τ_1 be an L -topology on X , τ_2 be an L -topology on Y , and $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ be semicontinuous. Then $\text{ccom}(f_L^{\rightarrow}(G)) \geq \text{sccom}(G)$ and $\text{Lp}(f_L^{\rightarrow}(G)) \geq \text{slp}(G)$.*

Theorem 4.10. *Let $f : X \rightarrow Y$ be a set map, τ_1 be an L -topology on X , τ_2 be an L -topology on Y , and $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ be strong irresolute. Then $\text{sccom}(f_L^{\rightarrow}(G)) \geq \text{ccom}(G)$ and $\text{slp}(f_L^{\rightarrow}(G)) \geq \text{Lp}(G)$.*

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