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Cubic subgroups

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Abstract. The notion of cubic subgroups is introduced, and related properties are investigated. Characterizations of a cubic subgroup are established, and how the images or inverse-images of cubic subgroups become cubic subgroups is studied.

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1. Introduction

Fuzzy sets, which were introduced by Zadeh [4], deal with possibilistic uncertainty, connected with imprecision of states, perceptions and preferences. Based on the (interval-valued) fuzzy sets, Jun et al. [1] introduced the notion of cubic subalgebras/ideals in BCK/BCI-algebras, and then they investigated several properties. They discussed relationship between a cubic subalgebra and a cubic ideal. Also, they provided characterizations of a cubic subalgebra/ideal, and considered a method to make a new cubic subalgebra from old one. Jun et al. [3] introduced the notion of cubic $\circ$-subalgebras and closed cubic ideals in BCK/BCI-algebras, and then they investigated several properties. They provided relations between a cubic ideal and a cubic $\circ$-subalgebra in a BCK-algebra, and the relation between a closed cubic ideal and a cubic subalgebra in a BCI-algebra. They also investigated a condition for a cubic set in a BCK-algebra with condition (S) to be a cubic ideal. Finally, they dealt with a characterization of cubic ideal in a BCK/BCI-algebra. Jun et al. [2] introduced the notion of cubic q-ideals in BCI-algebras. They discussed relationship between a cubic ideal and a cubic q-ideal, and provided conditions for a cubic ideal to be a cubic q-ideal. They also established characterizations of a cubic q-ideal, and considered the cubic extension property for a cubic q-ideal.

In this paper, we apply the notion of cubic sets to a group, and introduced the notion of cubic subgroups. We provide characterizations of a cubic subgroup, and study how the images or inverse-images of cubic subgroups become cubic subgroups.

2. Preliminaries

Let I be a closed unit interval, i.e., $I = [0,1]$. By an interval number we mean a closed subinterval $\tilde{a} = [a^-, a^+]$ of I, where $0 \le a^- \le a^+ \le 1$. Denote by $D[0,1]$ the set of all interval numbers. Let us define what is known as refined minimum (briefly, rmin) of two elements in $D[0,1]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $D[0,1]$. Consider two interval numbers $\tilde{a}_1 := \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 := \left[a_2^-, a_2^+\right]$. Then

$$\mathrm{rmin}\left\{\tilde{a}_1, \tilde{a}_2\right\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right],$$

$$\tilde{a}_1 \succeq \tilde{a}_2 \text{ if and only if } a_1^- \ge a_2^- \text{ and } a_1^+ \ge a_2^+,$$

and similarly we may have $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 \ne \tilde{a}_2$ (resp. $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 \ne \tilde{a}_2$). Let $\tilde{a}_i \in D[0,1]$ where $i \in \Lambda$. We define

$$\operatorname*{rinf}_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \operatorname*{rsup}_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

An interval-valued fuzzy set (briefly, IVF set) $\tilde{\mu}_A$ defined on a nonempty set X is given by

$$\tilde{\mu}_A := \left\{\left(x, \left[\mu_A^-(x), \mu_A^+(x)\right]\right) \mid x \in X\right\},$$

which is briefly denoted by $\tilde{\mu}_A = \left[\mu_A^-, \mu_A^+\right]$ where μ_A^- and μ_A^+ are two fuzzy sets in X such that $\mu_A^-(x) \le \mu_A^+(x)$ for all $x \in X$. For any IVF set $\tilde{\mu}_A$ on X and $x \in X$, $\tilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ is called the degree of membership of an element x to $\tilde{\mu}_A$, in which $\mu_A^-(x)$ and $\mu_A^+(x)$ are refereed to as the lower and upper degrees, respectively, of membership of x to $\tilde{\mu}_A$.

3. Cubic subgroups

In what follows let X denote a group unless otherwise specified.

Definition 3.1 ([1]). Let X be a nonempty set. A cubic set $\mathscr{A}$ in a set X is a structure

$$\mathscr{A} = \{\langle x, \tilde{\mu}_A(x), \lambda(x)\rangle : x \in X\}$$

which is briefly denoted by $\mathscr{A} = \langle \tilde{\mu}_A, \lambda\rangle$ where $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ is an IVF set in X and λ is a fuzzy set in X.

Denote by $\mathcal{C}(X)$ the family of cubic sets in a set X.

Definition 3.2. A cubic set $\mathscr{A} = \langle \tilde{\mu}_A, \lambda\rangle$ in X is called a cubic subgroup of X if it satisfies: for all $x, y \in X$,

(a) $\tilde{\mu}_A(xy) \succeq \mathrm{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$.
(b) $\tilde{\mu}_A(x^{-1}) \succeq \tilde{\mu}_A(x)$.
(c) $\lambda(xy) \le \max\{\lambda(x), \lambda(y)\}$.
(d) $\lambda(x^{-1}) \le \lambda(x)$.

Example 3.3. Let X be the Klein's four group. We have $X = \{e, a, b, ab\}$ where $a^2 = e = b^2$ and $ab = ba$. We define $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ and λ by

$$\tilde{\mu}_A = \begin{pmatrix} e & a & b & ab \\ [0.5, 0.8] & [0.4, 0.6] & [0.1, 0.5] & [0.1, 0.5] \end{pmatrix}$$

and

$$\lambda = \begin{pmatrix} e & a & b & ab \\ 0.2 & 0.3 & 0.6 & 0.6 \end{pmatrix}.$$

Then $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic subgroup of X.

Example 3.4. Let X be a non-trivial group and define an IVF set $\tilde{\mu}_B = [\mu_B^-, \mu_B^+]$ and a fuzzy set κ by $\tilde{\mu}_B(e) = [s_e, t_e]$ and $\tilde{\mu}_B(x) = [s, t]$ for all $x \neq e$ where $[s_e, t_e] \succ [s, t]$ in $D[0, 1]$, $\kappa(e) = r_e$ and $\kappa(x) = r$ for all $x \neq e$ where $r_e < r$ in $[0, 1]$ and e is the identity element of X. Then $\mathscr{B} = \langle \tilde{\mu}_B, \kappa \rangle$ is a cubic subgroup of X.

Proposition 3.5. *Let $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ be a cubic subgroup of X. Then $\tilde{\mu}_A(x^{-1}) = \tilde{\mu}_A(x)$ and $\lambda(x^{-1}) = \lambda(x)$ for all $x \in X$.*

Proof. For any $x \in X$, we have $\tilde{\mu}_A(x) = \tilde{\mu}_A((x^{-1})^{-1}) \succeq \tilde{\mu}_A(x^{-1}) \succeq \tilde{\mu}_A(x)$ and $\lambda(x) = \lambda((x^{-1})^{-1}) \leq \lambda(x^{-1}) \leq \lambda(x)$. Hence $\tilde{\mu}_A(x^{-1}) = \tilde{\mu}_A(x)$ and $\lambda(x^{-1}) = \lambda(x)$. □

Proposition 3.6. *Let $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ be a cubic subgroup of X. Then $\tilde{\mu}_A(e) \succeq \tilde{\mu}_A(x)$ and $\lambda(e) \leq \lambda(x)$ for all $x \in X$, where e is the identity element of X.*

Proof. Let $x \in X$. Using Proposition 3.5, we have

$$\tilde{\mu}_A(e) = \tilde{\mu}_A(xx^{-1}) \succeq \text{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(x^{-1})\} = \tilde{\mu}_A(x)$$

and $\lambda(e) = \lambda(xx^{-1}) \leq \max\{\lambda(x), \lambda(x^{-1})\} = \lambda(x)$. This completes the proof. □

Proposition 3.7. *Let $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ be a cubic subgroup of X. For any $x, y \in X$, if $\tilde{\mu}_A(xy^{-1}) = \tilde{\mu}_A(e)$ and $\lambda(xy^{-1}) = \lambda(e)$, then $\tilde{\mu}_A(x) = \tilde{\mu}_A(y)$ and $\lambda(x) = \lambda(y)$.*

Proof. Let $x, y \in X$ be such that $\tilde{\mu}_A\left(xy^{-1}\right) = \tilde{\mu}_A(e)$ and $\lambda\left(xy^{-1}\right) = \lambda(e)$. Using Proposition 3.6, we get $\tilde{\mu}_A(x) = \tilde{\mu}_A((xy^{-1})y) \succeq \text{rmin}\{\tilde{\mu}_A(e), \tilde{\mu}_A(y)\} = \tilde{\mu}_A(y)$ and $\lambda(x) = \lambda((xy^{-1})y) \leq \max\{\lambda(e), \lambda(y)\} = \lambda(y)$ for all $x, y \in X$. Similarly, $\tilde{\mu}_A(y) \succeq \tilde{\mu}_A(x)$ and $\lambda(y) \leq \lambda(x)$. Therefore we have the desired result. □

For a cubic subgroup $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ of X, we have the following question.

Question 3.8. *For any $x, y \in X$, if $\tilde{\mu}_A(y) \succ \tilde{\mu}_A(x)$ and $\lambda(y) < \lambda(x)$, then are the equalities $\tilde{\mu}_A(xy) = \tilde{\mu}_A(x) = \tilde{\mu}_A(yx)$ and $\lambda(xy) = \lambda(x) = \lambda(yx)$ true?*

The following example provide a negative answer to the Question 3.8.

Example 3.9. In the Klein's four group $X = \{e, a, b, ab\}$, we define $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ and λ by

$$\tilde{\mu}_A = \begin{pmatrix} e & a & b & ab \\ [0.3, 0.9] & [0.1, 0.7] & [0.1, 0.9] & [0.3, 0.7] \end{pmatrix}$$

and

$$\lambda = \begin{pmatrix} e & a & b & ab \\ 0.2 & 0.6 & 0.4 & 0.6 \end{pmatrix}.$$

Then $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic subgroup of X. Note that $\tilde{\mu}_A(b) = [0.1, 0.9] \succ [0.1, 0.7] = \tilde{\mu}_A(a)$ and $\lambda(b) = 0.4 < 0.6 = \lambda(a)$. But $\tilde{\mu}_A(ab) = [0.3, 0.7] \neq [0.1, 0.7] = \tilde{\mu}_A(a)$.

We provide characterizations of a cubic subgroup.

Theorem 3.10. *A cubic set $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ in X is a cubic subgroup of X if and only if it satisfies:*

(1) $\tilde{\mu}_A(xy^{-1}) \succeq \mathrm{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$,
(2) $\lambda(xy^{-1}) \leq \max\{\lambda(x), \lambda(y)\}$

for all $x, y \in X$.

Proof. Assume that $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic subgroup of X and let $x, y \in X$. Then $\tilde{\mu}_A(xy^{-1}) \succeq \mathrm{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y^{-1})\} = \mathrm{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ and

$$\lambda(xy^{-1}) \leq \max\{\lambda(x), \lambda(y^{-1})\} = \max\{\lambda(x), \lambda(y)\}$$

by Proposition 3.5.

Conversely, suppose that (1) and (2) are valid. If we take $y = x$ in (1) and (2), then $\tilde{\mu}_A(e) = \tilde{\mu}_A(xx^{-1}) \succeq \mathrm{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(x)\} = \tilde{\mu}_A(x)$ and $\lambda(e) = \lambda(xx^{-1}) \leq \max\{\lambda(x), \lambda(x)\} = \lambda(x)$. It follows from (1) and (2) that $\tilde{\mu}_A(y^{-1}) = \tilde{\mu}_A(ey^{-1}) \succeq \mathrm{rmin}\{\tilde{\mu}_A(e), \tilde{\mu}_A(y)\} = \tilde{\mu}_A(y)$ and $\lambda(y^{-1}) = \lambda(ey^{-1}) \leq \max\{\lambda(e), \lambda(y)\} = \lambda(y)$ so that

$$\tilde{\mu}_A(xy) = \tilde{\mu}_A(x(y^{-1})^{-1}) \succeq \mathrm{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y^{-1})\} \succeq \mathrm{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$$

and $\lambda(xy) = \lambda(x(y^{-1})^{-1}) \leq \max\{\lambda(x), \lambda(y^{-1})\} \leq \max\{\lambda(x), \lambda(y)\}$. Therefore $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic subgroup of X. □

Theorem 3.11. *If $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic subgroup of X, then the set*

$$S := \{x \in X \mid \tilde{\mu}_A(x) = \tilde{\mu}_A(e),\ \lambda(x) = \lambda(e)\}$$

is a subgroup of X.

Proof. Let $x, y \in S$. Then $\tilde{\mu}_A(x) = \tilde{\mu}_A(e) = \tilde{\mu}_A(y)$ and $\lambda(x) = \lambda(e) = \lambda(y)$. It follows from Theorem 3.10 that

$$\tilde{\mu}_A(xy^{-1}) \succeq \mathrm{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} = \tilde{\mu}_A(e)$$

and $\lambda(xy^{-1}) \leq \max\{\lambda(x), \lambda(y)\} = \lambda(e)$ so from Proposition 3.6 that $\tilde{\mu}_A(xy^{-1}) = \tilde{\mu}_A(e)$ and $\lambda(xy^{-1}) = \lambda(e)$. Hence $xy^{-1} \in S$, and so S is a subgroup of X. □

Let $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ be a cubic set in a set X, $r \in [0, 1]$ and $[s, t] \in D[0, 1]$. The set

$$U(\mathscr{A}; [s, t], r) := \{x \in X \mid \tilde{\mu}_A(x) \succeq [s, t],\ \lambda(x) \leq r\}$$

is called the cubic level set of $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$.

Theorem 3.12. *For a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ in X, the following are equivalent:*

(1) *$\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic subgroup of X.*
(2) *The nonempty cubic level set of $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a subgroup of X.*

Proof. Assume that $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic subgroup of X. Let $x, y \in U(\mathscr{A}; [s,t], r)$ for all $r \in [0,1]$ and $[s,t] \in D[0,1]$. Then $\tilde{\mu}_A(x) \succeq [s,t]$, $\lambda(x) \le r$, $\tilde{\mu}_A(y) \succeq [s,t]$ and $\lambda(y) \le r$. It follows from Theorem 3.10 that

$$\tilde{\mu}_A(xy^{-1}) \succeq \text{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} \succeq [s,t]$$

and $\lambda(xy^{-1}) \le \max\{\lambda(x), \lambda(y)\} \le r$ so that $xy^{-1} \in U(\mathscr{A}; [s,t], r)$. Therefore the nonempty cubic level set of $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a subgroup of X.

Conversely, let $r \in [0,1]$ and $[s,t] \in D[0,1]$ be such that $U(\mathscr{A}; [s,t], r) \neq \varnothing$, and $U(\mathscr{A}; [s,t], r)$ is a subgroup of X. Suppose that Theorem 3.10(1) is not true and Theorem 3.10(2) is valid. Then there exist $[s_0, t_0] \in D[0,1]$ and $a, b \in X$ such that

$$\tilde{\mu}_A(ab^{-1}) \prec [s_0, t_0] \preceq \text{rmin}\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}$$

and $\lambda(ab^{-1}) \le \max\{\lambda(a), \lambda(b)\}$. It follows that $a, b \in U(\mathscr{A}; [s_0, t_0], \max\{\lambda(a), \lambda(b)\})$ but $ab^{-1} \notin U(\mathscr{A}; [s_0, t_0], \max\{\lambda(a), \lambda(b)\})$. This is a contradiction. If Theorem 3.10(1) is true and Theorem 3.10(2) is not valid, then $\tilde{\mu}_A(ab^{-1}) \succeq \text{rmin}\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}$ and

$$\lambda(ab^{-1}) > r_0 \ge \max\{\lambda(a), \lambda(b)\}$$

for some $r_0 \in [0,1]$ and $a, b \in X$. Thus $a, b \in U(\mathscr{A}; \text{rmin}\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}, r_0)$ but $ab^{-1} \notin U(\mathscr{A}; \text{rmin}\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}, r_0)$, which is a contradiction. Assume that there exist $[s_0, t_0] \in D[0,1]$, $r_0 \in [0,1]$ and $a, b \in X$ such that

$$\tilde{\mu}_A(ab^{-1}) \prec [s_0, t_0] \preceq \text{rmin}\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}$$

and $\lambda(ab^{-1}) > r_0 \ge \max\{\lambda(a), \lambda(b)\}$. Then $a, b \in U(\mathscr{A}; [s_0, t_0], r_0)$ but $ab^{-1} \notin U(\mathscr{A}; [s_0, t_0], r_0)$. This is also a contradiction. Hence (1) and (2) of Theorem 3.10 are valid. Therefore $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic subgroup of X. □

Let X and Y be given classical sets. A mapping $f : X \to Y$ induces two mappings $\mathcal{C}_f : \mathcal{C}(X) \to \mathcal{C}(Y)$, $\mathscr{A} \mapsto \mathcal{C}_f(\mathscr{A})$, and $\mathcal{C}_f^{-1} : \mathcal{C}(Y) \to \mathcal{C}(X)$, $\mathscr{B} \mapsto \mathcal{C}_f^{-1}(\mathscr{B})$, where $\mathcal{C}_f(\mathscr{A})$ is given by

$$\mathcal{C}_f(\tilde{\mu}_A)(y) = \begin{cases} \text{rsup}_{y=f(x)} \tilde{\mu}_A(x) & \text{if } f^{-1}(y) \neq \varnothing \\ [0,0] & \text{otherwise} \end{cases}$$

$$\mathcal{C}_f(\lambda)(y) = \begin{cases} \inf_{y=f(x)} \lambda(x) & \text{if } f^{-1}(y) \neq \varnothing \\ 1 & \text{otherwise} \end{cases}$$

for all $y \in Y$; and $\mathcal{C}_f^{-1}(\mathscr{B})$ is defined by $\mathcal{C}_f^{-1}(\tilde{\mu}_B)(x) = \tilde{\mu}_B(f(x))$ and $\mathcal{C}_f^{-1}(\kappa)(x) = \kappa(f(x))$ for all $x \in X$. Then the mapping $\mathcal{C}_f$ (resp. $\mathcal{C}_f^{-1}$) is called a cubic transformation (resp. inverse cubic transformation) induced by f. A cubic set $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ in X has the cubic property if for any subset $T$of X there exists $x_0 \in T$ such that $\tilde{\mu}_A(x_0) = \text{rsup}_{x \in T} \tilde{\mu}_A(x)$ and $\lambda(x_0) = \inf_{x \in T} \lambda(x)$.

Theorem 3.13. *For a homomorphism $f : X \to Y$ of groups, let $\mathcal{C}_f : \mathcal{C}(X) \to \mathcal{C}(Y)$ and $\mathcal{C}_f^{-1} : \mathcal{C}(Y) \to \mathcal{C}(X)$ be the cubic transformation and inverse cubic transformation, respectively, induced by f.*

(1) *If* $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle \in \mathcal{C}(X)$ *is a cubic subgroup of* X *which has the cubic property, then* $\mathcal{C}_f(\mathscr{A})$ *is a cubic subgroup of* Y.
(2) *If* $\mathscr{B} = \langle \tilde{\mu}_B, \kappa \rangle \in \mathcal{C}(Y)$ *is a cubic subgroup of* Y, *then* $\mathcal{C}_f^{-1}(\mathscr{B})$ *is a cubic subgroup of* X.

Proof. (1) Given $f(x), f(y) \in f(X)$, let $x_0 \in f^{-1}(f(x))$ and $y_0 \in f^{-1}(f(y))$ be such that

$$\tilde{\mu}_A(x_0) = \operatorname*{rsup}_{a \in f^{-1}(f(x))} \tilde{\mu}_A(a), \ \lambda(x_0) = \inf_{a \in f^{-1}(f(x))} \lambda(a),$$

and

$$\tilde{\mu}_A(y_0) = \operatorname*{rsup}_{b \in f^{-1}(f(y))} \tilde{\mu}_A(b), \ \lambda(y_0) = \inf_{b \in f^{-1}(f(y))} \lambda(b),$$

respectively. Then

$$\begin{aligned} \mathcal{C}_f(\tilde{\mu}_A)(f(x)f(y)) &= \operatorname*{rsup}_{z \in f^{-1}(f(x)f(y))} \tilde{\mu}_A(z) \\ &\succeq \tilde{\mu}_A(x_0y_0) \succeq \operatorname{rmin}\{\tilde{\mu}_A(x_0), \tilde{\mu}_A(y_0)\} \\ &= \operatorname{rmin}\Big\{ \operatorname*{rsup}_{a \in f^{-1}(f(x))} \tilde{\mu}_A(a), \operatorname*{rsup}_{b \in f^{-1}(f(y))} \tilde{\mu}_A(b) \Big\} \\ &= \operatorname{rmin}\{\mathcal{C}_f(\tilde{\mu}_A)(f(x)), \mathcal{C}_f(\tilde{\mu}_A)(f(y))\}, \end{aligned}$$

$$\mathcal{C}_f(\tilde{\mu}_A)(f(x)^{-1}) = \operatorname*{rsup}_{z \in f^{-1}(f(x)^{-1})} \tilde{\mu}_A(z) \succeq \tilde{\mu}_A(x_0^{-1}) \succeq \tilde{\mu}_A(x_0) = \mathcal{C}_f(\tilde{\mu}_A)(f(x)),$$

$$\begin{aligned} \mathcal{C}_f(\lambda)(f(x)f(y)) &= \inf_{z \in f^{-1}(f(x)f(y))} \lambda(z) \\ &\leq \lambda(x_0y_0) \leq \max\{\lambda(x_0), \lambda(y_0)\} \\ &= \max\Big\{ \inf_{a \in f^{-1}(f(x))} \lambda(a), \inf_{b \in f^{-1}(f(y))} \lambda(b) \Big\} \\ &= \max\{\mathcal{C}_f(f(x)), \mathcal{C}_f(f(y))\} \end{aligned}$$

and $\mathcal{C}_f(\lambda)(f(x)^{-1}) = \inf_{z \in f^{-1}(f(x)^{-1})} \lambda(z) \leq \lambda(x_0^{-1}) \leq \lambda(x_0) = \mathcal{C}_f(\lambda)(f(x))$. Therefore $\mathcal{C}_f(\mathscr{A})$ is a cubic subgroup of Y.

(2) For any $x, y \in X$, we have

$$\begin{aligned} \mathcal{C}_f^{-1}(\tilde{\mu}_B)(xy) &= \tilde{\mu}_B(f(xy)) = \tilde{\mu}_B(f(x)f(y)) \\ &\succeq \operatorname{rmin}\{\tilde{\mu}_B(f(x)), \tilde{\mu}_B(f(y))\} \\ &= \operatorname{rmin}\{\mathcal{C}_f^{-1}(\tilde{\mu}_B)(x), \mathcal{C}_f^{-1}(\tilde{\mu}_B)(y)\}, \end{aligned}$$

$$\mathcal{C}_f^{-1}(\tilde{\mu}_B)(x^{-1}) = \tilde{\mu}_B(f(x^{-1})) = \tilde{\mu}_B(f(x)^{-1}) \succeq \tilde{\mu}_B(f(x)) = \mathcal{C}_f^{-1}(\tilde{\mu}_B)(x),$$

$$\begin{aligned} \mathcal{C}_f^{-1}(\kappa)(xy) &= \kappa(f(xy)) = \kappa(f(x)f(y)) \\ &\leq \max\{\kappa(f(x)), \kappa(f(y))\} \\ &= \max\{\mathcal{C}_f^{-1}(\kappa)(x), \mathcal{C}_f^{-1}(\kappa)(y)\}, \end{aligned}$$

and $\mathcal{C}_f^{-1}(\kappa)(x^{-1}) = \kappa(f(x^{-1})) = \kappa(f(x)^{-1}) \leq \kappa(f(x)) = \mathcal{C}_f^{-1}(\kappa)(x)$. Hence $\mathcal{C}_f^{-1}(\mathscr{B})$ is a cubic subgroup of X. □

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