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Fuzzy semiprime ideals and fuzzy irreducible ideals of Γ-semirings

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Abstract. In this paper we introduce the notions of fuzzy semiprime ideals, fuzzy irreducible ideals in Γ-semirings. We characterize regular Γ-semirings in terms of fuzzy semiprime ideals. We deduce that μ is a fuzzy prime ideal of a semiring if and only if μ is a fuzzy semiprime and a fuzzy irreducible ideal. Its Γ-semiring analogue is obtained by using operator semirings.

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1. Introduction

It is well known that operator semirings of a Γ-semiring play an important role in generalizing the results of semirings to Γ-semirings [4, 5, 6]. Dutta and Biswas studied semirings [2], [3] in terms of fuzzy subsets. In an attempt to transfer this technique from semiring to the general setting of Γ-semirings we have initiated the study of Γ-semirings in terms of fuzzy subsets [7], [12], [13] and [14]. This paper is a sequel to this study. We introduce here the notions of fuzzy semiprime ideals and fuzzy irreducible ideals of a Γ-semiring. We also introduce the notion of fuzzy irreducible ideals in a semiring. We obtain various relationship between the fuzzy semiprime ideals (fuzzy irreducible ideals) of a Γ-semiring and the corresponding types of ideals of the operator semirings. We obtain an inclusion preserving bijection between the set of all fuzzy semiprime ideals(fuzzy irreducible ideals) of a Γ-semiring and that of its left operator semiring. We also obtain various results as mentioned in the abstract.

2. Preliminaries

In this section we recall some notions and results of Γ-semirings to use in the sequel.

Definition 2.1 ([10])**.** Let S and Γ be two additive commutative semigroups. Then S is called a Γ-semiring if there exists a mapping $S \times \Gamma \times S \to S$ (images to be denoted by $a\alpha b$ for $a, b \in S$ and $\alpha \in \Gamma$) satisfying the following conditions:

(i) $(a+b)\alpha c = a\alpha c + b\alpha c$,
(ii) $a\alpha(b+c) = a\alpha b + a\alpha c$,
(iii) $a(\alpha+\beta)b = a\alpha b + a\beta b$,
(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$

for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Further, if in a Γ-semiring, $(S, +)$ and $(\Gamma, +)$ are both monoids that satisfies:

(i) $0_S\alpha x = 0_S = x\alpha 0_S$
(ii) $x0_\Gamma y = 0_S = y0_\Gamma x$

for all $x, y \in S$ and for all $\alpha \in \Gamma$ then we say that S is a Γ-semiring with zero.

Throughout this paper we consider Γ-semirings with zero. For simplicity we write 0 instead of 0_S.

Definition 2.2 ([4])**.** Let S be a Γ-semiring and F be the free additive commutative semigroup generated by $S \times \Gamma$. Now we define a relation ρ on F as follows:

$$\sum_{i=1}^{m}(x_i, \alpha_i)\rho\sum_{j=1}^{n}(y_j, \beta_j) \text{ if and only if } \sum_{i=1}^{m}x_i\alpha_i a = \sum_{j=1}^{n}y_j\beta_j a$$

for all $a \in S (m, n \in Z^+)$. Then ρ is a congruence on F. We denote the congruence class containing $\sum\limits_{i=1}^{m}(x_i, \alpha_i)$ by $\sum\limits_{i=1}^{m}[x_i, \alpha_i]$. Then F/ρ is an additive commutative semigroup. Now F/ρ forms a semiring with the multiplication defined by

$$\left(\sum_{i=1}^{m}[x_i, \alpha_i]\right)\left(\sum_{j=1}^{n}[y_j, \beta_j]\right) = \sum_{i,j}[x_i\alpha_i y_j, \beta_j]$$

We denote this semiring by L and call it the left operator semiring of the Γ-semiring S. Dually we define the right operator semiring R of the Γ-semiring S.

Throughout this paper S denotes a Γ-semiring, R denotes the right operator semiring and L denotes the left operator semiring of the Γ-semiring S.

Definition 2.3 ([4])**.** Let S be a Γ-semiring and L be the left operator semiring and R be the right one. If there exists an element $\sum\limits_{i=1}^{m}[e_i, \delta_i] \in L$ $\left(\sum\limits_{j=1}^{n}[\gamma_j, f_j] \in R\right)$

such that $\sum_{i=1}^{m} e_i\delta_i a = a \left(\sum_{j=1}^{n} a\gamma_j f_j = a\right)$ for all $a \in S$ then S is said to have the left unity $\sum_{i=1}^{m}[e_i, \delta_i]$ (resp. the right unity $\sum_{j=1}^{n}[\gamma_j, f_j]$).

Definition 2.4 ([4])**.** An additive subsemigroup I of a Γ-semiring S is called a left(right) ideal of S if SΓI$\subseteq$I (IΓS$\subseteq$I), where $S\Gamma I$ denotes the subset of S consisting of all finite sums of the form $\sum_i a_i\alpha_i b_i$ with $a_i \in S, b_i \in I$ and $\alpha_i \in \Gamma$. If I is both a left ideal and right ideal then I is called a two-sided ideal or simply an ideal of S.

Definition 2.5 ([7])**.** Let μ be a non empty fuzzy subset of a Γ-semiring S (i.e. $\mu(x) \neq 0$ for some $x \in S$). Then μ is called a fuzzy left ideal [resp. fuzzy right ideal] of S if

(i) $\mu(x+y) \geq min[\mu(x), \mu(y)]$,
(ii) $\mu(x\gamma y) \geq \mu(y)$ [resp. $\mu(x\gamma y) \geq \mu(x)$]

for all $x, y \in S$ and $\gamma \in \Gamma$.

A fuzzy ideal of a Γ-semiring S is a non empty fuzzy subset of S which is a fuzzy left ideal as well as a fuzzy right ideal of S.

Proposition 2.6 ([7])**.** *Let I be a left ideal (right ideal, ideal) of a Γ-semiring S and $\alpha \leq \beta \neq 0$ be any two elements in $[0, 1]$. Then the fuzzy subset μ of S, defined by*

$$\mu(x) = \begin{cases} \beta & if \;\; x \in I \\ \alpha & otherwise \end{cases}$$

is a fuzzy left ideal (resp. fuzzy right ideal, fuzzy ideal) of S.

Proposition 2.7 ([7])**.** *Let μ_1 and μ_2 be two fuzzy left ideals (fuzzy right ideals, fuzzy ideals) of a Γ-semiring S. Then $\mu_1 \cap \mu_2$ is a fuzzy left ideal (resp. fuzzy right ideal, fuzzy ideal) of S.*

In what follows $FLI(S)$, $FRI(S)$, $FI(S)$ respectively denote the set of all fuzzy left ideals, fuzzy right ideals, fuzzy ideals of S. Similar are the meanings of $FLI(L)$, $FRI(L)$, $FI(L)$ where L is the left operator semiring of S.

Definition 2.8 ([12])**.** Let S be a Γ-semiring and $\mu_1, \mu_2 \in FLI(S)$ $[FRI(S),\ FI(S)]$. Then the product $\mu_1\Gamma\mu_2$ and composition $\mu_1 \circ \mu_2$ of μ_1 and μ_2 are defined as follows:

$$(\mu_1\Gamma\mu_2)(x) = \begin{cases} \sup\limits_{x=u\gamma v} \{\min[\mu_1(u), \mu_2(v)] : u, v \in S; \gamma \in \Gamma\}, \\ 0 \;\; \text{if for any } u, v \in S \text{ and for any } \gamma \in \Gamma,\; u\gamma v \neq x. \end{cases}$$

$$(\mu_1 \circ \mu_2)(x) = \begin{cases} \sup\left\{\min\limits_{1\leq i\leq n}[\min[\mu_1(u_i), \mu_2(v_i)]] : u_i, v_i \in S, \gamma_i \in \Gamma\right\} \text{ if } x = \sum\limits_{i=1}^{n} u_i\gamma_i v_i \\ 0 \;\; \text{otherwise.} \end{cases}$$

Proposition 2.9 ([12])**.** *Let $\mu_1, \mu_2, \mu_3 \in FLI(S)[FRI(S), FI(S)]$. Then $\mu_1\Gamma\mu_2 \subseteq \mu_3$ if and only if $\mu_1 \circ \mu_2 \subseteq \mu_3$.*

Definition 2.10 ([12])**.** Let μ be a fuzzy subset of L, we define a fuzzy subset μ^+ of S by $\mu^+(x) = \inf_{\gamma\in\Gamma} \mu([x,\gamma])$ where $x \in S$. If σ is a fuzzy subset of S, we define a fuzzy subset $\sigma^{+'}$ of L by $\sigma^{+'}(\sum_i [x_i,\alpha_i]) = \inf_{s\in S} \sigma(\sum_i x_i\alpha_i s)$ where $\sum_i [x_i,\alpha_i] \in L$.

Definition 2.11 ([12])**.** If δ be a fuzzy subset of R, we define a fuzzy subset δ^* of S by $\delta^*(x) = \inf_{\gamma\in\Gamma} \delta([\gamma,x])$ where $x \in S$. If η is a fuzzy subset of S, we define a fuzzy subset $\eta*'$ of R by $\eta^{*'}(\sum_i [\alpha_i,x_i]) = \inf_{s\in S} \eta(\sum_i s\alpha_i x_i)$ where $\sum_i [\alpha_i,x_i] \in R$.

Proposition 2.12 ([12])**.** *Let S be a Γ-semiring with left and right unities and L be its operator semiring. Suppose $\sigma, \sigma_1, \sigma_2 \in FI(S)$ and $\mu \in FI(L)$. Then*

(1) $\sigma^{+'} \in FI(L)$. *Moreover, if σ is non constant then $\sigma^{+'}$ is non constant.*
(2) $(\sigma^{+'})^+ = \sigma$,
(3) $\sigma_1 \neq \sigma_2$ *implies that* $\sigma_1^{+'} \neq \sigma_2^{+'}$,
(4) $(\sigma_1 \oplus \sigma_2)^{+'} = \sigma_1^{+'} \oplus \sigma_2^{+'}$,
(5) $(\sigma_1 \cap \sigma_2)^{+'} = \sigma_1^{+'} \cap \sigma_2^{+'}$,
(6) $\sigma_1 \subseteq \sigma_2$ *implies that* $\sigma_1^{+'} \subseteq \sigma_2^{+'}$,
(7) $\mu^+ \in FI(S)$. *Moreover, if μ is non constant then μ^+ is non constant.*
(8) $(\mu^+)^{+'} = \mu$,
(9) $\mu_1 \subseteq \mu_2$ *implies that* $\mu_1^+ \subseteq \mu_2^+$.

Definition 2.13 ([7])**.** A function $f : R \to S$, where R and S are Γ-semirings, is said to be a Γ-morphism of Γ-semirings if

(i) $f(a+b) = f(a) + f(b)$,
(ii) $f(a\gamma b) = f(a)\gamma f(b)$

for all $a, b \in R$ and $\gamma \in \Gamma$.

Lemma 2.14 ([7])**.** *Let $f : S \to T$ be an onto Γ-morphism of Γ-semirings and μ be fuzzy ideal of S. Then $f(\mu)$ is a fuzzy ideal of T where $f(\mu))(x) := \sup_{f(y)=x} \mu(y)$ for all $x \in S$.*

Lemma 2.15 ([7])**.** *Let $f : S \to T$ be a Γ-morphism of Γ-semirings. If σ is a fuzzy ideal of T, then $f^{-1}(\sigma)$ is a fuzzy ideal of S.*

Definition 2.16 ([5])**.** Let S be a Γ-semiring. A proper ideal I of S is said to be semiprime if for any ideal H of S, $H\Gamma H \subseteq I$ implies that $H \subseteq I$.

Definition 2.17 ([4])**.** Let S be a Γ-semiring and L be the left operator semiring and R be the right one. For $P \subseteq L$ $(\subseteq R)$, $P^+ := \{a \in S : [a,\Gamma] \subseteq P\}$ (respectively $P^* := \{a \in S : [\Gamma,a] \subseteq P\}$). For $Q \subseteq S$,

$$Q^{+'} := \left\{ \sum_{i=1}^{m} [x_i,\alpha_i] \in L : \left(\sum_{i=1}^{m} [x_i,\alpha_i] \right) S \subseteq Q \right\}$$

where $\left(\sum_{i=1}^{m}[x_i,\alpha_i]\right)S$ denotes the set of all finite sums $\sum_{i,k} x_i\alpha_i s_k,\quad s_k \in S$ and

$$Q^{*'} := \left\{\sum_{i=1}^{m}[\alpha_i,x_i] \in R : \left(\sum_{i=1}^{m}S[\alpha_i,x_i]\right) \subseteq Q\right\}$$

where $S\left(\sum_{i=1}^{m}[x_i,\alpha_i]\right)$ denotes the set of all finite sums $\sum_{i,k} s_k\alpha_i x_i,\quad s_k \in S$.

Proposition 2.18 ([1])**.** *A fuzzy ideal μ of a semiring S is fuzzy semiprime if and only if each level ideal μ_t is semiprime for $t \in Im\ \mu$.*

Proposition 2.19 ([1])**.** *Let μ be a fuzzy semiprime ideal of a semiring S. Then $\mu_0 = \{x \in S : \mu(x) = \mu(0)\}$ is a semiprime ideal of S.*

Definition 2.20 ([9])**.** A proper ideal I of a semiring S is said to be strongly irreducible if for any two ideals H and K of S, $H \cap K \subseteq I$ implies that $H \subseteq I$ or $K \subseteq I$.

Proposition 2.21 ([9])**.** *An ideal I of a semiring S is prime if and only if it is semiprime and strongly irreducible.*

Definition 2.22 ([13])**.** A fuzzy ideal μ of a Γ-semiring S is said to be fuzzy prime if μ is not a constant function and for any two fuzzy ideals σ and θ of S, $\sigma\Gamma\theta \subseteq \mu$ implies that either $\sigma \subseteq \mu$ or $\theta \subseteq \mu$.

The set of fuzzy prime ideals of a Γ-semiring S and the set of fuzzy prime ideals of its left operator semiring L are denoted by $FPI(S), FPI(L)$ respectively.

Proposition 2.23 ([14])**.** *If $\sigma \in FPI(S)$ and $\mu \in FPI(L)$ then*
(1) $\mu^+ \in FPI(S)$,
(2) $\sigma^{+'} \in FPI(L)$,
(3) $(\mu^+)^{+'} = \mu$,
(4) $(\sigma^{+'})^+ = \sigma$.

Definition 2.24 ([7])**.** A fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of a Γ-semiring S is said to be a fuzzy left semi k-ideal (resp. fuzzy right semi k-ideal, fuzzy semi k-ideal) of S if for any element a of S there exists a unique element b of S such that $a + b = 0$ then $\mu(a) = \mu(b)$.

3. Fuzzy semiprime ideals of Γ-semirings.

In this section we define fuzzy semiprime ideals in a Γ-semiring and obtain its various characterizations. We obtain an inclusion preserving bijection between the fuzzy semiprime ideals of a Γ-semiring and that of its operator semirings. We also obtain a characterization of a regular Γ-semiring in terms of fuzzy semiprime ideals.

Definition 3.1. A fuzzy ideal μ of a Γ-semiring S is said to be a fuzzy semiprime ideal if μ is non constant and for any fuzzy ideal θ of S, $\theta\Gamma\theta \subseteq \mu$ implies $\theta \subseteq \mu$.

The following result follows easily.

Proposition 3.2. *Every fuzzy prime ideal of S is a fuzzy semiprime ideal of S.*

Proposition 3.3. *A fuzzy ideal μ of S is fuzzy semiprime if and only if μ is non constant and for any fuzzy ideal θ of S, $\theta \circ \theta \subseteq \mu$ implies $\theta \subseteq \mu$.*

Proof. Let $\theta \in FI(S)$. By Proposition 2.9 we have $\theta \circ \theta \subseteq \mu$ if and only if $\theta\Gamma\theta \subseteq \mu$. Hence the proposition holds. □

Proposition 3.4. *Let μ be a fuzzy semiprime ideal of S. Then for any fuzzy ideal θ of S, $\theta^m \subseteq \mu$, $m \in Z^+$ implies that $\theta \subseteq \mu$ where θ^m denotes that $\theta \circ \theta \circ \cdots \theta$ in which θ appears m times.*

Proof. The result is true for $m = 1$ and $m = 2$. Now suppose the result is true for $m = n$ i.e., for any fuzzy ideal θ of S, $\theta^n \subseteq \mu \Rightarrow \theta \subseteq \mu$. Let $\theta^{n+1} \subseteq \mu$. Then $(\theta^n)^2 = \theta^n \circ \theta^n \subseteq \theta^n \circ \theta$(using definition of $\circ$)$= \theta^{n+1} \subseteq \mu$ which together with the above proposition (applied for θ^n) implies that $\theta^n \subseteq \mu$. Hence by induction hypothesis $\theta \subseteq \mu$. Consequently, the result follows by the principle mathematical induction. □

Proposition 3.5. *Let I be an ideal of S. Then I is a semiprime ideal of S if and only if the characteristic function λ_I of I is a fuzzy semiprime ideal of S.*

Proof. Let I be a semiprime ideal of S. Since $I \neq S$, λ_I is a non constant fuzzy ideal of S. Let θ be a fuzzy ideal of S such that $\theta\Gamma\theta \subseteq \lambda_I$ and $\theta \not\subseteq \lambda_I$. Then there exists an element $x \in S$ such that $\theta(x) > \lambda_I(x)$. Since $\theta(x) \ngtr 1$, $\lambda_I(x) = 0$. Hence $\theta(x) \neq 0$ and $x \notin I$. Since I is a semiprime ideal of S, $x\gamma_1 s\gamma_2 x \notin I$ for some $s \in S$, $\gamma_1, \gamma_2 \in \Gamma$. Let $a = x\gamma_1 s\gamma_2 x$. Now $\lambda_I(a) = 0$. Thus $(\theta\Gamma\theta)(a) = 0$. But

$$\begin{aligned}(\theta\Gamma\theta)(a) &= \sup_{a=y\gamma z}[\min[\theta(y), \theta(z)]] \geq \min[\theta(x), \theta(s\gamma_2 x)] \\ &\geq \min[\theta(x), \theta(x)] = \theta(x) \neq 0\end{aligned}$$

a contradiction. Thus λ_I is a fuzzy semiprime ideal of S.

Conversely, let λ_I be a fuzzy semiprime ideal of S. Clearly $I \neq S$. Let H be an ideal of S such that $H\Gamma H \subseteq I$. Let $x \in S$. If $(\lambda_H\Gamma\lambda_H)(x) = 0$ then $(\lambda_H\Gamma\lambda_H)(x) \leq \lambda_I(x)$. If $(\lambda_H\Gamma\lambda_H)(x) \neq 0$ then $(\lambda_H\Gamma\lambda_H)(x) = \sup_{x=y\gamma z}[\min[\lambda_H(y), \lambda_H(z)]] \neq 0$. Thus there exist $y, z \in S$ such that $x = y\gamma z$ and $\lambda_H(y) \neq 0$ and $\lambda_H(z) \neq 0$. So $y \in H$ and $z \in H$. Hence $x = y\gamma z \in H\Gamma H \subseteq I$. Thus $\lambda_I(x) = 1$. Therefore $(\lambda_H\Gamma\lambda_H)(x) \leq \lambda_I(x)$ for all $x \in S$. Therefore $\lambda_H\Gamma\lambda_H \subseteq \lambda_I$. Consequently, $\lambda_H \subseteq \lambda_I$, as λ_I is a fuzzy semiprime ideal of S. So $H \subseteq I$. Hence I is a semiprime ideal of S. □

Proposition 3.6. *Let μ be a non constant fuzzy ideal of S. Then the following conditions are equivalent:*

(1) *μ is a fuzzy semiprime ideal of S.*
(2) *For any $x \in S$, $\inf\{\mu(x\gamma_1 s\gamma_2 x) : s \in S, \gamma_1, \gamma_2 \in \Gamma\} = \mu(x)$.*

Proof. (1) ⇒ (2) Let μ be a fuzzy semiprime ideal of S. Then μ is a fuzzy ideal of S. Hence $\mu(x\gamma_1 s\gamma_2 x \geq \mu(x)$ for all $x \in S$ and $\gamma_1, \gamma_2 \in \Gamma$. Thus

$$\inf\{\mu(x\gamma_1 s\gamma_2 x) : s \in S, \gamma_1, \gamma_2 \in \Gamma\} \geq \mu(x).$$

If possible, let $\inf\{\mu(x\gamma_1 s\gamma_2 x) : s \in S, \gamma_1, \gamma_2 \in \Gamma\} > \mu(x)$. Let $t \in [0, 1)$ be so chosen that $\mu(x) < t < \inf\{\mu(x\gamma_1 s\gamma_2 x) : s \in S, \gamma_1, \gamma_2 \in \Gamma\}$. Let θ be the fuzzy ideal of S defined by $\theta = t\lambda_{<x>}$ where $< x >$ is the principal ideal generated by x. Also let y be an element of S which cannot be expressed in the form $y = u\gamma v$ where $u, v \in < x >$. Then $(\theta\Gamma\theta)(y) = 0$ and hence $(\theta\Gamma\theta)(y) \leq \mu(y)$. Otherwise $(\theta\Gamma\theta)(y) = \sup\limits_{y=u\gamma v} [\min[\theta(u), \theta(v)] : u, v \in < x >] = t$. Now $u \in < x >$ is of the form

$$u = \sum_{i=1}^{p} s_i^{'}\gamma_i^{'}x\gamma_i^{''}s_i^{''}$$

where $s_i^{'}, s_i^{''} \in S$, $\gamma_i^{'}, \gamma_i^{''} \in \Gamma$ and $p \in Z^+$. Similarly

$$v = \sum_{j=1}^{q} s_j^{'''}\gamma_j^{'''}x\gamma_j^{''''}s_j^{''''}$$

where $s_j^{'''}, s_j^{''''} \in S$, $\gamma_j^{'''}, \gamma_j^{''''} \in \Gamma$ and $q \in Z^+$. Now

$$u\gamma v = \left(\sum_{i=1}^{p} s_i^{'}\gamma_i^{'}x\gamma_i^{''}s_i^{''}\right)\gamma\left(\sum_{j=1}^{q} s_j^{'''}\gamma_j^{'''}x\gamma_j^{''''}s_j^{''''}\right).$$

Since μ is a fuzzy ideal of S, it follows that, for some $s^1 \in S$ and $\xi_1, \xi_2 \in \Gamma$,

$$\begin{aligned}\mu(y) &= \mu(u\gamma v) \geq \mu(x\xi_1 s^1\xi_2 x)\\ &\geq \inf\{\mu(x\gamma_1 s\gamma_2 x) : s \in S,\ \gamma_1, \gamma_2 \in \Gamma\}\\ &> t = (\theta\Gamma\theta)(y).\end{aligned}$$

Therefore $(\theta\Gamma\theta)(y) \leq \mu(y)$ for all $y \in S$ and so $\theta\Gamma\theta \subseteq \mu$ which implies that $\theta \subseteq \mu$, as μ is a fuzzy semiprime ideal of S. Hence $t = \theta(x) \leq \mu(x)$, a contradiction. Thus $\inf\{\mu(x\gamma_1 s\gamma_2 x) : s \in S, \gamma_1, \gamma_2 \in \Gamma\} = \mu(x)$.

(2) $\Rightarrow$ (1) Let μ be a fuzzy ideal of S satisfying

$$\inf\{\mu(x\gamma_1 s\gamma_2 x) : s \in S, \gamma_1, \gamma_2 \in \Gamma\} = \mu(x)$$

for any $x \in S$. Let θ be a fuzzy ideal of S such that $\theta\Gamma\theta \subseteq \mu$ and $\theta \nsubseteq \mu$. Then there exists $y \in S$ such that $\theta(y) > \mu(y)$. Now

$$\mu(y\gamma_1 s\gamma_2 y) \geq (\theta\Gamma\theta)(y\gamma_1 s\gamma_2 y) = \sup_{(y\gamma_1 s\gamma_2 y)=u\gamma v} [\min[\theta(u), \theta(v)]] \geq \theta(y).$$

Thus $\mu(y) = \inf\{\mu(y\gamma_1 s\gamma_2 y) : s \in S,\ \gamma_1, \gamma_2 \in \Gamma\} \geq \theta(y) > \mu(y)$, a contradiction. Hence μ is a fuzzy semiprime ideal of S. □

Proposition 3.7. *Every fuzzy semiprime ideal of an additively cancellative* Γ-*semiring is a fuzzy semi* k-*ideal.*

Proof. Let μ be a fuzzy semiprime ideal of an additively cancellative Γ-semiring S. Assume that for some $a \in S$ there exists unique $b \in S$ such that $a + b = 0$. Let $s \in S$ and $\gamma_1, \gamma_2 \in \Gamma$. Then $b\gamma_1 s\gamma_2 b + a\gamma_1 s\gamma_2 b = 0$. Also $a\gamma_1 s\gamma_2 a + a\gamma_1 s\gamma_2 b = 0$. Since S is additively cancellative, we have, $a\gamma_1 s\gamma_2 a = b\gamma_1 s\gamma_2 b$ for all $s \in S$ and for all $\gamma_1, \gamma_2 \in \Gamma$. Then by using Proposition 3.6 we obtain

$$\begin{aligned}\mu(a) &= \inf\{\mu(a\gamma_1 s\gamma_2 a) : s \in S, \gamma_1, \gamma_2 \in \Gamma\}\\ &= \inf\{\mu(b\gamma_1 s\gamma_2 b) : s \in S, \gamma_1, \gamma_2 \in \Gamma\} = \mu(b).\end{aligned}$$

Hence μ is a fuzzy semi k-ideal of S. $\square$

Proposition 3.8. *Let I be a semiprime ideal of S. Then the fuzzy subset μ of S defined by*

$$\mu(x) = \begin{cases} 1 & if \quad x \in I \\ a & if \quad x \in S \setminus I, \ a \in [0,1) \end{cases}$$

is a fuzzy semiprime ideal of S.

Proof. By Proposition 2.6, μ is a fuzzy ideal of S. Let θ be a fuzzy ideal of S such that $\theta \not\subseteq \mu$. Then there exists $x \in S$ such that $\theta(x) > \mu(x)$. So by definition of μ, $\mu(x) = a$. i.e., $x \notin I$. So by Theorem 3.6 of [5], there exist $s \in S$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $x\gamma_1 s\gamma_2 x \notin I$. Thus $\mu(x\gamma_1 s\gamma_2 x) = a = \mu(x) < \theta(x) \leq (\theta\Gamma\theta)(x\gamma_1 s\gamma_2 x)$. Thus $\theta\Gamma\theta \not\subseteq \mu$. Hence μ is a fuzzy semiprime ideal of S. $\square$

Proposition 3.9. *A fuzzy ideal μ of S is fuzzy semiprime if and only if each level ideal μ_t is a semiprime ideal of S.*

Proof. Let μ be a fuzzy semiprime ideal of S. Let $x \in S$, $t \in Im\ \mu$ be such that $x\Gamma S\Gamma x \subseteq \mu_t$. Then $\mu(x\gamma_1 s\gamma_2 x) \geq t$ for all $s \in S$ and for all $\gamma_1, \gamma_2 \in \Gamma$. Hence $\inf\{\mu(x\gamma_1 s\gamma_2 x) : s \in S, \ \gamma_1, \gamma_2 \in \Gamma\} \geq t$. i.e., $\mu(x) \geq t$, by Proposition 3.6. So $x \in \mu_t$. Hence μ_t is a semiprime ideal of S.

Conversely, let each level ideal μ_t be a semiprime ideal of S. Also let

$$\inf\{\mu(x\gamma_1 s\gamma_2 x) : s \in S, \ \gamma_1, \gamma_2 \in \Gamma\} = t \ \text{ for } \ x \in S.$$

Now

$$\mu(x\gamma_1 s\gamma_2 x) \geq \inf\{\mu(x\gamma_1 s\gamma_2 x) : s \in S, \ \gamma_1, \gamma_2 \in \Gamma\} = t$$

which implies that $x\gamma_1 s\gamma_2 x \in \mu_t$ for all $s \in S$ and for all $\gamma_1, \gamma_2 \in \Gamma$. i.e., $x\Gamma S\Gamma x \subseteq \mu_t$. Since μ_t is semiprime, it follows that $x \in \mu_t$. Thus

$$\mu(x) \geq t = \inf\{\mu(x\gamma_1 s\gamma_2 x) : s \in S, \ \gamma_1, \gamma_2 \in \Gamma\}. \tag{3.1}$$

Again $\mu(x\gamma_1 s\gamma_2 x) \geq \mu(x)$ for all $s \in S$ and for all $\gamma_1, \gamma_2 \in \Gamma$. So

$$\inf\{\mu(x\gamma_1 s\gamma_2 x) : s \in S, \ \gamma_1, \gamma_2 \in \Gamma\} \geq \mu(x). \tag{3.2}$$

Combining (3.1) and (3.2) we obtain $\inf\{\mu(x\gamma_1 s\gamma_2 x) : s \in S, \ \gamma_1, \gamma_2 \in \Gamma\} = \mu(x)$. Consequently, by Proposition 3.6, μ is a fuzzy semiprime ideal of S. $\square$

Corollary 3.10. *Let μ be a fuzzy semiprime ideal of S. Then*

$$\mu_0 = \{x \in S : \mu(x) = \mu(0)\}$$

is a semiprime ideal of S.

Lemma 3.11. *If $\mu \in FI(L)$ then $(\mu_t)^+ = (\mu^+)_t$ where $t \in Im\ \mu$.*

Proof. We have

$$\begin{aligned} s \in (\mu_t)^+ &\Longleftrightarrow [s,\gamma] \in \mu_t, \ \text{ for all } \ \gamma \in \Gamma \\ &\Longleftrightarrow \mu([s,\gamma]) \geq t, \ \text{ for all } \ \gamma \in \Gamma \\ &\Longleftrightarrow \inf_{\gamma \in \Gamma} \mu([s,\gamma]) \geq t \Leftrightarrow \mu^+(s) \geq t \\ &\Longleftrightarrow s \in (\mu^+)_t. \end{aligned}$$

Hence $(\mu_t)^+ = (\mu^+)_t$. □

Theorem 3.12. *Let μ be a fuzzy semiprime ideal of L. Then μ^+ is a fuzzy semiprime ideal of S.*

Proof. By Proposition 3.9, μ_t is semiprime ideal of L for all $t \in Im\ \mu$. Then by Lemma 3.3 of [5], $(\mu_t)^+$ is semiprime ideal of S for all $t \in Im\ \mu$ and so by Lemma 3.11, $(\mu^+)_t$ is semiprime ideal of S for all $t \in Im\ \mu$. Consequently, by Proposition 3.9, μ^+ is a fuzzy semiprime ideal of S. □

Lemma 3.13. *If $\sigma \in FI(S)$ then $(\sigma_t)^{+'} = (\sigma^{+'})_t$ where $t \in Im\ \sigma$.*

Proof. We have

$$\begin{aligned}
\sum_i [x_i, \alpha_i] \in (\sigma_t)^{+'} &\iff \sum_i x_i \alpha_i s \in \sigma_t \text{ for all } s \in S \\
&\iff \sigma(\sum_i x_i \alpha_i s) \geq t \text{ for all } s \in S \\
&\iff \inf_{s \in S} \sigma(\sum_i x_i \alpha_i s) \geq t \\
&\iff \sigma^{+'}(\sum_i [x_i, \alpha_i]) \geq t \\
&\iff \sum_i [x_i, \alpha_i] \in (\sigma^{+'})_t.
\end{aligned}$$

Hence $(\sigma_t)^{+'} = (\sigma^{+'})_t$. □

Theorem 3.14. *Let σ be a fuzzy semiprime ideal of S. Then $\sigma^{+'}$ is a fuzzy semiprime ideal of L.*

Proof. By Proposition 3.9, σ_t is a semiprime ideal of S for all $t \in Im\ \sigma$. Again from Lemma 3.4 of [5], $(\sigma_t)^{+'}$ is a semiprime ideal of L for all $t \in Im\ \sigma$. Therefore, by Lemma 3.13, $(\sigma^{+'})_t$ is a semiprime ideal of L for all $t \in Im\ \sigma$. By Proposition 2.18, $\sigma^{+'}$ is a fuzzy semiprime ideal of L. □

Theorem 3.15. *Let S be a Γ-semiring with left and right unities and L be its left operator semiring. The mapping $\mu \mapsto \mu^{+'}$ defines a one-one correspondence between the set of all fuzzy semiprime ideals of S and the set of all fuzzy semiprime ideals of L where μ is a fuzzy semiprime ideal of L.*

Proof. Let μ be a fuzzy semiprime ideal of S. Then by Theorem 3.14, $\mu^{+'}$ is fuzzy semiprime ideal of L and hence by Theorem 3.12, $(\mu^{+'})^+$ is a fuzzy semiprime ideal of S. Now by Proposition 2.12(2), $\mu = (\mu^{+'})^+$. Again, let σ be a fuzzy semiprime ideal of L. Then σ^+ is a fuzzy semiprime ideal of S and $(\sigma^+)^{+'}$ is a fuzzy semiprime ideal of L by Theorem 3.12 and Theorem 3.14 respectively. Also by Proposition 2.12(8), $\sigma = (\sigma^+)^{+'}$. Hence the theorem holds. □

Definition 3.16. Let S and T be two Γ-semirings and f be a mapping from S onto T. Let $\mu \in FI(S)$. μ is said to be f-invariant if $f(x) = f(y)$ implies that $\mu(x) = \mu(y)$ for all $x, y \in S$.

Proposition 3.17. *Let $f : S \to T$ be an onto Γ-morphism of Γ-semirings and let μ and θ be fuzzy semiprime ideals of S and T respectively and μ be f-invariant. Then*

(1) *$f(\mu)$ is a fuzzy semiprime ideal of T.*
(2) *$f^{-1}(\theta)$ is a fuzzy semiprime ideal of S.*
(3) *The mapping $\mu \mapsto f(\mu)$ establishes a one-one correspondence between the set of all f-invariant fuzzy semiprime ideals of S and the set of all fuzzy semiprime ideals of T.*

Proof. By Lemma 2.14 and Lemma 2.15, we see that $f(\mu)$ and $f^{-1}(\theta)$ are fuzzy ideals of T and S respectively.

(1) Let $t \in T$. Then there exists $s \in S$ such that $f(s) = t$. Since since μ is both f-invariant and fuzzy semiprime, we have

$$\begin{aligned}
&\inf\{f(\mu)(t\gamma_1 t_1 \gamma_2 t) : t_1 \in T, \gamma_1, \gamma_2 \in \Gamma\}\\
&= \inf\left\{\sup_{f(z)=t\gamma_1 t_1\gamma_2 t} \mu(z) : t_1 \in T, \gamma_1, \gamma_2 \in \Gamma\right\}\\
&= \sup_{f(z)=t\gamma_1 t_1 \gamma_2 t = f(s\gamma_1 s_1\gamma_2 s)} \{\inf \mu(s\gamma_1 s_1 \gamma_2 s) : s_1 \in S, \gamma_1, \gamma_2 \in \Gamma\}\\
&= \sup_{f(s)=t} \mu(s) = f(\mu)(t).
\end{aligned}$$

Hence $f(\mu)$ is a fuzzy semiprime ideal of T.

(2) Let $s \in S$. Then

$$\begin{aligned}
&\inf\{f^{-1}(\theta)(s\gamma_1 s_1\gamma_2 s) : s_1 \in S, \gamma_1, \gamma_2 \in \Gamma\}\\
&= \inf\{\theta(f(s\gamma_1 s_1 \gamma_2 s)) : s_1 \in S, \gamma_1, \gamma_2 \in \Gamma\}\\
&= \inf\{\theta(f(s)\gamma_1 f(s_1)\gamma_2 f(s)) : s_1 \in S, \gamma_1, \gamma_2 \in \Gamma\}\\
&= \theta(f(s)), \quad \text{as } \theta \text{ is a fuzzy semiprime ideal}\\
&= f^{-1}(\theta)(s).
\end{aligned}$$

Hence $f^{-1}(\theta)$ is a fuzzy semiprime ideal of S.

(3) We shall first prove that $f^{-1}(f(\mu)) = \mu$ and $f(f^{-1}(\theta)) = \theta$. Let $s \in S$. Then $(f^{-1}(f(\mu)))(s) = (f(\mu))(f(s)) = \sup_{f(z)=f(s)} \mu(z) = \mu(s)$, as μ is f-invariant. Thus $f^{-1}(f(\mu)) = \mu$ and so the mapping $\mu \mapsto f(\mu)$ is well defined. Again, let $t \in T$. Then $(f(f^{-1}(\theta)))(t) = \sup_{f(z)=t} (f^{-1}(\theta))(z) = \sup_{f(z)=t} \theta(f(z)) = \theta(t)$ for all $t \in T$. Hence $f(f^{-1}(\theta)) = \theta$ and consequently the mapping $\mu \mapsto f(\mu)$ is onto. Next, let μ_1 and μ_2 be two f-invariant fuzzy semiprime ideals of S such that $\mu_1 \neq \mu_2$. If possible, let $f(\mu_1) = f(\mu_2)$. Then $f^{-1}(f(\mu_1)) = f^{-1}(f(\mu_2))$. i.e., $\mu_1 = \mu_2$, a contradiction. So the mapping $\mu \mapsto f(\mu)$ is one-one. Hence the theorem holds. □

Now in order to obtain a characterization of regular Γ-semirings in terms of fuzzy subsets we recall the following two results.

Theorem 3.18 ([8]). *A Γ-semiring S is multiplicatively regular ([10]) if and only if $\mu_1\Gamma\mu_2 = \mu_1 \cap \mu_2$ for every fuzzy right ideal μ_1 and every fuzzy left ideal μ_2 of S.*

Proposition 3.19 ([8]). *Let $\mu_1, \mu_2 \in FI(S)$. Then*

$$\mu_1 \Gamma \mu_2 \subseteq \mu_1 \circ \mu_2 \subseteq \mu_1 \cap \mu_2 \subseteq \mu_1, \mu_2.$$

In view of Theorem 3.18 and Proposition 3.19 we obtain the following.

Theorem 3.20. *A Γ-semiring S is multiplicatively regular if and only if $\mu_1 \circ \mu_2 = \mu_1 \cap \mu_2$ for every fuzzy right ideal μ_1 and every fuzzy left ideal μ_2 of S.*

Theorem 3.21. *A commutative Γ-semiring S is multiplicatively regular if and only if for every fuzzy ideal μ of S, $\mu \circ \mu = \mu$.*

Proof. Let S be a regular commutative Γ-semiring and μ be a fuzzy ideal of S. Then by Theorem 3.20, $\mu \circ \mu = \mu$.

Conversely, assume that $\sigma \circ \sigma = \sigma$ for every fuzzy ideal σ of a commutative Γ-semiring S. Let μ_1 and μ_2 be two fuzzy ideals of S. Then $\mu_1 \cap \mu_2$ is a fuzzy ideal of S, by Proposition 2.7. Now

$$\begin{aligned}
&((\mu_1 \cap \mu_2) \circ (\mu_1 \cap \mu_2))(x) \\
&= \sup_{x=\sum_{i=1}^{n} u_i \gamma_i v_i} \left[\min_i [\min[(\mu_1 \cap \mu_2)(u_i), (\mu_1 \cap \mu_2)(v_i)]]\right] \\
&= \sup_{x=\sum_{i=1}^{n} u_i \gamma_i v_i} \left[\min_i [\min[\min[\mu_1(u_i), \mu_2(u_i)], \min[\mu_1(v_i), \mu_2(v_i)]]]\right] \\
&\leq \sup_{x=\sum_{i=1}^{n} u_i \gamma_i v_i} \left[\min_i [\min[\mu_1(u_i), \mu_2(v_i)]]\right] \\
&= (\mu_1 \circ \mu_2)(x).
\end{aligned}$$

So $((\mu_1 \cap \mu_2) \circ (\mu_1 \cap \mu_2)) \subseteq \mu_1 \circ \mu_2$. Hence $\mu_1 \cap \mu_2 \subseteq \mu_1 \circ \mu_2$. Again $\mu_1 \circ \mu_2 \subseteq \mu_1 \cap \mu_2$, by Proposition 3.19. Thus $\mu_1 \circ \mu_2 = \mu_1 \cap \mu_2$. Hence by Theorem 3.20, we conclude that S is regular. □

Proposition 3.22. *Let S be a commutative Γ-semiring. Then μ is a fuzzy semiprime ideal of S if and only if $\mu(x\gamma x) = \mu(x)$ for all $x \in S$ and for all $\gamma \in \Gamma$.*

Proof. Let $\mu(x\gamma x) = \mu(x)$ for all $x \in S$ and for all $\gamma \in \Gamma$. Let θ be a fuzzy ideal of S such that $\theta \Gamma \theta \subseteq \mu$ and let $\theta \nsubseteq \mu$. Then there exists $y \in S$ such that $\theta(y) > \mu(y)$. Now $(\theta\Gamma\theta)(y\gamma y) \geq \theta(y) > \mu(y)$. Again $\mu(y) = \mu(y\gamma y) \geq (\theta\Gamma\theta)(y\gamma y)$, which is a contradiction and hence $\theta \subseteq \mu$. Thus μ is a fuzzy semiprime ideal of S.

Conversely, let μ be a fuzzy semiprime ideal of a commutative Γ-semiring S. Let $x \in S$. It follows from Proposition 3.6 that

$$\mu(x) = \inf\{\mu(x\gamma s\delta x) : s \in S,\ \gamma, \delta \in \Gamma\} \geq \mu(x\gamma x) \geq \mu(x).$$

Thus $\mu(x\gamma x) = \mu(x)$ for all $s \in S$ and $\gamma, \delta \in \Gamma$. □

Theorem 3.23. *A commutative Γ-semiring S is multiplicatively regular if and only if every non-constant fuzzy ideal of S is fuzzy semiprime.*

Proof. Let μ be a non-constant fuzzy ideal of a commutative regular Γ-semiring S and θ be a fuzzy ideal of S such that $\theta \circ \theta \subseteq \mu$. Since S is regular, it follows from Theorem 3.21 that $\theta = \theta \circ \theta \subseteq \mu$. Hence μ is fuzzy semiprime.

Conversely, let every non-constant fuzzy ideal of the commutative Γ-semiring S be fuzzy semiprime. So $\lambda_{<x\Gamma x>}$ is a fuzzy semiprime ideal of S. Then from Proposition 3.22, $\lambda_{<x\Gamma x>}(s\gamma s) = \lambda_{<x\Gamma x>}(s)$ for all $s \in S$ and $\gamma \in \Gamma$. Thus $\lambda_{<x\Gamma x>}(x\gamma x) = 1 = \lambda_{<x\Gamma x>}(x)$ for all $x \in S$ and $\gamma \in \Gamma$. Therefore $x \in < x\Gamma x > \subseteq x\Gamma S\Gamma x$ as S is commutative. Hence S is regular. □

4. Fuzzy Irreducible ideals.

In this section we introduce the notion of a fuzzy irreducible ideal in semirings as well as in Γ-semirings. We obtain various results on fuzzy irreducible ideals of semirings. We then obtain their analogues in Γ-semirings. The proof of these Γ-semiring analogues are obtained by using the interplay between a Γ-semiring and its operator semirings which are established in Lemmas 4.2, 4.3, 4.4 and Theorem 4.5. This method of proof once again illustrates the efficacy of operator semirings in the study of Γ-semiring in terms of fuzzy subsets.

Definition 4.1. A fuzzy ideal μ of a(semiring) Γ-semiring S is said to be fuzzy irreducible if it is not an intersection of two fuzzy ideals of S properly containing μ. Otherwise, μ is called fuzzy reducible.

Throughout this section S denotes a Γ-semiring with left and right unities.

Lemma 4.2. *Suppose μ is a fuzzy irreducible ideal of L. Then μ^+ is also fuzzy irreducible ideal of S.*

Proof. If possible, let μ^+ be not irreducible. Then there exist $\theta_1, \theta_2 \in FI(S)$ such that $\mu^+ = \theta_1 \cap \theta_2$ and $\theta_1 \supset \mu^+$, $\theta_2 \supset \mu^+$. Now $(\mu^+)^{+'} = (\theta_1 \cap \theta_2)^{+'}$ implies that $\mu = \theta_1^{+'} \cap \theta_2^{+'}$ and $\theta_1^{+'} \supset \mu$, $\theta_2^{+'} \supset \mu$, by Proposition 2.12, which contradicts that μ is a fuzzy irreducible ideal. Thus μ^+ is a fuzzy irreducible ideal. □

Lemma 4.3. *If $\mu, \sigma \in FI(S)$ then $(\mu \cap \sigma)^+ = \mu^+ \cap \sigma^+$.*

Proof. Let $s \in S$. Then by definition 2.10,

$$\begin{aligned}(\mu \cap \sigma)^+(s) &= \inf_{\gamma \in \Gamma} (\mu \cap \sigma)([s,\gamma]) = \inf_{\gamma \in \Gamma} [\min[\mu([s,\gamma]), \sigma([s,\gamma])]] \\ &= \min\left[\inf_{\gamma \in \Gamma} \mu([s,\gamma]), \inf_{\gamma \in \Gamma} \sigma([s,\gamma])\right] \\ &= \min[\mu^+(s), \sigma^+(s)] = (\mu^+ \cap \sigma^+)(s).\end{aligned}$$

Hence $(\mu \cap \sigma)^+ = \mu^+ \cap \sigma^+$. □

Lemma 4.4. *If μ is a fuzzy irreducible ideal of S then $\mu^{+'}$ is a fuzzy irreducible ideal of L.*

Proof. If possible, let $\mu^{+'}$ be reducible. Then there exist $\sigma, \theta \in FI(L)$ such that $\mu^{+'} = \sigma \cap \theta$ and $\sigma \supset \mu^{+'}$, $\theta \supset \mu^{+'}$. Then $(\mu^{+'})^+ = (\sigma \cap \theta)^+$, i.e., $\mu = \sigma^+ \cap \theta^+$, by Proposition 2.12 and Lemma 4.3 with $\sigma^+ \supset \mu$, $\theta^+ \supset \mu$ which contradicts the fact that μ is irreducible. Hence the lemma is true. □

Combining Lemma 4.2 and 4.4 we deduce the following theorem.

Theorem 4.5. *The mapping $\mu \mapsto \mu^{+'}$ define a one-one correspondence between the set of all fuzzy irreducible ideals of S and the set of all fuzzy irreducible ideals of L where μ is a fuzzy irreducible ideal of L.*

Theorem 4.6. *Suppose μ is a fuzzy prime ideal of a semiring M. Then μ is a fuzzy irreducible ideal.*

Proof. Let us suppose that there exist fuzzy ideals σ and θ of M such that $\mu = \sigma \cap \theta,\ \sigma \supset \mu,\ \theta \supset \mu$. Then

$$\sigma(x) > \mu(x) \text{ and } \theta(y) > \mu(y) \tag{4.1}$$

for all $x, y \in S$. Since μ is fuzzy prime we have from Theorem 4.7 of [3],

(a) $Im\mu = \{1, \alpha\}$, where $\alpha \in [0, 1)$,
(b) μ_0 is a prime ideal of S.

Clearly in view of (4.1) and (a), $x \notin \mu_0$ and $y \notin \mu_0$. Hence $xy \notin \mu_0$ and so $\mu(x) = \mu(y) = \mu(xy) = \alpha$. Now

$$(\sigma \cap \theta)(xy) \geq (\sigma\theta)(xy) \geq \min[\sigma(x), \theta(y)] > \min[\mu(x), \mu(y)] = \mu(xy)$$

which is untenable. This completes the proof. □

If μ is a fuzzy prime ideal of a Γ-semiring S then by Proposition 2.23, $\mu^{+'}$ is fuzzy prime ideal of L. Hence by Theorem 4.6, $\mu^{+'}$ is a fuzzy irreducible ideal of L. Hence by Theorem 4.5, $\mu = (\mu^{+'})^{+}$ is a fuzzy irreducible ideal of S. Hence we obtain the following analogue of Theorem 4.6.

Theorem 4.7. *Suppose μ is a fuzzy prime ideal of a Γ-semiring S. Then μ is a fuzzy irreducible ideal.*

Theorem 4.8. *If μ is a fuzzy irreducible ideal of a semiring M then there exists $\alpha \in [0, 1)$ such that*

(1) $Im\ \mu = \{1, \alpha\}$,
(2) $\mu_0 = \{x \in M : \mu(x) = \mu(0)\}$ *is irreducible.*

Proof. (1) Let $\mu(0) = t$. If possible, let $1 \notin Im\ \mu$. Then $t < 1$. Let us define fuzzy subsets σ and θ of M by

$$\sigma(x) = \begin{cases} 1 & \text{if } x \in \mu_t \\ \mu(x) & \text{otherwise} \end{cases}$$

and $\theta(x) = t$ for all $x \in M$. Then σ and θ are fuzzy ideals of M and $\mu = \sigma \cap \theta,\ \sigma \supset \mu,\ \theta \supset \mu$ which contradicts the fact that μ is fuzzy irreducible. Hence $1 \in Im\ \mu$ and consequently $\mu(0) = 1$. We shall now show that μ_1 is the only proper level ideal of M. To prove this by contradiction, we assume that there exists a level ideal $\mu_s,\ s \in [0, 1)$ of μ such that $\mu_1 \subset \mu_s \subset M$. Then for some $\beta \in [0, 1)$, we have

$$\mu(x) = \begin{cases} 1 & \text{if } x \in \mu_1 \\ s & \text{if } x \in \mu_s \setminus \mu_1 \\ \beta & \text{if } x \in M \setminus \mu_s. \end{cases}$$

Now we define fuzzy subsets $\sigma^{'}$ and $\theta^{'}$ of M by

$$\sigma^{'}(x) = \begin{cases} 1 & \text{if } x \in \mu_1 \\ \mu(x) & \text{if } x \in M \setminus \mu_1 \end{cases}$$

and

$$\theta^{'}(x) = \begin{cases} 1 & \text{if } x \in \mu_1 \\ s & \text{if } x \in \mu_s \setminus \mu_1 \\ r & \text{if } x \in M \setminus \mu_s \end{cases}$$

where $\beta < r < s$. Then $\sigma^{'}$ and $\theta^{'}$ are fuzzy ideals of M and $\mu = \sigma^{'} \cap \theta^{'}$, $\sigma^{'} \supset \mu$, $\theta^{'} \supset \mu$, which contradicts the fact that μ is fuzzy irreducible. Thus μ_1 is the only proper level ideal of M. So there exists $\alpha \in [0, 1)$ such that $Im\ \mu = \{1, \alpha\}$.

(2) If possible, let μ_0 be not irreducible. Then there exist ideals H and K of M such that $\mu_0 = H \cap K$, $H \supset \mu_0$, $K \supset \mu_0$. Clearly, neither H is contained in K nor K is contained in H and $(H \setminus \mu_0) \cap (K \setminus \mu_0)$ is an empty set. Also μ is given by

$$\mu(x) = \begin{cases} 1 & \text{if } x \in \mu_0 \\ \alpha & \text{if } x \in M \setminus \mu_0 \end{cases}$$

for some $\alpha < 1$. Next define fuzzy subsets σ^* and θ^* of M by

$$\sigma^*(x) = \begin{cases} 1 & \text{if } x \in \mu_0 \\ t^* & \text{if } x \in H \setminus \mu_0 \\ \alpha & \text{if } x \in M \setminus H \end{cases} \quad \text{and} \quad \theta^*(x) = \begin{cases} 1 & \text{if } x \in \mu_0 \\ t^* & \text{if } x \in K \setminus \mu_0 \\ \alpha & \text{if } x \in M \setminus K. \end{cases}$$

Then σ^* and θ^* are fuzzy ideals of M and $\mu = \sigma^* \cap \theta^*$, $\sigma^* \supset \mu$, $\theta^* \supset \mu$–which contradicts the fact that μ is fuzzy irreducible. Hence μ_0 is irreducible. □

Note 1. What Golan[9] called a strongly irreducible ideal we call it here an irreducible ideal.

Now we obtain below the Γ-semiring analogue of the above result.

Theorem 4.9. *Suppose μ is a fuzzy irreducible ideal of a Γ-semiring S. Then there exists $\alpha \in [0, 1)$ such that*

(1) $Im\ \mu = \{1, \alpha\}$
(2) $\mu_0 = \{x \in S : \mu(x) = \mu(0)\}$ *is irreducible.*

Proof. Let μ be a fuzzy irreducible ideal of a Γ-semiring S. By Theorem 4.5, $\mu^{+^{'}}$ is a fuzzy irreducible ideal of the left operator semiring L. Then by Theorem 4.8, $Im\ \mu^{+^{'}} = \{0, 1\}$ and $(\mu^{+^{'}})_0$ is irreducible ideal of L whence $Im\ \mu = \{0, 1\}$ and $(\mu_0)^{+^{'}}$ is an irreducible ideal of L. Now by Theorem 2.2.31 of [11], we deduce that μ_0 is an irreducible ideal of S. □

Theorem 4.10. *If μ is a fuzzy ideal of a semiring M which is both fuzzy semiprime and fuzzy irreducible then μ is fuzzy prime.*

Proof. It follows from Theorem 4.8 that

(i) there exists $\alpha \in [0, 1)$ such that $Im\ \mu = \{1, \alpha\}$
(ii) μ_0 is irreducible.

Also since μ is fuzzy semiprime, by Proposition 2.19 μ_0 is a semiprime ideal of M. Hence by Proposition 2.21 and in view of Note 1 we deduce that μ_0 is prime. Then by Theorem 4.7 of [3], μ is fuzzy prime. □

Now combining Theorem 4.6 and 4.10 we obtain the following theorem.

Theorem 4.11. *Let μ be a fuzzy ideal of a semiring M. Then μ is fuzzy prime if and only if it is both fuzzy semiprime and fuzzy irreducible ideal of M.*

The following is the Γ-semiring analogue of the above result which is obtained by the correspondence between the prime (irreducible) ideals of a Γ-semiring and that of its operator semirings.

Theorem 4.12. *Let μ be a fuzzy ideal of a Γ-semiring S. Then μ is fuzzy prime if and only if it is both fuzzy semiprime and fuzzy irreducible ideal of S.*

Proof. Let μ be a fuzzy prime ideal of a Γ-semiring S. Then from Proposition 3.2, μ is fuzzy semiprime. Again since μ is fuzzy prime, then by Proposition 2.23, $\mu^{+'}$ is a fuzzy prime ideal of L. By Theorem 4.6, $\mu^{+'}$ is a fuzzy irreducible ideal of L. Now by Lemma 4.2, $(\mu^{+'})^{+} = \mu$ is fuzzy irreducible.

Conversely, let μ be a fuzzy ideal of a Γ-semiring S which is both fuzzy semiprime and fuzzy irreducible. Then it follows from Theorem 3.14 and Lemma 4.4 that $\mu^{+'}$ is fuzzy semiprime and fuzzy irreducible ideal of L. Hence by Theorem 4.10, $\mu^{+'}$ is fuzzy prime ideal of L. Thus $(\mu^{+'})^{+} = \mu$ is fuzzy prime ideal of S. Hence the theorem holds. □

Combining Theorems 3.23 and 4.12 we obtain the following theorem which is a partial converse of Theorem 4.7.

Theorem 4.13. *In a multiplicatively regular Γ-semiring, every non-constant fuzzy irreducible ideal is fuzzy prime.*

Remark 4.14. Theorem 4.12 can be proved directly by using Theorem 4.7 and Theorem 4.9.

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References

[1] B. K. Biswas, Fuzzy ideals of semirings and near-rings, Ph. D. Dissertation, University of Calcutta, India 2000.

[2] T. K. Dutta and B. K. Biswas, Structure of fuzzy ideals of semirings; Bull. Calcutta Math. Soc. 89(4) (1997) 271–284.

[3] T. K. Dutta and B. K. Biswas, Fuzzy prime ideals of a semiring, Bull. Malaysian Math. Soc.(Second series) 17 (1994) 9–16.

[4] T. K. Dutta and S. K. Sardar, On the operator semirings of a Γ-semiring, Southeast Asian Bull. Math. 26 (2002) 203–213.

[5] T. K. Dutta and S. K. Sardar, Semiprime ideals and irreducible ideals of Γ-semiring, Novi Sad J. Math. 30(1) (2000) 97–108.

[6] T. K. Dutta and S. K. Sardar, On prime ideals and prime radicals of a Γ-semiring, ANALELE STINTIFICE ALE UNIVERSITATII "AL.I.CUZA" IASI, 46 (2000) 319–329.

[7] T. K. Dutta, S. K. Sardar and S. Goswami, An introduction to fuzzy ideals of Γ-semirings, Proceedings of National seminar on Algebra, Analysis and Discrete Mathematics, University of Kerala, India (To appear).

[8] T. K. Dutta, S. K. Sardar and S. Goswami, Operations on fuzzy ideals of Γ-semirings, Communicated.

[9] J. S. Golan, Semirings and their applications, Kluwer Academic Publishers, 1999.
[10] M. M. K. Rao, Γ-semiring-1, Southeast Asian Bull. Math. 19 (1995) 49–54.
[11] S. K. Sardar, Some problems associated with Γ-semirings, Ph. D. Dissertation, University of Calcutta, India 2003.
[12] S. K. Sardar and S. Goswami, Role of operator semirings in characterizing Γ-semirings in terms of fuzzy subsets, Communicated.
[13] S. K. Sardar and S. Goswami, Fuzzy prime ideals of Γ-semirings, Bull. Calcutta Math. Soc. (to appear).
[14] S. K. Sardar and S. Goswami, A note on characterization of fuzzy prime ideals of Γ-semirings via operator semirings, International Journal of Algebra 4(18) (2010), 867–873.

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