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Measurement error effects on the performance of the process capability index based on fuzzy tolerance interval

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Abstract. In this paper, we discuss on the fuzzy Process capability index such that the tolerance interval is a fuzzy set. We also discuss on the fuzzy capability analysis in case of measurement error occurrence. Measurement error can be due to insufficient gauge calibration or external influences on the measurement device. We consider two case systematic and random measurement errors.

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1. Introduction

Statistical techniques can be helpful throughout the product cycle, including development activities prior to manufacturing, in quantifying process variability, in analyzing this variability relative to product requirements or specifications, and in assisting development and manufacturing in eliminating or greatly reducing this variability. This generally activity is called process capability analysis. Process capability refers to the uniformity of process. Obviously, the variability in the process is a measure of the uniformity of output. There may not exist an exact definition of the term "process capability" but in the literature there is an agreement to consider a process as capable (e.g. [7, 8, 9]) if with high probability the (real-valued) quality characteristic X of the produced items lies between some lower and upper specification limits LSL and USL (or tolerance interval limits). Therefore the idea of process capability implies that the fraction p of produced non conforming items should be small if the process is said to be capable.

In the traditional quality management, the most commonly used capability indices like C_p, CPU, CPL, C_{pk}, C_{pm} and C_{pmk} are employed to indicate process capability (we recall their definitions in the next section). An underlying assumption is that the output process measurements are distributed as normal random variables. When normal distributions are assumed, however, different distributions are present such as skew, heavy-tailed, and short-tailed distributions and the percentages on non-conforming parts are significantly different from the computed process capability indices indication. Experience shows that the normality assumption is often not met in real world application (see [6]).

Process capability indices are used to indicate to what extent production process can meet the technical requirements. In fact, it is not quite reasonable to define technical standard as a clear interval, for there actually exist gradual change and transition on the boundaries of technical standard. Therefore to define technical standard as a fuzzy subset more appropriately accords with the actual production, to use a fuzzy subset to indicate technical standard is totally in line with the suitability quality outlook. So the traditional capability indices can not be used to represent process capability any more. Also, in traditional computing method of process capability index, the position of the distribution centre of characteristic X and that of standard centre do not always coincide and their formula also vary. We can also say that in the traditional quality management, process capability index and percentage of substandard products are two different concepts, but they are closely related to each other. After the introduction of the formula C_p, we can see that when the distribution centre of quality index is consistent with the standard centre, the percentage of substandard products is in inverse proportion to the process capability index. This accords with the fact that process capability index is a reflection of how process capability meet technical demands. However, when the distribution centre is inconsistent with the standard centre, we shall see this phenomenon if we still use the computing formula; there are great differences in the percentage of substandard products corresponding to the numerical value of the same process capability index.

2. Traditional Process Capability Indices

In the literature, one of the proposed definitions on process capability index consider that as the ratio of the real performance of process to requested performance (see [8]), that is,

$$\text{Capability index} = \frac{\text{The width of tolerance interval}}{\text{The width of process dispersion}}.$$

But, just as we recalled in the introduction section, for a given tolerance interval $[LSL, USL]$ and a risk α, a process with the quality characteristic X is said to be capable if,

$$P\left(X \in [LSL, USL]\right) \geq 1 - \alpha. \tag{2.1}$$

If the center of the distribution of X ($\mu = E(X)$) and the mid-point of the tolerance interval (standard center) be equal ($\mu = M = \frac{(LSL+USL)}{2}$), then we have

$$P\left(X \in [LSL, USL]\right) = P\left(|X - \mu| \leq \frac{USL - LSL}{2}\right). \tag{2.2}$$

Let

$$r = \min\left\{c : P\left(|X-\mu| \le c\right) \ge 1-\alpha\right\}. \tag{2.3}$$

Since $\{|X-\mu| \le c\}$ is monotone with respect to c, we can say that r is the unique solution of the equation $P\left(|X-\mu| \le c\right) = 1-\alpha$, and the process will be capable if,

$$\frac{USL-LSL}{2} \ge r, \tag{2.4}$$

or

$$\frac{USL-LSL}{2r} \ge 1. \tag{2.5}$$

The ratio $\frac{USL-LSL}{2r}$, is called capability index.

If X be normally distributed, that is,

$$X \rightsquigarrow N(\mu,\sigma^2),\ \ \mu \in \mathbb{R} \ \text{ and } \sigma \in \mathbb{R}^+, \tag{2.6}$$

then, we have

$$P\left(|X-\mu| \le 3\sigma\right) = 0.9973. \tag{2.7}$$

Therefore, with a risk $\alpha = 0.0027$, the simplest capability index which is called the process potential index defined by

$$C_p = \frac{USL-LSL}{6\sigma}. \tag{2.8}$$

For more information one can study [3, 7, 8].

In the case $\mu = M$ (that is perfect location of the process) a C_p value of at least 1 assures that at most 0.27% of the produced items fall outside the tolerance interval. But if $\mu \neq M$ a C_p value of 1 does not all guarantee a small fraction of non confirming items because the C_p index does not take into consideration the location μ of the distribution. In order to reflect departures from the target value as well as changes in the process variation several order indices have been proposed such as C_{pk} ([1, 2, 3, 5, 10, 11]).

The two indices

$$CPU = \frac{USL-\mu}{3\sigma} \ \text{ and } \ CPL = \frac{\mu-LSL}{3\sigma}, \tag{2.9}$$

measure the performance of the process with respect to the upper and lower specification limits. They are used in unilateral tolerance situations where only one single specification limit is given. Again a process e.g. with upper specification limit is considered as capable if $CPU \ge 1$.

From these definition we get for the specification case the C_{pk} index,

$$C_{pk} = \min(CPU, CPL). \tag{2.10}$$

C_{pk} measures the distance between the process mean and the closest specification limit relation to the one-side actual process spread 3σ. Kane [3] gave an equivalent representation:

$$C_{pk} = C_p(1-k), \quad k = \frac{|\mu-M|}{\frac{USL-LSL}{2}}. \tag{2.11}$$

For fixed μ, C_{pk} decreases with increasing σ and for fixed σ, C_{pk} decrease with increasing difference $|\mu - M|$. Because $k \geq 0$ it holds that $C_{pk} \leq C_p$, where equality is given if and only if $\mu = M$. Though C_{pk} take into account deviation from the target as change in variations, a high value of C_{pk} does not assure that the process is located near to target. There is also no exact relation between C_{pk} and the fraction p of non conforming items.

Departures from the target value carry more weight with the other well-known capability indices C_{pm} and C_{pmk} defined by

$$C_{pm} = \frac{USL - LSL}{6\sigma'}, \quad C_{pmk} = C_p(1-k) \;\; ; \; \sigma' = \sqrt{\sigma^2 + (\mu - M)^2}. \tag{2.12}$$

In principal, C_{pm} behaved like C_{pk} but C_{pm} is bounded above as $\sigma \to 0$ and $\mu \neq M$. It holds

$$C_{pm} \leq \frac{USL - LSL}{6\,|\mu - M|}, \tag{2.13}$$

which implies that if μ falls outside the middle third of the tolerance interval (LSL, USL), then C_{pm} is smaller than 1 in spite of how small σ might be. For $\mu = M$ it holds $Cp = CPU = CPL = C_{pk} = C_{pm} = C_{pmk}$.

Because the capability indices depend on the unknown parameters μ and σ. These parameters have to be estimated from random samples $(X_1, ..., X_n)$. Common practice is to use as estimators for μ, σ and σ' respectively:

(2.14)

$$\widehat{\mu} = \overline{X}, \widehat{\sigma} = S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2} \;\text{ and }\; \widehat{\sigma'} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2 + (\overline{X} - M)^2}.$$

3. Process Capability Index Based on Fuzzy Tolerance Interval

In this section, we recall some notions of fuzzy sets theory and the probability of a fuzzy event. Then, we try to define a process capability index as the probability of a fuzzy event which present the tolerance interval.

Let $\mathbb{X}$ be a given universal set, a set can be defined by a function, usually called a characteristic function, that declares which elements of $\mathbb{X}$ are members of the set and which are not. Set A is defined by its characteristic function, χ_A, as follows:

$$\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \in \mathbb{X} - A. \end{cases} \tag{3.1}$$

That is, the characteristic function maps elements of $\mathbb{X}$ to element of the set $\{0, 1\}$, which is formally expressed by $\chi_A : \mathbb{X} \to \{0, 1\}$.

The concept of characteristic function can be generalized such that the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set in question. Larger values denote higher degrees of membership. Such a function is called a membership function, and the set defined by it a fuzzy set. The most commonly used range of values of membership functions is the unit interval $[0, 1]$. In this case, each membership function maps elements of a given universal set $\mathbb{X}$, which is always a crisp set, into real numbers in $[0, 1]$. Two distinct notation are most commonly employed in the literature to denote membership functions (see [4]). In one of them, the membership function of a fuzzy set $\widetilde{A}$ is denoted by $\mu_{\widetilde{A}}$: that is, $\mu_{\widetilde{A}} : \mathbb{X} \to [0,1]$. In the other one, the function is denoted by $\widetilde{A}$ and has, of course, the same form: that is $\widetilde{A} : \mathbb{X} \to [0,1]$. In this paper, we use the second notation.

Now, we recall the definition of the probability of a fuzzy event which has been given by Zadeh ([13, 14]). Let $(\mathbb{R}^n, \mathcal{B}^n, P)$ be a probability space, in which $\mathcal{B}^n$ is the σ-field of Borel set in $\mathbb{R}^n$ and P is the probability measure over $\mathbb{R}^n$. Then, a fuzzy event in $\mathbb{R}^n$ is a fuzzy set $\widetilde{A}$ ($\widetilde{A} : \mathbb{R}^n \to [0,1]$, is Borel measurable). The probability of a fuzzy event $\widetilde{A}$ is defined by the Lebesgue-Stieltjes integral:

$$P(\widetilde{A}) = \int_{\mathbb{R}^n} \widetilde{A}(x) dP. \tag{3.2}$$

The existence of the Lebesgue-Stieltjes integral is insured by the assumption that $\widetilde{A}$ is Borel measurable.

According to the traditional definition of process capability index (C_p), a process is said to be capable, if for a given risk α ($\alpha \in (0,1)$):

$$P\left(X \in [LSL, USL]\right) = \int_{\mathbb{R}} \chi_{[LSL,USL]}(x) dP \geq 1 - \alpha. \tag{3.3}$$

To modify the ambiguity which exist in the definition of tolerance interval (especially in the limits) and for the mentioned reasons in the introduction section, in the following we realize a relation between the characteristic function of a tolerance interval and the membership function of a fuzzy set which employ to present the tolerance interval. Then, we define the fuzzy process capability index as the probability of fuzzy tolerance interval. Figure 1 shows the characteristic function of tolerance interval. As is shown in Figure 1, the quality standard curve is just a reflection of the traditional concept.

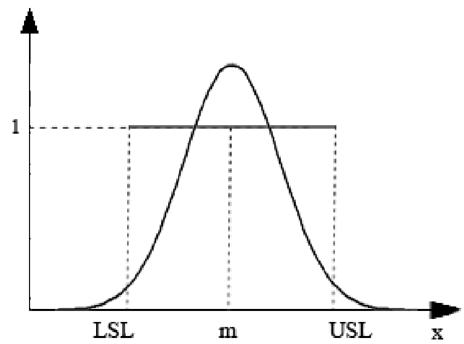

FIGURE 1. Characteristic function of tolerance interval

At the range of LSL and USL, there exists a jump. Products that fall within the range of $[LSL, ULS]$ are judged as qualified, while products out of the range as unqualified. Figure 2 provides fuzzy quality, when $x = m$, $\widetilde{A}(m) = 1$, which shows products are completely up-to-standard, when $x = LSL$ or USL, $\widetilde{A}(LSL) = \widetilde{A}(USL) = 0$, it shown product are totally below standard, and is declining, with both sides symmetrical. The curve shown in Figure 2 is worked out through the fuzzy statistical, processing of users suitability appraisals. However, in actual calculation, it can sometimes be simplified: using a linear line segment rather than a curve.

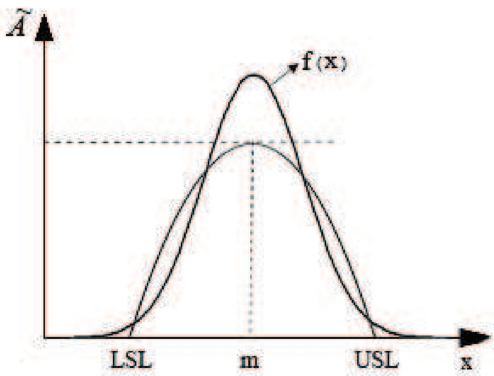

Figure 2. Fuzzy up-to standard

Now, the fuzzy process capability index can be defined as the probability of fuzzy up-to-standard products turned out in the process of production it is labeled as $C_{\widetilde{p}}$ (for more information see [12]). When the quality index of process products is a continuous random variable,

$$C_{\widetilde{p}} = \int_{-\infty}^{+\infty} \widetilde{A}(x) f(x) dx, \tag{3.4}$$

where $f(x)$ is the probability density function of quality characteristic X, $\widetilde{A}(x)$ is the membership function of fuzzy up-to-standard products.

When the quality characteristic of process is a discrete random variable X and the values of X is x_i $(i = 1, , n)$,

$$C_{\widetilde{p}} = \sum_{i=1}^{n} \widetilde{A}(x_i) P_X(x_i), \tag{3.5}$$

where $P_X(x_i)$ is the probability when $X = x_i$, $\widetilde{A}(x_i)$ is the membership function of fuzzy up-to-standard products.

Obviously, we have to know the value of the probability density of quality characteristic and the membership function of fuzzy up-to-standard products, before we can work out $C_{\widetilde{p}}$.

The membership function of fuzzy up-to-standard products generally can help to realize the fuzzy nature of clear technical standard boundaries.

Figures 3 and 4 show two common membership functions.

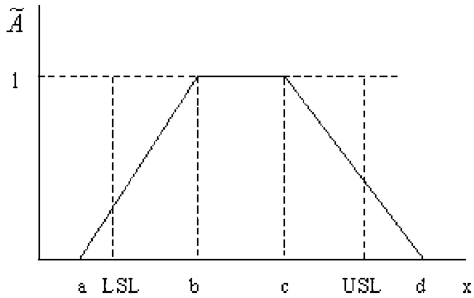

FIGURE 3. Trapezoidal membership function

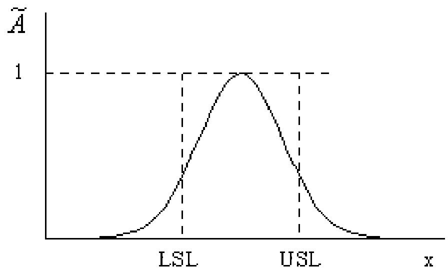

FIGURE 4. Exponential (Normal) membership function

Example 3.1 Let the distribution of quality characteristic X be normal (i.e. $X \rightsquigarrow N(\mu, \sigma^2)$). The degree of membership of fuzzy up-to-standard products can be work out through the formula $C_{\widetilde{p}}$ under the condition that the diagram of membership function is in the form of trapezoidal as provided in Figure 3 We have

$$\begin{aligned} C_{\widetilde{p}(trap)} \quad = \quad & -\frac{\sigma}{\sqrt{2\pi}(a-b)}\left[\exp\left(\frac{-(a-\mu)^2}{2\sigma^2}\right)-\exp\left(\frac{-(b-\mu)^2}{2\sigma^2}\right)\right] \\ & +\frac{\sigma}{\sqrt{2\pi}(c-d)}\left[\exp\left(\frac{-(c-\mu)^2}{2\sigma^2}\right)-\exp\left(\frac{-(d-\mu)^2}{2\sigma^2}\right)\right] \\ & +\frac{\mu-a}{b-a}\left[\Phi(\frac{b-\mu}{\sigma})-\Phi(\frac{a-\mu}{\sigma})\right] \\ & +\frac{\mu-d}{c-d}\left[\Phi(\frac{d-\mu}{\sigma})-\Phi(\frac{c-\mu}{\sigma})\right] \\ & +\left[\Phi(\frac{c-\mu}{\sigma})-\Phi(\frac{b-\mu}{\sigma})\right]. \end{aligned} \tag{3.6}$$

In the case of $b = c = \mu$ (i.e. in the case of triangular membership function),

$$C_{\widetilde{p}(tri)} = -\frac{\sigma}{\sqrt{2\pi}(a-\mu)}\left[e^{-(a-\mu)^2/2\sigma^2} - 1\right] + \frac{\sigma}{\sqrt{2\pi}(\mu-d)}\left[1 - e^{-(d-\mu)^2/2\sigma^2}\right] + \left[\Phi(\frac{d-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})\right]. \tag{3.7}$$

And when the membership function is exponential (i.e. $\widetilde{A}(x) = \exp\left[-\frac{(x-m)^2}{2\lambda^2}\right]$), We have

$$C_{\widetilde{p}(\exp)} = \int_{-\infty}^{+\infty} \exp\left(\frac{-(x-m)^2}{2\lambda^2}\right)\frac{1}{\sqrt{2\pi\sigma}}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)dx = \left(\frac{\lambda}{\sqrt{\sigma^2+\lambda^2}}\right)\exp\left(\frac{-(\mu-m)^2}{2(\sigma^2+\lambda^2)}\right), \tag{3.8}$$

where for fixed μ, m and σ, it is a function of modification parameter of tolerance interval $(\lambda > 0)$.

4. Process capability analysis in case of measurement error occurrence

Until now it was implicitly assumed that the measurement of the characteristic X did not contain measurement errors. This assumption will now be abandoned. Measurement error can be due to insufficient gauge calibration or external influences on the measuring device. In case of measurement error occurrence one needs to distinguish between latent or true variables X and observable or empirical variables X^e. The latent variables are those which would be observed in case of measurement error absence. One typically can distinguish between systematic and random measurement errors.

With systematic errors the variable X^e is a deterministic function of the variable X. In the following the measurement error effect analysis referring to the systematic error is restricted to the constant measurement error which represents the most important special case, Instead of X one observes

$$X^e = X + c, \tag{4.1}$$

where c is a real-valued number. A constant measurement error leads to a virtual shift of the process mean by c units.

A random measurement error results when the error is modeled by a random variable ε. It will be assumed that X and ε, analogously to (4.1), are additively linked according to

$$X^e = X + \varepsilon. \tag{4.2}$$

It is further assumed that the random variables X and ε are stochastically independent and the $E(\varepsilon) = 0$. A measurement error of this type enlarge the process variance by adding the error variance without affecting the process mean $\mu = E(X)$.

4.1. **Constant measurement errors.** If a constant measurement error occurs, one measures instead of the $N(\mu, \sigma^2)$-distributed latent variable X the $N(\mu^e, \sigma^2)$-distributed variable X^e given in (4.1) with $\mu^e = \mu + c$, the constant measurement error has influence on the value of the fuzzy capability index $C_{\widetilde{p}}$. In fact, if we consider an exponential membership function for the quality characteristic, we have

$$\begin{aligned} C^e_{\widetilde{p}(\exp)} &= \int_{-\infty}^{\infty} \exp\left(\frac{-(x-m)^2}{2\lambda^2}\right) \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(x-\mu-c)^2}{2\sigma^2}\right) dx \\ &= \left(\frac{\lambda}{\sqrt{\lambda^2+\sigma^2}}\right) \exp \frac{-(\mu+c-m)^2}{2(\lambda^2+\sigma^2)} \qquad (4.3) \end{aligned}$$

In order to quantitatively evaluation the measurement errors induced changes, one can look at the ratio of the empirical and true process capability indices:

$$\begin{aligned} \frac{C^e_{\widetilde{p}(\exp)}}{C_{\widetilde{p}(\exp)}} &= \frac{\left(\frac{\lambda}{\sqrt{\lambda^2+\sigma^2}}\right) \exp\left(\frac{-(\mu+c-m)^2}{2(\lambda^2+\sigma^2)}\right)}{\left(\frac{\lambda}{\sqrt{\lambda^2+\sigma^2}}\right) \exp\left(\frac{-(\mu-m)^2}{2(\lambda^2+\sigma^2)}\right)} \\ &= \exp\left[-\frac{y^2}{2}\left(1+\frac{2z}{y}\right)\right], \qquad (4.4) \end{aligned}$$

where $y = \frac{c}{\sqrt{\lambda^2+\sigma^2}}$ and $z = \frac{\mu-m}{\sqrt{\lambda^2+\sigma^2}}$.

Figure 5 shows the quotient (4.4) as a function of the relation process mean z and the relative constant measurement error y for the range defined by $|y| \leq 0.1$ and $|z| \leq 1$.

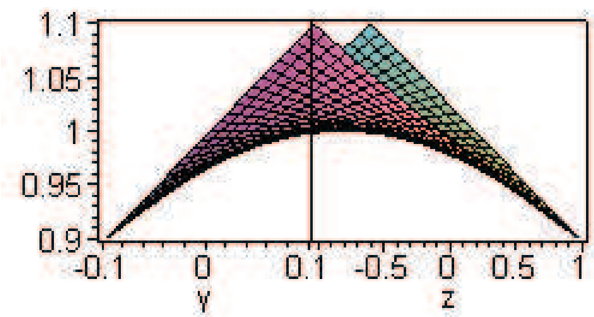

FIGURE 5. Graph of $\frac{C^e_{\widetilde{p}(exp)}}{C_{\widetilde{p}(exp)}}$ in the case of constant error

For fixed error contamination degree y, the quotient (4.4) is studied as a function of z, this function is decreasing for $y \in [0, 0.1]$ and fixed, it is increasing for fixed $y \in [-0.1, 0]$, it takes its minimum values in $(0.1, 1)$ and $(-0.1, -1)$ it also takes its maximum values in $(0.1, -1)$ and $(-0.1, 1)$. It is equal to 1 in $(0, 0)$. In case of exact process adjustment, i.e. $z = 0$, the ratio $\frac{C^e_{\widetilde{p}(\exp)}}{C_{\widetilde{p}(\exp)}}$ takes the value $e^{-y^2/2}$. This means that in case of $\mu = m$,

$$C^e_{\widetilde{p}(\exp)} = \exp\left(-\frac{y^2}{2}\right) C_{\widetilde{p}(\exp)}\ , \qquad (4.5)$$

that is, $C^e_{\widetilde{p}(\exp)}$ will be decreased by the coefficient $\exp\left(-\frac{y^2}{2}\right)$.

But in the traditional method, we have

$$\frac{C^e_p}{C_p} = 1 \ \text{ and } \ \frac{C^e_{pk}}{C_{pk}} = \frac{1-|z+y|}{1-|z|}, \qquad (4.6)$$

where $z = \frac{\mu - M}{\frac{USL-LSL}{2}}$ and $y = \frac{c}{\frac{USL-LSL}{2}}$.

Figure 6 shows the quotient (4.6) as a function of the process mean z and the relative constant measurement error y for the range defined by $|y| \leq 0.1$ and $|z| \leq 0.9$.

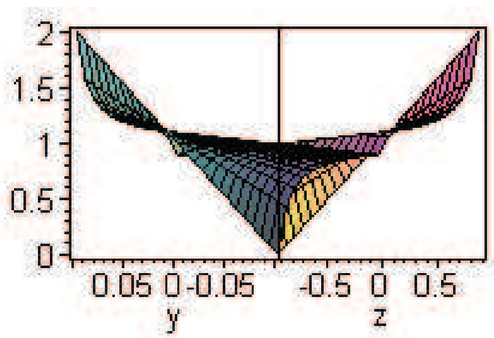

FIGURE 6. Graph of $\frac{C^e_p}{C_p}$ in the case of constant error

For fixed error contamination degree y, the quotient (4.6) is studied as a function of z. Its curve can be obtained by vertically intersecting the $3D$ graph in figure 4.2 parallel to the z- $\frac{C^e_{pk}}{C_{pk}}$ -plane. This curve has two poles in $y = \mp 0.1$, i.e. at the limits of the tolerance interval. This implies that the error induced relative change in C_{pk} is not bounded.

In $z = -\frac{y}{2}$ the function take on value 1. Hence at $z = -\frac{y}{2}$ the C_{pk} index will not be influenced by the constant error c. For $z < -\frac{y}{2}$ there will be depending on the sign of c, an error induced increase or decrease of C_{pk}. For the $z > -\frac{y}{2}$ the distortion points to the opposite direction. In the case of exact process centredness, i.e. $z = 0$, the ratio $\frac{C^e_{pk}}{C_{pk}}$ takes on the value $1 - |y|$. This means that in case of $\mu = M$ the index will be decreased by $100 \cdot |y|\,\%$.

4.2. **Random measurement errors.** It is now assumed that an additive $N(0, \sigma_e^2)$-distributed measurement error ε occurs which is supposed to be stochastically independent from the quality characteristic X. Thus, according to relation (4.2) one observes instead of the $N(\mu, \sigma^2)$-distributed variable X the $N\left(\mu, (\sigma^e)^2\right)$-distributed variable X^e with $(\sigma^e)^2 = \sigma^2 + \sigma_e^2$.

A quantitative analysis of the effects of a random error can again be performed by evaluating the ratios of empirical and true process capability indices based on a exponential membership function for tolerance interval. We have :

$$\begin{aligned} C^e_{\tilde{p}(\exp)} &= \int_{-\infty}^{\infty} \exp\left(\frac{-(x-m)^2}{2\lambda^2}\right) \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(x-c)^2}{2(\sigma^2+\sigma_e^2)}\right) dx \\ &= \left(\frac{\lambda}{\sqrt{\lambda^2+\sigma^2+\sigma_e^2}}\right) \exp\left(\frac{-(\mu-m)^2}{2(\lambda^2+\sigma^2+\sigma_e^2)}\right). \end{aligned} \tag{4.7}$$

Therefore

$$\begin{aligned} \frac{C^e_{\tilde{p}(\exp)}}{C_{\tilde{p}(\exp)}} &= \frac{\left(\frac{\lambda}{\sqrt{\lambda^2+\sigma^2+\sigma_e^2}}\right) \exp\left(\frac{-(\mu-m)^2}{2(\lambda^2+\sigma^2+\sigma_e^2)}\right)}{\left(\frac{\lambda}{\sqrt{\lambda^2+\sigma^2}}\right) \exp\left(\frac{-(\mu-m)^2}{2(\lambda^2+\sigma^2)}\right)} \\ &= \frac{1}{\sqrt{1+y^2}} \exp\left[-\frac{z^2}{2}\left(\frac{y^2}{1+y^2}\right)\right], \end{aligned} \tag{4.8}$$

where $y = \frac{\sigma_e}{\sqrt{\lambda^2+\sigma^2}}$ and $z = \frac{\mu-m}{\sqrt{\lambda^2+\sigma^2}}$.

If one consider the error contamination degree y fixed and examine (4.8) as a function of the relative process level z only, one gets a $2D$ curve which is symmetric about $z = 0$, This graph results by vertically intersecting the $3D$ graph in Figure 7 parallel to the z, $\frac{C^e_{\tilde{p}(\exp)}}{C_{\tilde{p}(\exp)}}$- plane.

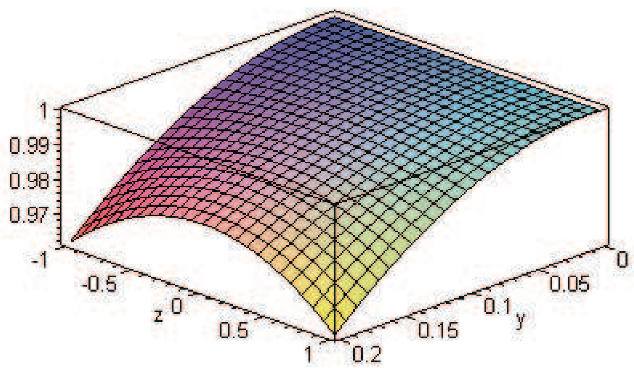

FIGURE 7. Graph of $\frac{C^e_{\tilde{p}(exp)}}{C_{\tilde{p}(exp)}}$ in the case of random error

At $z = 0$, this curve take on its minimum value $\frac{1}{\sqrt{1+y^2}}$.

$$C^e_{\tilde{p}(\exp)} = \frac{1}{\sqrt{1+y^2}} C_{\tilde{p}(\exp)}\ , \tag{4.9}$$

i.e. when $\mu = m$, $C_{\widetilde{p}(\exp)}$ will be decreasing by the coefficient $\left(1 - \frac{1}{\sqrt{1+y^2}}\right)$.

But in the traditional method, we have

$$\frac{C_p^e}{C_p} = \frac{C_{pk}^e}{C_{pk}} = \frac{1}{\sqrt{1+x^2}}, \tag{4.10}$$

where $x = \dfrac{\sigma_e}{\sigma}$. It is independent of μ and the tolerance interval. Figure 8 shows the quotient $\dfrac{C_{pk}^e}{C_{pk}}$ as a function of x.

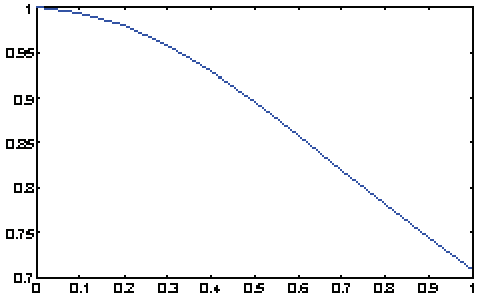

FIGURE 8. Graph of $\frac{C_p^e}{C_p}$ in the case of constant error

5. Numerical Examples

In this section, we calculate the presented fuzzy process capability index with some random generated data. We apply two common membership functions (i.e. Trapezoidal and exponential) for showing the fuzzy up-to-standard productions.

Example 5.1: In this example, we generate $N = 1000$ random numbers corresponding to a normal distribution with parameters $\mu = 74$ and $\sigma = 0.0125$. Figure 9 shows the histogram of generated numbers.

Let $[73.95, 74.05]$ be the standard range of quality characteristic (tolerance interval), we present it by the following exponential and trapezoidal membership functions

$$\widetilde{A}_{\exp}(x) = \exp\left[-\frac{1}{2}\left(\frac{x-74}{0.042}\right)^2\right],$$

and

$$\widetilde{A}_{\text{trap}}(x) = \begin{cases} \frac{x-73.92}{0.06} & \text{if} \quad 73.92 \le x < 73.98 \\ 1 & \text{if} \quad 73.98 \le x \le 74.02 \\ \frac{74.08-x}{0.06} & \text{if} \quad 74.02 < x \le 74.08 \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$$

Figure 10 shows the curves of $\widetilde{A}_{\exp}(x)$ and $\widetilde{A}_{trap}(x)$.

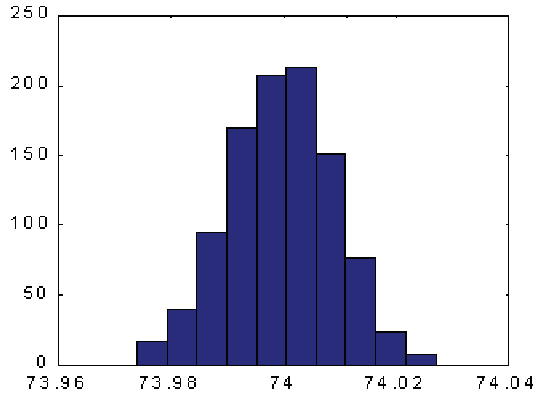

FIGURE 9. Histogram of generated data

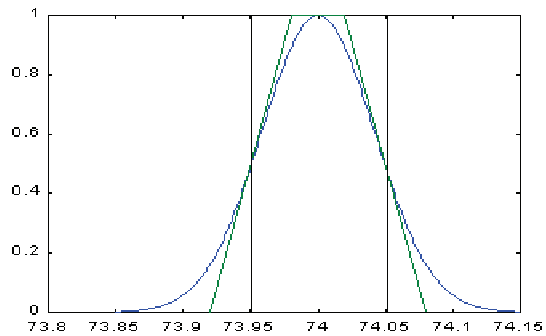

FIGURE 10. Curves of $\widetilde{A}_{exp}(x)$ and $\widetilde{A}_{trap}(x)$

The estimated values of the capability index $C_{\widetilde{p}}$ are given as follows:

$$\begin{aligned}
\widehat{C}_{\widetilde{p}(\text{trap})1} &= -\frac{s}{\sqrt{2\pi}(a-b)}\left[\exp\left(-\frac{(a-\overline{x})^2}{2s^2}\right) - \exp\left(-\frac{(b-\overline{x})^2}{2s^2}\right)\right] \\
&\quad + \frac{s}{\sqrt{2\pi}(c-d)}\left[\exp\left(-\frac{(c-\overline{x})^2}{2s^2}\right) e - \exp\left(-\frac{(d-\overline{x})^2}{2s^2}\right)\right] \\
&\quad + \frac{\overline{x}-a}{b-a}\left[\Phi(\frac{b-\overline{x}}{s}) - \Phi(\frac{a-\overline{x}}{s})\right] \\
&\quad + \frac{\overline{x}-d}{c-d}\left[\Phi(\frac{d-\overline{x}}{s}) - \Phi(\frac{c-\overline{x}}{s})\right] \\
&\quad + \left[\Phi(\frac{c-\overline{x}}{s}) - \Phi(\frac{b-\overline{x}}{s})\right]. \\
&= 0.9903,
\end{aligned}$$

$$\begin{aligned} \widehat{C}_{\widetilde{p}(\text{trap})2} &= \frac{1}{N}\sum_{i=1}^{N}\widetilde{A}_{\text{trap}}(x_i) \\ &= 0.9901, \end{aligned}$$

$$\begin{aligned} \widehat{C}_{\widetilde{p}(\exp)1} &= \frac{\lambda}{\sqrt{\lambda^2+\sigma^2}}\exp\left[-\frac{(\overline{x}-m)^2}{2\left(\lambda^2+\sigma^2\right)}\right] \\ &= 0.9585, \end{aligned}$$

$$\begin{aligned} \widehat{C}_{\widetilde{p}(\exp)2} &= \frac{1}{N}\sum_{i=1}^{N}\widetilde{A}_{\exp}(x_i) \\ &= 0.9585, \end{aligned}$$

and

$$\begin{aligned} \widehat{C}_{p(\text{classic})} &= \frac{Ts-Ti}{6s} \\ &= 1.3333. \end{aligned}$$

In this example the position of the distribution centre of characteristic X and the centre of tolerance interval coincide and the calculated indices express a capable process, that is, the percentage of non-conforming products is small.

Example 5.2 In this example, we generate $N = 1000$ random numbers corresponding to a normal distribution with parameters $\mu = 74.025$ and $\sigma = 0.0125$. Figure 11 shows the histogram of generated numbers.

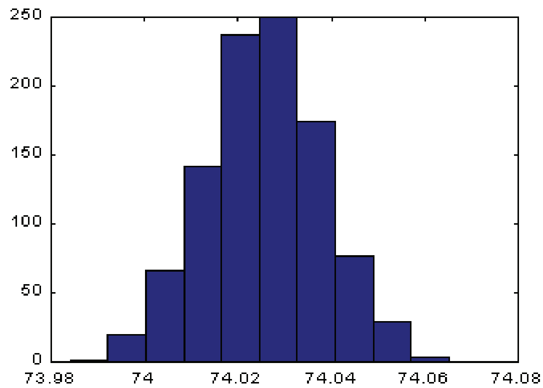

FIGURE 11. Histogram of generated data

By same tolerance interval and membership functions, the estimated values of capability index $C_{\widetilde{p}}$ are calculated as follows:

$$
\begin{aligned}
\widehat{C}_{\widetilde{p}(\mathrm{trap})1} &= -\frac{s}{\sqrt{2\pi}(a-b)}\left[\exp\left(-\frac{(a-\overline{x})^2}{2s^2}\right)-\exp\left(-\frac{(b-\overline{x})^2}{2s^2}\right)\right] \\
&\quad +\frac{s}{\sqrt{2\pi}(c-d)}\left[\exp\left(-\frac{(c-\overline{x})^2}{2s^2}\right)-\exp\left(-\frac{(d-\overline{x})^2}{2s^2}\right)\right] \\
&\quad +\frac{\overline{x}-a}{b-a}\left[\Phi(\frac{b-\overline{x}}{s})-\Phi(\frac{a-\overline{x}}{s})\right] \\
&\quad +\frac{\overline{x}-d}{c-d}\left[\Phi(\frac{d-\overline{x}}{s})-\Phi(\frac{c-\overline{x}}{s})\right] \\
&\quad +\left[\Phi(\frac{c-\overline{x}}{s})-\Phi(\frac{b-\overline{x}}{s})\right]. \\
&= 0.8687,
\end{aligned}
$$

$$
\begin{aligned}
\widehat{C}_{\widetilde{p}(\mathrm{trap})2} &= \frac{1}{N}\sum_{i=1}^{N}\widetilde{A}_{\mathrm{trap}}(x_i) \\
&= 0.8612,
\end{aligned}
$$

$$
\begin{aligned}
\widehat{C}_{\widetilde{p}(\exp)1} &= \frac{\lambda}{\sqrt{\lambda^2+\sigma^2}}\exp\left[-\frac{(\overline{x}-m)^2}{2\left(\lambda^2+\sigma^2\right)}\right] \\
&= 0.8145,
\end{aligned}
$$

$$
\begin{aligned}
\widehat{C}_{\widetilde{p}(\exp)2} &= \frac{1}{N}\sum_{i=1}^{N}\widetilde{A}_{\exp}(x_i) \\
&= 0.8078,
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{C}pi &= \frac{\overline{x}-Ti}{3s} \qquad \text{and} \qquad \widehat{C}ps = \frac{\overline{x}-Ti}{3s} \\
&= 2.0162, \qquad\qquad\qquad\quad = 0.6505
\end{aligned}
$$

$$
\begin{aligned}
\widehat{C}pk &= \min(\widehat{C}pi, \widehat{C}pu) \\
&= 0.6505.
\end{aligned}
$$

In this example the position of the distribution centre of characteristic X and the centre of tolerance interval do not coincide and the calculated indices express a non-capable process, that is, the percentage of non-conforming products is high.

6. Conclusions

In this paper we discussed on the process capability indices, specially on fuzzy process of capability which hasn't the shortcoming of the traditional method. Our essential goal is representing a method to describe the treatment of the produced items, where are approximately up-to-standard. It means, the measures which are in the neighborhood of the tolerance interval limits. These measures could be have an admissible interpretation by considering a membership function for representing. We have shown that like to traditionally method, random and constant measurement errors can considerably falsify the results of the fuzzy process capability analysis. This fact underlines the importance of ensuring gauge capability before evaluating process capability. The discussion also illustrates that the C_{pk} and $C_{\widetilde{p}}$ indices have very sensitively to normally distributed measurement errors.

One of the best applications of the fuzzy process of capability is comparing the quality of two production systems.

We can use this method for computing the capability index of a production system that its quality characteristic (X) is multivariate.

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