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In search of the root of fuzziness: The measure theoretic meaning of partial presence

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Abstract. Two laws of randomness are necessary and sufficient to describe a normal fuzzy number. Trying to impose one single probability law on an interval on which a possibility law has been defined is absolutely illogical. This is the reason why the attempts of establishing a principle of consistency between probability and possibility have not yielded any fruitful result till this day. We are hereby nullifying all heuristic works that had been published in the last forty five years in this context the world over. For a normal fuzzy number $[a, b, c]$, the partial presence of an element x in the interval $[a, c]$ is either a probability distribution function $F(x)$ defined in $a \leq x < b$, or a complementary probability distribution function $(1-G(x))$ where $G(x)$ is a probability distribution function defined in $b \leq x < c$. In other words, possibility is indeed probability in disguise. Thus fuzziness is measure theoretic on its own right. We need not define a fuzzy measure, in the entire interval $[a, c]$, which is not actually a measure in the classical sense. We need to correct this mathematical blunder as early as possible to fetch the mathematics of fuzziness back into the right path.

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1. Introduction

Lotfi Zadeh's discovery of the concept of fuzziness forty five years ago as a competitor of randomness was an epoch making event in the history of mathematics. However, into the mathematics of fuzziness there entered a blunder right at the start. Researchers had tried to infer a probability law from a given possibility law, and also to infer a possibility law from a given probability law. In course of time, it was concluded that an inference of that kind is not possible. Hence it was decided that probability and possibility are not conceptually related. Three probability - possibility consistency principles have meanwhile been proposed, none of them actually leading to any definite conclusion. Indeed had there been any logic in trying to infer one single probability law from a given possibility law, there should have been just one such consistency principle, and not three. In the process, it was declared that fuzzy sets do not conform to measure theoretic formalisms because of non-fulfillment of the additivity postulate required to define a measure, and therefore a definition of a fuzzy measure was constructed. What had followed thereafter sometimes defied earthly logic. All sorts of weird mathematical formalisms started to appear in the literature thereafter. Mathematics should follow logic; it must never be the other way around. In the case of defining a fuzzy measure, logic has unfortunately been forced to follow mathematics. In the meantime, the mathematics fraternity got divided into two distinct groups of people: those who work on fuzzy mathematics, and those who do not believe a word of it. A time came when the nonbelievers of fuzzy mathematics started to accept that the mathematics of fuzziness defining the fuzzy measure was here to stay, and that nothing can be done about it anyway. It is time that we look into the matters newly, and reconstruct the theory of fuzzy sets correctly. Using an operation that we had named superimposition of sets [3], we have been successful in establishing that two, and not one, probability laws are needed to define one possibility law [2]. We have further been successful in showing that our perspective of looking at a normal fuzzy number helps in fuzzy arithmetic too ([8, 12]). We have recently recently shown that the correct *randomness-fuzziness consistency principle* [5] should actually be

$$\text{Poss } [x] = \theta \text{ Prob } [a \le y \le x] + (1-\theta)\,\{1 - \text{Prob } [b \le y \le x]\}$$

where $\theta = 1$ if $a \le x \le b$, and $\theta = 0$ if $b \le x \le c$. So, possibility of $[X = x]$ for $a \le x \le c$ is expressible as nothing but a probability only, either as Prob $[a \le X \le x]$ or as $\{1 - \text{Prob } [b \le X \le x]\}$, whichever is the case. We have used the term probability in the broader measure theoretic sense, and not in the narrow statistical sense, here. We have very recently started this process of correcting the aforesaid mathematical blunder. We understand that the process of making the earlier workers to see reason would not be easy. Hundreds of books and thousands of research articles have meanwhile been published the world over. Disproving mathematical results is not a new phenomenon. However, to challenge the world of fuzzy mathematics single handedly is a very tough proposition indeed as we have been made to realize in recent years. This sort of a thing has probably never happened earlier in the history of mathematics. After our perspective starts to get accepted, a lot of materials published till this day would have to be automatically nullified. Two opposing concepts can not be simultaneously correct. It should be quite obvious to the readers that there have been objections in accepting our findings from those who have built up their kind of mathematics based on certain incorrect assumptions. In fact, we have used the concept of set superimposition in recognizing periodic patterns ([10, 11, 13]). Only then could we confidently speak out that if the application of that operation was found to be correct, how the theory itself could be wrong! An incorrect concept has been prevailing for so many years. The new generation of workers would hopefully realize that in the name of mathematics of fuzziness all sorts of illogical things have entered into the literature. We hope, workers would come forward to remove the unwanted formalisms from the theory of fuzzy sets to pull it from the quagmire it is currently in. In fact, there is yet another much more serious blunder in the theory of fuzzy sets [6]. It is said that intersection of a fuzzy set and its complement is not null! It looks very strange that the fuzzy mathematics fraternity accepted this illogical statement, and the mathematics of fuzziness proceeded accordingly. Everyone concerned started to build up some formalisms based on this strange idea. Logically speaking, how can it be true that anything and its complement can have something in common? Obviously the definition of complementation of a fuzzy set has to be incorrect. However, this definition of complementation has been used in so many publications that it not really feasible to count them. As we have said earlier, here too logic has been forced to follow mathematics. It is really very surprising to see that till this day no one has objected to this wrong notion. In fact, this definition of complement of a fuzzy set has been used in software too causing a mess in the application of the concept of fuzziness.

In this article, we are going to discuss how to *define* a normal fuzzy number properly. We would desist from referring to any earlier work of any other author because we are in a process of correcting a blunder, and citing mistakes would in fact be an unnecessary burden. We are hereby nullifying all other works done in this context by all previous workers the world over. We insist that a mistake remains a mistake even though it is believed to be true for long forty five years. Mathematics should be based on pure logic; it must not be based on popular beliefs. In the next section, we would discuss about randomness. We need to explain why in our randomness . fuzziness consistency principle, we have mentioned that we have used the term probability in the broader measure theoretic sense. Thereafter we would proceed to show how to define a normal fuzzy number in its true perspective. We shall thereafter with the help of numerical examples show how such probability laws can be found. We shall further discuss the construction of a fuzzy number using probabilistic interpretations.

2. Definition of Randomness

Randomness is one term which is very widely misunderstood [4]. Ask a person from Statistics the definition of randomness and he would, in most cases, reply that a random variable is one that is associated with some probability law of *errors*. However, from the measure theoretic standpoint, if we can associate a *density* function $f(x)$ with the variable X defined in some interval $[a, b]$ such that

$$\int_a^b f(x)dx = 1,$$

then X is said to be a random variable with reference to $f(x)$. In other words, if a variable is random in the measure theoretic sense, it need not be random in the statistical sense. For example,

$$\int_0^1 2xdx = 1,$$

and therefore x here is random by definition, but not necessarily probabilistic following some probability law of errors in the statistical sense. We would first like to quote some excerpts from some standard publications how people define randomness. We would like to state that the confusions are there not for nothing.

Rohatgi and Saleh [14], in page 41 of their book, have defined the randomness related matters in the following way. 'Let (Ω, S) be a sample space. A finite, single valued function that maps Ω into $\mathbb{R}$ is called a *random variable* if the inverse images under X of all Borel sets in $\mathbb{R}$ are events, that is, if $X^{-1}(B) = \{\omega : X(\omega) \in B\} \in S$, for all $B \in \mathbf{B}$. Here $\mathbf{B}$ is a Borel Field.' They have added the remark that *the notion of probability does not enter into the definition of a random variable.* They have of course mentioned further, in page 43, that *in practice random variables are of interest only when they are defined on a probability space.* Rohatgi and Saleh have directly noted that randomness does not necessarily imply that a probability law must be associated with it. As we have said, *if a variable is random, in the measure theoretic sense it need not actually be probabilistic in the statistical sense.*

We would like to quote the following lines from another standard text book on Probability and Measure Theory. Ash and Doleans-Dade [1], in page 5 of their book, have defined a probability measure as follows: 'A measure on a σ-field $\mathscr{F}$ is a nonnegative, extended real valued function Ψ on $\mathscr{F}$ such that whenever $A_1, A_2, \cdots,$ form a finite or countably infinite collection of disjoint sets in $\mathscr{F}$, we have $\Psi(\cup A_n) = \sum \Psi(A_n)$, the union and summation being taken over n. If $\Psi(\Omega) = 1$, Ψ is called a probability measure. A measure space with a triple $(\Omega, \mathscr{F}, \Psi)$ where Ω is a set, $\mathscr{F}$ is a σ-field of subsets of Ω, and Ψ is a measure on $\mathscr{F}$. If Ψ is a probability measure, $(\Omega, \mathscr{F}, \Psi)$ is called a probability space. From the same book, from pages $173--174$, we would like to quote some lines again. 'Intuitively, a random variable is a quantity that is measured in connection with a random experiment.' In other words, they have also retorted like Rohatgi and Saleh that randomness is however, intuitively though occurs with reference to a probability law of errors. According to them, a random variable X on a probability space $(\Omega, \mathscr{F}, P)$ is a Borel measurable function from Ω to $\mathbf{R}$, the set of real numbers. The distribution function of a random variable X is a function given by $F(x) = P\{\omega : X(\omega) \le x\}$, for real x. Since for $a < b$, $F(b) - F(a) = P\{\omega : a < X(\omega) \le b\} = P_X(a, b]$, F is a distribution function corresponding to the Lebesgue - Stieltjes measure P_X. Under a caption Fundamental Statistical Concepts, Gibbons and Chakraborti [9] in page 9 of their book, have clearly mentioned as follows. 'A sample space is the set of all possible outcomes of a random experiment. A random variable is a set function whose domain is the elements of a sample space on which a probability function has been defined and whose range is the set of all real numbers. Alternatively, X is a random variable if for every real number x there exists a probability that the value assumed by the random variable does not exceed x, denoted by $P(X \le x)$ or $F_X(x)$, and called the cumulative distribution function of X. It may be noted that Gibbons and Chakraborti have however not said anything about the measure theoretic matters regarding the definition of a random variable. The book concerned was not written in measure theoretic language anyway. Bhat [7] has gone one step further. In his book, in pages $1-2$, he has mentioned very clearly as follows. 'Ideal situation envisaged in a deterministic model hardly exists in everyday life. Also the model may not fit the observation well because some essential features have been ignored. Many times improvement can be achieved by introducing random variables or chance factors in the model.' In other words, he seems to agree with Gibbons and Chakraborti that random variables occur due to chance factors. In the same book, in page 7, Bhat has defined Measurable Functions as follows. 'Let X be a mapping from Ω to $\mathbb{R}$ such that $X^{-1}(B) \in \mathbf{A}$, for any Borel set B of $\mathbb{R}$. Then X is said to be $\mathbf{A}$-measurable function or a random variable. The class $\{X^{-1}(B), B \in \mathbf{B}\}$, where $\mathbf{B}$ is the Borel Field, is a σ-field and is called the σ-field induced by X.' It is clear that Bhat has made the flaw of giving two different definitions of randomness. On one hand he says that randomness is synonymous to chance factors, while on the other hand he cites the measure theoretic definition where there is no reference of a chance factor. By now, it should be clear as to why there is a confusion regarding the very definition of randomness. In the measure theoretic definition of randomness, the question of probability simply does not come. This has been clearly noted by Rohatgi and Saleh, though according to them in practice the question of randomness appears with reference to a probability space. Ash and Doleans-Dade have started with the comment that intuitively a random variable is connected with a random experiment, though the measure theoretic definition does not really require that. Gibbons and Chakraborti have simply declared that a random variable must be associated with probability, without which as a condition it would not be called a random variable. Bhat has plainly given two contradictory explanations of randomness. If the literature is searched fully, many such opposing and confusing statements regarding randomness would be found. This has created a lot of confusion among the readers. We would like to state it as follows. First, the names probability measure and probability space are to be understood properly. Indeed, had these two things would have been named without using the word probability, there would not have been any confusion. Most people do not try to understand that a probability measure does not have anything to do with probability at all. We shall hereafter use the term randomness in the measure theoretic sense. Further, we shall use the terminology probability distribution to mean the distribution function with reference to a density function stated above.

3. The Operation of Set Superimposition

We have discussed the operation of set superimposition in our earlier works already. However for easy readability of this article we are going to discuss about it in short. The operation of set superimposition is expressed as follows: if the set A is superimposed over the set B, we get

$$A(S)B = (A - B) \cup (A \cap B)^{(2)} \cup (B - A)$$

where S represents the operation of superimposition, and $(A \cap B)^{(2)}$ represents the elements of $(A \cap B)$ occurring twice, provided that $(A \cap B)$ is not void. We have defined this operation keeping in view the fact that when we overwrite, the overwritten portion looks darker for such a double representation.

It can be seen that for n fuzzy intervals $[a_1, b_1]^{\frac{1}{n}}$, $[a_2, b_2]^{\frac{1}{n}}, \cdots, [a_n, b_n]^{\frac{1}{n}}$ all with membership value equal to $\frac{1}{n}$ everywhere, we shall have

$$\begin{aligned}&[a_1, b_1]^{\frac{1}{n}}(S)[a_2, b_2]^{\frac{1}{n}}(S)\cdots(S)[a_n, b_n]^{\frac{1}{n}}\\ &= [a_{(1)}, a_{(2)}]^{\frac{1}{n}} \cup [a_{(2)}, a_{(3)}]^{\frac{2}{n}} \cup \cdots \cup [a_{(n-1)}, a_{(n)}]^{\frac{n-1}{n}}\\ &\quad \cup [a_{(n)}, b_{(1)}]^{(1)} \cup [b_{(1)}, b_{(2)}]^{\frac{n-1}{n}} \cup \cdots \cup [b_{(n-2)}, b_{(n-1)}]^{\frac{2}{n}}\\ &\quad \cup [b_{(n-1)}, b_{(n)}]^{\frac{1}{n}},\end{aligned}$$

where, for example, $[b_{(1)}, b_{(2)}]^{\frac{n-1}{n}}$ represents the uniformly fuzzy interval $[b_{(1)}, b_{(2)}]$ with membership $\frac{n-1}{n}$ in the entire interval, $a_{(1)}, a_{(2)}, \cdots, a_{(n)}$ being values of a_1, a_2, $\cdots$, a_n arranged in increasing order of magnitude, and $b_{(1)}, b_{(2)}, \cdots, b_{(n)}$ being values of $b_1, b_2, \cdots, b_n$ arranged in increasing order of magnitude. Thus for the fuzzy intervals $[x_1, y_1]^{\frac{1}{n}}$, $[x_2, y_2]^{\frac{1}{n}}$, $\cdots$, $[x_n, y_n]^{\frac{1}{n}}$, all with uniform membership $\frac{1}{n}$, the values of membership of the superimposed fuzzy intervals are $\frac{1}{n}$, $\frac{2}{n}$, $\cdots$, $\frac{n-1}{n}$, 1, $\frac{n-1}{n}$, $\cdots$, $\frac{2}{n}$, and $\frac{1}{n}$. These values of membership considered in two halves as

$$(0, \tfrac{1}{n}, \tfrac{2}{n}, \cdots, \tfrac{n-1}{n}, 1),$$

and

$$(1, \tfrac{n-1}{n}, \cdots, \tfrac{2}{n}, \tfrac{1}{n}, 0),$$

would suggest that they can define an empirical distribution and a complementary empirical distribution on $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$, respectively. In other words, for realizations of the values of $x_{(1)}, x_{(2)}, \cdots, x_{(n)}$ in increasing order and of $y_{(1)}, y_{(2)}, \cdots, y_{(n)}$ again in increasing order, we can see that if we define

$$\Psi_1(x) = \begin{cases} 0 & \text{if } x < x_{(1)}, \\ \frac{r-1}{n} & \text{if } x_{(r-1)} \le x \le x_{(r)},\ r = 2, 3, \cdots, n, \\ 1 & \text{if } x \ge x_{(n)}, \end{cases}$$

$$\Psi_2(y) = \begin{cases} 1 & \text{if } y < y_{(1)}, \\ 1 - \frac{r-1}{n} & \text{if } y_{(r-1)} \le y \le y_{(r)},\ r = 2, 3, \cdots, n, \\ 0 & \text{if } y \ge y_{(n)}, \end{cases}$$

then the Glivenko - Cantelli Lemma on Order Statistics assures that

$$\Psi_1(x) \to \prod\nolimits_1 [\alpha, x],\ \alpha \le x \le \beta,$$

$$\Psi_2(y) \to 1 - \prod\nolimits_2 [\beta, y],\ \beta \le y \le \gamma,$$

where $\prod_1 [\alpha, x]$, $\alpha \le x \le \beta$ and $\Psi_2(y)$, $\beta \le y \le \gamma$ are two probability distributions. Thus

$$\text{Poss } [x] = \theta \text{ Prob } [a \le y \le x] + (1 - \theta)\{1 - \text{Prob } [b \le y \le x]\},$$

where

$$\theta = \begin{cases} 1 & \text{if } a \le x \le b, \\ 0 & \text{if } b \le x \le c. \end{cases}$$

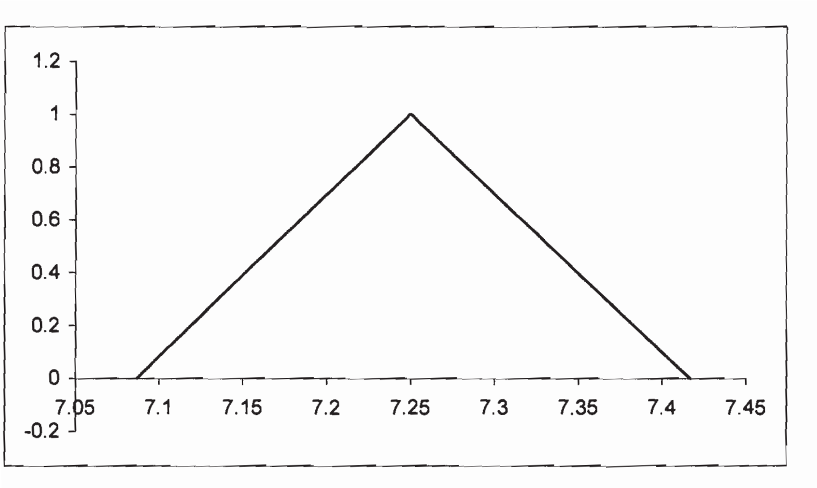

FIGURE 1. Membership Function: Example 1

And therefore there is no need to define a fuzzy measure; fuzziness is measure theoretic in the classical sense already. We therefore propose to define a normal fuzzy number as follows. If $\Phi_1(x)$ and $(1 - \Phi_2(x))$ are two independent probability distribution functions defined in $[\alpha, \beta]$ and $[\beta, \gamma]$ respectively, then the membership function of a normal fuzzy number $N = [\alpha, \beta, \gamma]$ can be expressed as

$$\mu_N(x) = \begin{cases} \Phi_1(x) & \text{if } \alpha \le x \le \beta, \\ \Phi_2(x) & \text{if } \beta \le x \le \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

This definition does not defy the one given by Lotfi Zadeh. In addition, this definition embodies the necessary and sufficient mathematical explanation of partial presence of an element in the fuzzy set. Here we are defining probability distribution functions with reference to the measure theoretic definition of randomness. There should be no confusion in that respect. As we have said earlier, we are using the term probability here as it is used in defining a probability measure. This need not have any relation with error variables and such other statistical matters.

4. Numerical Examples

Let the fuzzy membership function (Example 1) in a particular case be [4]

$$\mu(x) = \begin{cases} 0 & \text{if } x < 7.08683, \\ \frac{8.0802x - 57.263}{0.737 + 0.0802x} & \text{if } 7.08683 \le x \le 7.25, \\ \frac{58.743 - 7.9202x}{0.743 + 0.0798x} & \text{if } 7.25 \le x \le 7.416865, \\ 0 & \text{if } x > 7.416865. \end{cases}$$

The possibility distribution in this case is as shown in Figure 1. Observe that in the denominators in both of the cases, the multipliers of x are much smaller in comparison with the multipliers of x in the numerators. The values of x being small, the possibility distribution looks nearly triangular which is however not the case. Here,

$$L(x) = \frac{8.0802x - 57.263}{0.737 + 0.0802x}$$

is the Left Reference Function and

$$R(x) = \frac{58.743 - 7.9202x}{0.743 + 0.0798x}$$

is the Right Reference Function as defined by Dubois and Prade. $L(x)$ is non-decreasing, increasing from 0 to 1, and $R(x)$ is non-increasing, decreasing from 1 to 0, for $7.08683 \leq x \leq 7.25$ and $7.25 \leq x \leq 7.416865$ respectively. According to our findings,

$$F(x) = L(x) = \frac{8.0802x - 57.263}{0.737 + 0.0802x}$$

is a distribution function for $7.08683 \leq x \leq 7.25$, satisfying all the properties of a distribution function and

$$G(x) = 1 - R(x) = 1 - \frac{58.743 - 7.9202x}{0.743 + 0.0798x}$$

is another distribution function for $7.25 \leq x \leq 7.416865$. The density for $F(x)$ is given by

$$f(x) = dF(x)/dx = \frac{10.5476}{(0.737 + 0.0802x)^2},\ 7.08683 \leq x \leq 7.25.$$

Similarly, the density for $G(x)$ is given by

$$g(x) = dG(x)/dx = \frac{10.5724}{(0.743 + 0.0798x)^2},\ 7.25 \leq x \leq 7.416865.$$

This means the value of

$$\int \frac{10.5476}{(0.737 + 0.0802x)^2}\, dx,$$

the integral taken from 7.08683 to x, $7.08683 \leq x \leq 7.25$, is the possibility of occurrence of x, which is nothing but Dubois - Prade's $L(x)$. Similarly, the value of

$$1 - \int \frac{10.5476}{(0.737 + 0.0802x)^2}\, dx,$$

the integral taken from 7.25 to x, $7.25 \leq x \leq 7.416865$, is the possibility of occurrence of x, which is nothing but Dubois - Prade's $R(x)$. Therefore,

$$\text{Poss } [x] = \begin{cases} \text{Prob } [7.08683 \leq y < x] & \text{if } 7.08683 \leq x \leq 7.25, \\ 1 - \text{Prob } [7.25 \leq y < x] & \text{if } 7.25 \leq x \leq 7.416865. \end{cases}$$

Therefore possibility is measure theoretic, with one probability space defining the interval for the Left Reference Function and another probability space defining the interval for the Right Reference Function. In other words, two probability laws, one on the interval $[7.08683, 7.25]$ and the other on the interval $[7.25, 7.416865]$ would define one possibility law on the triad $[7.08683, 7.25, 7.416865]$. It is obvious that if we try to associate one probability law to get some kind of consistency with a possibility law on the same interval $[7.08683, 7.416865]$, we would fail in our attempt. Let us

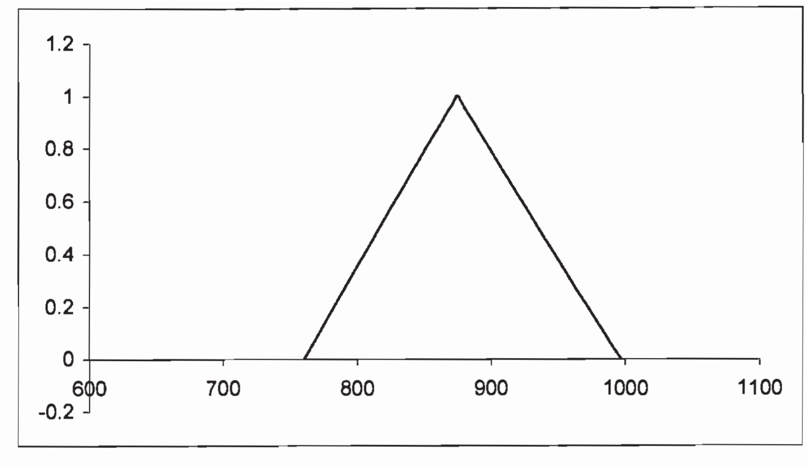

FIGURE 2. Membership Function: Example 2

consider another case (Example 2). Let the membership function in a particular case be

$$\mu(x) = \begin{cases} 0 & \text{if } x < 760.38, \\ \frac{-110.3+4\sqrt{x}}{8} & \text{if } 760.38 \leq x \leq 874.68, \\ \frac{126.3-4\sqrt{x}}{8} & \text{if } 874.68 \leq x \leq 996.98, \\ 0 & \text{if } x > 996.98. \end{cases}$$

The possibility distribution in this case is as shown in Figure 2. This diagram too, like the earlier one, looks nearly triangular, which is however not the case. In this case, the spreads from 760.38 to 874.68, and from 874.68 to 996.98, are small in comparison with the values. This is why, here too a nearly triangular shape has been seen. Here,

$$L(x) = \frac{-110.3+4\sqrt{x}}{8}$$

is the Left Reference Function and

$$R(x) = \frac{126.3-4\sqrt{x}}{8}$$

is the Right Reference Function as defined by Dubois and Prade. $L(x)$ is non-decreasing, increasing from 0 to 1, and $R(x)$ is non-increasing, decreasing from 1 to 0, for $760.38 \leq x \leq 874.68$ and $874.68 \leq x \leq 996.98$ respectively. According to our findings,

$$F(x) = L(x) = \frac{-110.3+4\sqrt{x}}{8}$$

is a distribution function for $760.38 \leq x \leq 874.68$, satisfying all the properties of a distribution function and

$$G(x) = 1 - R(x) = 1 - \frac{126.3-4\sqrt{x}}{8}$$

Dates	Minimum Temperature in Degrees Celsius	Maximum Temperature in Degrees Celsius
May 4	24.3	33.2
May 5	23.8	34.1
May 6	18.8	33.4
May 7	24.8	34.5
May 8	23.9	34.0

Table 1. Data on Maximum and Minimum Temperature

is another distribution function for $874.68 \leq x \leq 996.98$. The density for $F(x)$ is given by

$$f(x) = dF(x)/dx = \frac{1}{4\sqrt{x}},\ 760.38 \leq x \leq 874.68.$$

Similarly, the density for $G(x)$ is given by

$$g(x) = dG(x)/dx = \frac{1}{4\sqrt{x}},\ 874.68 \leq x \leq 996.98.$$

This means the value of

$$\int \frac{1}{4\sqrt{x}} dx,$$

the integral taken from 760.38 to x, $760.38 \leq x \leq 874.68$, is the possibility of occurrence of x, which is nothing but Dubois - Prade's $L(x)$. Similarly, the value of

$$1 - \int \frac{1}{4\sqrt{x}} dx,$$

the integral taken from 874.68 to x, $874.68 \leq x \leq 996.98$, is the possibility of occurrence of x, which is nothing but Dubois - Prade's $R(x)$. Therefore,

$$\text{Poss } [x] = \begin{cases} \text{Prob } [760.38 \leq y < x] & \text{if } 760.38 \leq x \leq 874.68, \\ 1 - \text{Prob } [874.68 \leq y < x] & \text{if } 874.68 \leq x \leq 996.98. \end{cases}$$

In other words, two probability laws, one on the interval $[760.38, 874.68]$ and the other on the interval $[874.68, 996.98]$ would define one possibility law on the triad $[760.38, 874.68, 996.98]$. We would like to mention that randomness as used in the two examples here is in the measure theoretic sense. What we mean is: we are not saying that some kind of statistical chance factor must be associated with the random variables used here. The reverse can also be true. We are going to cite an example for that. The following data in Table 1 are minimum and maximum temperatures for five consecutive days, May 4, 5, 6, 7 and 8, 2010, in the city of Guwahati†. We are going to use only five data points as an exercise only. Obviously for an actual construction, we would need a lot more data, and that can be taken care of in actual situations to construct fuzzy numbers wherever necessary. These values are due to chance occurrences. With many more days taken into consideration, frequency distributions for minimum temperature and maximum temperature could be fixed. There from, the cumulative frequency distributions can be constructed. Thereafter, the numerically fitted probability distribution for minimum temperature would form the left reference function, and the complementary probability distribution would

†Refer: The Sentinel, May 4,5,6,7 and 8, 2010, Guwahati, Assam, India

form the right reference function of a fuzzy number. We proceed to construct the fuzzy number as follows:

$$[24.3, 33.2]^{(1/5)}(S)[23.8, 34.1]^{(1/5)}(S)[18.8, 33.4]^{(1/5)}(S)[24.8, 34.5]^{(1/5)}(S)[23.9, 34.0]^{(1/5)}$$
$$= [18.8, 23.8]^{(1/5)} \cup [23.8, 23.9]^{(2/5)} \cup [23.9, 24.3]^{(3/5)} \cup [24.3, 24.8]^{(4/5)}$$
$$\cup [24.8, 33.2]^{(5/5)} \cup [33.2, 33.4]^{(4/5)} \cup [33.4, 34.0]^{(3/5)}$$
$$\cup [34.0, 34.1]^{(2/5)} \cup [34.1, 34.5]^{(1/5)}.$$

Accordingly, the fuzzy membership values of fuzzy temperature x in the city of Guwahati during those five days would be

$$\mu(x) = \begin{cases} 1/5 & \text{if} \quad 18.8 \le x \le 23.8, \\ 2/5 & \text{if} \quad 23.8 \le x \le 23.9, \\ 3/5 & \text{if} \quad 23.9 \le x \le 24.3, \\ 4/5 & \text{if} \quad 24.3 \le x \le 24.8, \\ 5/5 & \text{if} \quad 24.8 \le x \le 33.2, \\ 4/5 & \text{if} \quad 33.2 \le x \le 33.4, \\ 3/5 & \text{if} \quad 33.4 \le x \le 34.0, \\ 2/5 & \text{if} \quad 34.0 \le x \le 34.1, \\ 1/5 & \text{if} \quad 34.1 \le x \le 34.5. \end{cases}$$

Now fitting a probability distribution and a complementary probability distribution, possibly with more data, would be a statistical exercise ultimately to get a fuzzy membership function. However, in a discrete form this is how one can construct a fuzzy number. It is said that the triangular fuzzy number is easy to use, and therefore we use it in fuzzy arithmetical calculations. Actually, just as the uniform law is the simplest and the easiest to handle among all probability distributions, the triangular fuzzy number is the simplest and the easiest to handle in fuzzy arithmetical calculations. After all, two uniform probability laws can give rise to one triangular fuzzy number. Assuming that on both sides of the point of maximum possibility we have two uniform probability laws in action, we get the much used triangular fuzzy number.

5. Conclusions

All sorts of heuristic results have appeared in the literature on fuzziness in the last forty five years stating that fuzziness is not measure theoretic. Trying to establish a consistency between a possibility law defined in an interval with a probability law defined in the same interval bore no fruit till this day simply because it was not logical to do so. In fact, for a normal fuzzy number, the partial presence of an element is expressible either as a probability distribution function, or as a complementary probability distribution function. We have shown that we can arrive at the definition of normal fuzziness from two laws of randomness naturally. Normal fuzzy numbers must therefore be explained in this way only. Fuzziness therefore is expressible in terms of randomness in the measure theoretic sense. In the case of a probability law of errors following a probability density function the concerned random variable takes values around a location parameter and in the case of a normal possibility law in an interval around a point we define a membership function with maximum possibility allotted to the point concerned. Two different probability densities, one to the left of the aforesaid point and one to the right of that point, leads to defining a normal fuzzy number. Right at the start, people did suspect that possibility is actually probability in disguise. But they thought so with a view to establishing a relationship between the fuzzy membership function and a probability density function over the same interval. To arrive at the correct conclusion, one needs to look into the matters through the spectacles of set superimposition, and of course one needs to know that the Glivenko - Cantelli Lemma on Order Statistics exists in the statistical literature.

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