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Strong intuitionistic fuzzy topological groups

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Abstract. The fuzzy sets were introduced by Zadeh in 1965 and were extended to intuitionistic fuzzy sets by Atanassov (1986). A notion of a fuzzy topological group was proposed by Foster in 1979 especially he took a group and furnished it with a fuzzy topological structure. An equivalent notion of a fuzzy topological group was introduced by Ma and Yu (1984) by replacing points by fuzzy points. In this paper we extend the notion of strong fuzzy topological group by Lakshmana (2008) to strong intuitionistic fuzzy topological groups for a given group using the induced topology. We study properties of strong intuitionistic fuzzy topological groups, analyzing such entities as its connection with the previous notions, subgroups, images and products of strong fuzzy topological groups.

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1. Introduction

The concept of fuzzy sets was first introduced by Zadeh [22]. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups by Rosenfeld [20] as a natural generalization of the concept of subgroup. Anthony and Sherwood [1] redefined a fuzzy subgroup of a group using the concept of triangular norm. Several mathematicians have followed the Rosenfeld-Anthony-Sherwood approach in investigating the fuzzy subgroup theory. Fuzzy subgroups and anti-fuzzy subgroups were studied by Biswas [3]. Foster [5] introduced the concept of a fuzzy topological group using the Lowen's [17] definition of a fuzzy topological space. Ma and Yu [18] changed the definition of a fuzzy topological group in order to make sure that an ordinary topological group is a special case of a fuzzy topological group. Also Yu and Ma [18], [19], [21] characterized fuzzy topological groups, fuzzy normal subgroups, fuzzy quotient groups and the direct group of a finite family of fuzzy topological groups. Coker [4] and Hur et al. [10] studied the concept of intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets and investigated some of its properties. In particular, Hur et.al [11] applied the notion of intuitionistic fuzzy set to topological group. Hur et al. [6], [7], [8], [9] studied various properties of intuitionistic fuzzy subgroupoids, intuitionistic fuzzy subgroups, intuitionistic fuzzy sub-rings and intuitionistic fuzzy topological groups. The induced topology on fuzzy singletons was introduced in [14] and extended to the induced topology on intuitionistic fuzzy singletons by Lakshmana et al. [15]. The properties of fuzzy translation invariant spaces and intuitionistic fuzzy translation invariant spaces were studied by Lakshmana et al.[12], [16]. This paper is organized as follows: In section 2, the preliminaries are discussed. In section 3, we introduce few definitions, lemmas, theorems on the strong intuitionistic fuzzy topological groups. Finally the conclusions are drawn in section 4.

2. Preliminaries

Here we give a brief review of some preliminaries.

Definition 2.1 ([2]). Let X be a nonempty set. An intuitionistic fuzzy set (IFS) A of X is defined by $A = (\mu_A, \nu_A)$, where $\mu_A : X \to [0\,,\ 1]$ and $\nu_A : X \to [0\,,\ 1]$ with the condition $\mu_A(x) + \nu_A(x) \leq 1$. The numbers $\mu_A(x),\ \nu_A(x) \in [0\,,\ 1]$ denote the degree of membership and non-membership of x to lie in A respectively.

Definition 2.2 ([2]). Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFSs of X. Then

(i) $A \bigcap B\,(x) = (\min(\mu_A(x), \mu_B\,(x)), \max(\nu_A(x), \nu_B(x)))$
(ii) $A \bigcup B\,(x) = (\max(\mu_A(x), \mu_B\,(x)), \min(\nu_A(x), \nu_B(x)))$
(iii) $(A\,\times\,B)(x, y) = (\min\{\mu_A(x), \mu_B(y)\}, \max\{\nu_A(x), \nu_B(y)\})$.

Definition 2.3 ([4]). Let $\{A_i = (\mu_{A_i}, \nu_{A_i}) | i \in J\}$ be an arbitrary family of IFSs in X. Then (a) $\bigcap_{i\in J} A_i = (\wedge_{i\in J}\mu_{A_i},\ \vee_{i\in J}\nu_{A_i})$; (b) $\bigcup_{i\in J} A_i = (\vee_{i\in J}\mu_{A_i},\ \wedge_{i\in J}\nu_{A_i})$.

Definition 2.4 ([4]). Let $f : X \to Y$ be any function. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy subsets of X and Y respectively. Then

(i) The image $f(A) = (\mu_{f(A)}, \nu_{f(A)})$ of A under f is given by

$$\mu_{f(A)}(z) = \begin{cases} \sup\limits_{x\in f^{-1}(z)} \mu_A(x) & \text{if } f^{-1}(z) \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu_{f(A)}(z) = \begin{cases} \inf\limits_{x\in f^{-1}(z)} \nu_A(x) & \text{if } f^{-1}(z) \neq \varnothing, \\ 1 & \text{otherwise,} \end{cases}$$

(ii) The inverse image $f^{-1}(B) = (\mu_{f^{-1}(B)}\ ,\ \nu_{f^{-1}(B)})$ of B under f is given by $\mu_{f^{-1}(B)}\,(x) = \mu_B(f(x))$ and $\nu_{f^{-1}(B)}\,(x) = \nu_B(f(x))$.

Definition 2.5 ([7]). An intuitionistic fuzzy set (IFS) $A = (\mu_A, \nu_A)$ of a group G is called an intuitionistic fuzzy subgroup of G if, for all $x, y \in G$, the following conditions are satisfied:

(i) $\mu_A(xy) \geq\ \min\ (\mu_A(x), \mu_A(y))$ and $\nu_A(xy) \leq\ \max\ (\nu_A(x), \nu_A(y))$;
(ii) $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \leq \nu_A(x)$.

Example 2.6 ([9])**.** Let A be an intuitionistic fuzzy subgroup of G and let $x \in G$. The intuitionisitic fuzzy left coset xA of G determined by x and A, is given by $\mu_{xA}(z) = \mu_A(x^{-1}z)$ and $\nu_{xA}(z) = \nu_A(x^{-1}z)$, for every $z \in G$.

The notion of intuitionistic fuzzy points (Gallego (2003)) can be generalized to intuitionistic fuzzy singletons as follows

Definition 2.7 ([15])**.** Let X be a non empty set. An intuitionistic fuzzy singleton $p = \{(\mu_{p_x}, \nu_{p_x})\}$ defined on x is defined as

$$\mu_{p_x}(z) = \begin{cases} \alpha & \text{if } z = x, \\ 0 & \text{if } z \neq x, \end{cases} \quad \text{and} \quad \nu_{p_x}(z) = \begin{cases} \beta & \text{if } z = x, \\ 1 & \text{if } z \neq x, \end{cases}$$

such that $0 < \alpha, \beta \leq 1$ and $\alpha + \beta \leq 1$.

Note 2.8. If $p_x = \{(\mu_{p_x}, \nu_{p_x})\}$ defined on x is an intuitionistic fuzzy singleton, then support $\mu_{p_x} = \{x\}$, $0 < \mu_{p_x}(x) \leq 1$, $\nu_{p_x}(z) = 1$, for every $z \neq x$ and $\mu_{p_x}(x) + \nu_{p_x}(x) \leq 1$.

Definition 2.9 ([15])**.** Let X be any nonempty set. An intuitionistic fuzzy singleton p_x defined on x is said to lie in an intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ ($p_x \in A$) if $\mu_{p_x} \leq \mu_A$ and $\nu_A \leq \nu_{p_x}$.

Definition 2.10 ([4])**.** Let X be any nonempty set. An intuitionistic fuzzy topology on X is a collection δ of intuitionistic fuzzy subsets of X satisfying

1. $\tilde{0} = (0,\ 1)$, $\tilde{1} = (1,\ 0) \in \delta$.
2. Intersection of members of any finite sub-collection $\sigma \subseteq \delta$ is also a member of δ.
3. Union of members of any sub-collection $\sigma_o \subseteq \delta$ is again a member of δ.

The pair (X, δ) is called an intuitionistic fuzzy topological space. Elements of δ are called intuitionisitic fuzzy open sets.

Definition 2.11 ([15])**.** Let (G, δ) be an intuitionistic fuzzy topological space. The induced topology τ_δ on the collection $\wp(G)$ of all intuitionistic fuzzy singletons of G is defined as the topology generated by $B = \{V_A \mid A \in \delta\}$, where

$$V_A = \{p_x \in \wp(G) \mid p_x \in A\}$$

and hence $(\wp(G), \tau_\delta)$ is called an induced topological space on intuitionistic fuzzy singleton.

Definition 2.12 ([12])**.** Let (G_i, δ_i), $(i \in I)$ be an indexed family of intuitionistic fuzzy topological spaces. The product intuitionistic fuzzy topology $\prod \delta_i$ on $\prod G_i$ is defined as the smallest intuitionistic fuzzy topology in which all projection maps $\pi_i : \prod G_i \to G_i, \forall i$, are intuitionistic fuzzy continuous.

Theorem 2.13 ([15])**.** *Let $(G_1, \delta_1), (G_2, \delta_2)$ be intuitionistic fuzzy topological spaces. Then the induced topological space on $\wp(G_1 \times G_2)$ by the product intuitionistic fuzzy topological space $(G_1 \times G_2, \delta_1 \times \delta_2)$ is imbedded in the product topological space $(\wp(G_1) \times \wp(G_2), \tau_{\delta_1} \times \tau_{\delta_2})$*

Definition 2.14 ([4]). An intuitionistic fuzzy topological space (G, δ) is said to be intuitionistic fuzzy hausdorff if for every pair of points $x, y \in X$ such that $x \neq y$, there exists intuitionistic fuzzy open sets $A = (\mu_A, \nu_A)$,$B = (\mu_B, \nu_B) \in \delta$ such that $\mu_A(x) = 1$, $\mu_B(y) = 1$, $\nu_A(x) = 0$ and $\nu_B(y) = 0$ and $A \bigcap B = 0$.

Definition 2.15 ([13]). Let G be any group. A fuzzy hausdorff space (G, δ) is said to be a strong fuzzy topological group if

(i) $M : \wp(G) \times \wp(G) \to \wp(G)$ defined by $M(p_x, q_y) = p_x q_y$, for every $(p_x, q_y) \in \wp(G) \times \wp(G)$, is continuous.

(ii) $I : \wp(G) \to \wp(G)$ defined by $I(p) = p^{-1}$, for every $p \in \wp(G)$, is continuous.

3. Strong intuitionistic fuzzy topological groups

In this section, a new notion of strong intuitionistic fuzzy topological group is introduced and studied.

Definition 3.1. Let G be any group. An intuitionistic fuzzy hausdorff space (G, δ) is said to be a strong intuitionistic fuzzy topological group if

(i) $M : \wp(G) \times \wp(G) \to \wp(G)$ defined by $M(p_x, q_y) = p_x q_y$, for every $(p_x, q_y) \in \wp(G) \times \wp(G)$ is continuous where

$$p_x q_y(xy) = (\mu_{p_x q_y}(xy), \nu_{p_x q_y}(xy)) = (\min(\mu_{p_x}(x), \mu_{q_y}(y)), \max(\nu_{p_x}(x), \nu_{q_y}(y))).$$

(ii) $I : \wp(G) \to \wp(G)$ defined by $I(p_x) = p_x^{-1}$, for every $p_x \in \wp(G)$, is continuous.

Example 3.2. Let $G = \{e, x, y, xy\}$ be a Klein's four group. Let $a > \frac{1}{2}$. Let δ be the collection of intuitionistic fuzzy sets $A = (\mu_A,\, \nu_A)$ where $\mu_A(z) \in [0, a] \bigcup \{1\}$, $\nu_A(z) \in \{0\} \bigcup [1 - a,\, 1]$ respectively for every $z \in G$. Clearly (G, δ) is an intuitionistic fuzzy topological space. We claim that $M : \wp(G) \times \wp(G) \to \wp(G)$ defined by $M(p, q) = (\mu_{pq},\, \nu_{pq})(t_1 t_2) = (\min(\mu_p(t_1), \mu_q(t_2)), \max(\nu_p(t_1), \nu_q(t_2)))$, for every (μ_p, μ_q), $(\nu_p, \nu_q) \in \wp(G) \times \wp(G)$ is continuous. Let $(p_1, p_2) \in \wp(G) \times \wp(G)$ where supp $p_1 = \{t_1\}$ and supp $p_2 = \{t_2\}$. Let $V_A \in \sigma$ such that $M(p_1, p_2) = (\mu_{p_1 p_2},\, \nu_{p_1 p_2}) \in V_A$ and σ is a base for the induced topology τ_δ. Clearly supp $p_1 p_2 = \{t_1 t_2\}$.

Case 1: Intuitionistic Fuzzy value $\mu_{p_1 p_2} \leq a$ and $\nu_{p_1 p_2} \geq 1 - a$.
To prove this case, there arise the following subcases to prove

(i) $\mu_{p_1}(t_1) \leq a$, $\mu_{p_2}(t_2) > a$ and $\nu_{p_1}(t_1) \geq 1 - a$, $\nu_{p_2}(t_2) < 1 - a$.

(ii) $\mu_{p_1}(t_1) > a$, $\mu_{p_2}(t_2) \leq a$ and $\nu_{p_1}(t_1) < 1 - a$, $\nu_{p_2}(t_2) \geq 1 - a$.

(iii) $\mu_{p_1}(t_1) \leq a$, $\mu_{p_2}(t_2) \leq a$ and $\nu_{p_1}(t_1) \geq 1 - a$, $\nu_{p_2}(t_2) \geq 1 - a$

(iv) $\mu_{p_1}(t_1) \leq a$, $\mu_{p_2}(t_2) \leq a$ and $\nu_{p_1}(t_1) \geq 1 - a$, $\nu_{p_2}(t_2) < 1 - a$.

Now we shall prove subcase (i). The other subcases are similar. Now we have $\mu_{p_1}(t_1) \leq a$, $\nu_{p_1}(t_1) \geq 1 - a$ and $\mu_{p_2}(t_2) > a$, $\nu_{p_2}(t_2) < 1 - a$. Let $\mu_{A_1} = \mu_{p_1}$, $\nu_{A_1} = \nu_{p_1}$. Clearly $A_1 \in \delta$. Let $\mu_{A_2}(t) = 1$ if $t = t_2$ and $\mu_{A_2}(t) = 0$ if $t \neq t_2$. let $\nu_{A_2}(t) = 0$ if $t = t_2$ and $\nu_{A_2}(t) = 1$ if $t \neq t_2$. Clearly $A_2 \in \delta$ and hence V_{A_1}, $V_{A_2} \in \sigma$ and clearly $p_i \in V_{A_i}$ for $i = 1, 2$. Now we claim that $V_{A_1} V_{A_2} \subseteq V_A$. Let $p \in V_{A_1}$, $q \in V_{A_2}$ and hence $p \in A_1$ and $q \in A_2$. Clearly the only possibility of supports of p, q are t_1, t_2 respectively. Now $pq = (\mu_{pq}, \nu_{pq})$ is a intuitionistic fuzzy singleton defined on $t_1 t_2$ with value $\mu_{pq}(t_1, t_2) \leq \min\{\mu_p(t_1),\, \mu_q(t_2)\} \leq \mu_{p_1}(t_1)$ and $\nu_{pq}(t_1, t_2) \geq \max\{\nu_p(t_1),\, \nu_q(t_2)\} \geq \max\{\nu_{A_1}(t_1),\, \nu_{A_2}(t_2)\} = \nu_{A_1}(t_1) =$

$\nu_{p_1}(t_1)$. Since $p_1p_2 \in V_A$, $\mu_A(t_1t_2) \geq \mu_{p_1p_2}(t_1t_2) = \mu_{p_1}(t_1) \geq \mu_{pq}(t_1t_2)$ and $\nu_A(t_1t_2) \leq \nu_{p_1p_2}(t_1t_2) = \nu_{p_1}(t_1) \leq \nu_{pq}(t_1t_2)$ and hence $pq \in A$ and hence $pq \in V_A$.

Case 2 : Intuitionistic fuzzy value $\mu_{p_1p_2} > a$ and $\nu_{p_1p_2} < 1-a$.

Here $\mu_A(t_1t_2) \geq \min\{p_1(t_1), p_2(t_2)\} > a$ and so $\mu_A(t_1t_2) = 1$ and $\nu_A(t_1t_2) \leq 1-\mu_A(t_1t_2) < 1-a$ and hence $\nu_A(t_1t_2) = 0$. Let $\mu_{A_1}(t) = 1$ if $t = t_1$ and $\mu_{A_1}(t) = 0$ if $t \neq t_1$ and $\mu_{A_2}(t) = 1$ if $t = t_2$ and $\mu_{A_2}(t) = 0$ if $t \neq t_2$. Let $\nu_{A_1}(t) = 0$ if $t = t_1$ and $\nu_{A_1}(t) = 1$ if $t \neq t_1$ and $\nu_{A_2}(t) = 0$ if $t = t_2$ and $\nu_{A_2}(t) = 1$ if $t \neq t_2$. Clearly $A_1, A_2 \in \delta$ and V_{A_1} , $V_{A_2} \in \sigma$ and clearly $p_i \in V_{A_i}$ for $i = 1, 2$. Now we claim that $V_{A_1}V_{A_2} \subseteq V_A$. Let $p \in V_{A_1}$, $q \in V_{A_2}$. Now $\mu_{p_x}(t_1) \leq 1$ and $\mu_{q_y}(t_2) \leq 1$ and hence $\mu_{pq}(t_1t_2) \leq \min(\mu_p(t_1), \mu_q(t_2)) \leq 1 = \mu_A(t_1t_2)$. Now $\nu_p(t_1) \geq 0$ and $\nu_q(t_2) \geq 0$ and hence $\nu_{pq}(t_1t_2) \geq \max(\nu_p(t_1), \nu_q(t_2)) \geq 0 = \nu_A(t_1t_2)$ and hence $V_{A_1}V_{A_2} \subseteq V_A$. Hence M is continuous.

Now we prove I is continuous. In Klein's group, the inverse $x^{-1}$of every element x is itself. Hence $p^{-1} = p$. So I is the identity map, which is continuous. Clearly (G, δ) is a Hausdorff intuitionistisc fuzzy topological space and hence (G, δ) is a strong intuitionistic fuzzy topological group.

Theorem 3.3. *If* $M : \wp(G) \times \wp(G) \to \wp(G)$ *defined by* $M(p_x, q_y) = p_xq_y$*, for every* $(p_x, q_y) \in \wp(G) \times \wp(G)$ *is continuous, then* $m : G \times G \to G$ *defined by* $m(x, y) = xy$ *is intuitionistic fuzzy continuous.*

Proof. Let M be continuous. To prove that m is an intuitionistic fuzzy continuous, let $r = (\mu_r, \nu_r)$ be an intuitionistic fuzzy point in $G \times G$ with support $\{(x, y)\}$. Let $\mu_r(x, y) \in (0, 1)$ and $\nu_r(x, y) \in (0, 1)$. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy open set in G containing $m(r) = (\mu_{m(r)}, \nu_{m(r)})$. We have $\mu_{m(r)}(z) = \sup_{z_1z_2=z} r(z_1, z_2)$ and $\nu_{m(r)}(z) = \inf_{z_1z_2=z} r(z_1, z_2)$. So $m(r)$ is an intuitionistic fuzzy point defined on xy with value $r(x, y)$. Now define intuitionistic fuzzy singletons $r_1 = (\mu_{r_1}, \nu_{r_1})$ and $r_2 = (\mu_{r_2}, \nu_{r_2})$ on x, y respectively. An intuitionistic fuzzy singleton (μ_{r_1}, ν_{r_1}) has intuitionistic fuzzy values $\mu_A(xy)$ and $\nu_A(xy)$. An intuitionistic fuzzy singleton (μ_{r_2}, ν_{r_2}) has intuitionistic fuzzy values 1 and 0. Now $r_1r_2(xy) = (\mu_{r_1r_2}(xy)\,\nu_{r_1r_2}(xy)) = \{\min(\mu_{r_1}(x)\ \mu_{r_2}(y)),\ \max(\nu_{r_1}(x),\ \nu_{r_2}(y))\} = (\mu_A(xy), \nu_A(xy))$. Hence $r_1r_2 \in A$ and $r_1r_2 \in V_A$. Since M is continuous and $r_1, r_2 \in V_A \in \tau_\delta$, there exists $A_1 = (\mu_{A_1}, \nu_{A_1}) \in \tau_\delta$ and $A_2 = (\mu_{A_2}, \nu_{A_2}) \in \tau_\delta$ such that $r_1 \in V_{A_1}$, $r_2 \in V_{A_2}$ and $V_{A_1}V_{A_2} \subseteq V_A$. Since $r_1 = (\mu_{r_1}, \nu_{r_1}) \in A_1$, $r_2 = (\mu_{r_2}, \nu_{r_2}) \in A_2$, we have to prove that $r \in A_1 \times A_2$ i.e., $(\mu_r, \nu_r) \in (\mu_{A_1 \times A_2}, \nu_{A_1 \times A_2})$. Now

$$\begin{aligned}\mu_r(x, y) = \mu_{m(r)}(xy) < \mu_A(xy) &= \min(\mu_{r_1}(x),\ \mu_{r_2}(y)) \\ &\leq \min(\mu_{A_1}(x),\ \mu_{A_2}(y)) = \mu_{A_1 \times A_2}(x, y)\end{aligned}$$

and

$$\begin{aligned}\nu_r(x, y) = \nu_{m(r)}(xy) > \nu_A(xy) &= \max(\nu_{r_1}(x),\ \nu_{r_2}(y)) \\ &\geq \max(\nu_{A_1}(x)\,\nu_{A_2}(y)) = \nu_{A_1 \times A_2}(x, y).\end{aligned}$$

Therefore $r \in A_1 \times A_2$. Now we prove that $m(A_1 \times A_2) \subseteq A$ and by definition

$$\mu_m(A_1 \times A_2)(t) = \sup_{t_1t_2=t} \min(\mu_{A_1}(t_1),\ \mu_{A_2}(t_2)) = \mu_{A_1 \times A_2}(t_1, t_2).$$

$\nu_m(A_1 \times A_2)(t) = \inf_{t_1 t_2 = t} \max(\nu_{A_1}(t_1), \nu_{A_2}(t_2)) = \nu_{A_1 \times A_2}(t_1, t_2)$. $V_{A_1} V_{A_2} \subseteq V_A$ implies $\min\{\mu_{A_1}(x), \mu_{A_2}(y)\} \leq \mu_A(xy)$ and $\max\{\nu_{A_1}(x), \nu_{A_2}(y)\} \geq \nu_A(xy)$ $\forall x, y \in G$. So $\mu_{(A_1 \times A_2)}(t_1, t_2) \leq \mu_A(t_1 t_2) = \mu_A(t)$ and $\nu_{(A_1 \times A_2)}(t_1, t_2) \geq \nu_A(t_1 t_2) = \nu_A(t)$. Hence $\mu_m(A_1 \times A_2)(t) \leq \mu_A(t)$ and $\nu_m(A_1 \times A_2)(t) \geq \nu_A(t)$. Hence by the definition of intuitionistic fuzzy continuous, m is an intuitionistic fuzzy continuous. □

The proof of following theorem is easy.

Theorem 3.4. *If $M : \wp(G) \times \wp(G) \to \wp(G)$ defined by $M(p_x, q_y) = p_x q_y$, for every $(p_x, q_y) \in \wp(G) \times \wp(G)$ is intuitionistic fuzzy continuous, then $m : G \times G \to G$ defined by $m(x, y) = xy$ is fuzzy continuous.*

Definition 3.5 ([15])**.** Let X and Y be two nonempty sets. Let $f : X \to Y$ be any map. For any intuitionistic fuzzy singleton p_x defined on $x \in X$, if the function $i_f : \wp(X) \to \wp(Y)$ is defined by $i_f(p_x) = q_y$, where q_y is an intuitionistic fuzzy singleton defined on $f(x) \in Y$ with $q_y(f(x)) = p_x(x)$, then i_f is called the induced function of f.

Theorem 3.6 ([15])**.** *Let (X, δ) and (Y, σ) be an intuitionistic fuzzy topological spaces. A function $f : (X, \delta) \to (Y, \sigma)$ is a intuitionistic fuzzy continuous function if and only if the induced function $i_f : (\wp(X), \tau_\delta) \to (\wp(Y), \tau_\sigma)$ is continuous.*

Theorem 3.7. *The continuity of $I : \wp(G) \to \wp(G)$ defined by $I(p) = p^{-1}$ for every $p \in \wp(G)$ and the intuitionistic fuzzy continuity of $i : G \to G$ defined by $i(x) = x^{-1}$ are equivalent.*

Proof. Since I is the induced function of i, then by the above theorem, the continuity of I and the intuitionistic fuzzy continuity i are equivalent. □

Theorem 3.8. *If (G, δ) is a strong intuitionistic fuzzy topological group, then it is a intuitionistic fuzzy topological group.*

Proof. By Theorem 3.3, the continuity of M implies the intuitionistic fuzzy continuity of m, the condition of Definition 3.1. By Theorem 3.7, the continuity of I and the intuitionistic fuzzy continuity of i , the condition (ii) of the Definition 3.1 are equivalent. □

But the converse need not be true as seen in the following remark.

Remark 3.9. By Theorem 3.6, the intuitionistic fuzzy continuity of m and the continuity of i_m are equivalent. But i_m is defined on $\wp(G \times G) \to \wp(G)$. By Theorem 2.13, $\wp(G \times G) \to \wp(G)$ is intuitionistic fuzzy homeomorphic to a subspace of $\wp(G) \times \wp(G)$. So continuity of $M : \wp(G) \times \wp(G) \to \wp(G)$ need not be equivalent to the continuity of i_m which is equivalent to the intuitionistic fuzzy continuity of m.

Theorem 3.10. *Let (G, δ) be an intuitionistic fuzzy topological space on a group G. Then (G, δ) is a strong intuitionistic fuzzy topological group iff $M_I : \wp(G) \times \wp(G) \to \wp(G)$ defined by $M_I(p_x, q_y) = p_x q_y^{-1}$, for every $(p_x, q_y) \in \wp(G) \times \wp(G)$ is continuous.*

Proof. First we prove $\Rightarrow$ part. Let (G, δ) be a strong intuitionistic fuzzy topological group. Let $p_x = (\mu_{p_x}, \nu_{p_x})$ and $q_y = (\mu_{q_y}, \nu_{q_y})$ be an intuitionistic fuzzy singletons defined on x and y respectively. Let $f_1, f_2 : \wp(G) \to \wp(G)$ defined by $f_1(p_x) = p_x$, $f_2(q_y) = q_y^{-1}$, respectively. Clearly f_1 and f_2 are continuous. Hence the function $f : \wp(G) \times \wp(G) \to \wp(G) \times \wp(G)$ defined by $f(p_x, q_y) = (p_x, q_y^{-1})$ for every $(p_x, q_y) \in \wp(G) \times \wp(G)$ is continuous. Now $M_I(p_x, q_y) = p_x q_y = M(f(p_x, q_y)) = (M \circ f)\ (p_x, q_y)$ and hence M_I is continuous.
$\Leftarrow$ part. Let us assume that $M_I(p_x, q_y) = p_x q_y^{-1}$, for every $(p_x, q_y) \in \wp(G) \times \wp(G)$ is continuous. Clearly the function $g : \wp(G) \to \wp(G) \times \wp(G)$ defined by $g(p_x) = (1_e, p_x)$ is continuous and hence $I(p_x) = p_x^{-1} = M_I(g(p_x)) = (M_I \circ g)\,(p_x)$ is continuous. If $f_1 : \wp(G) \to \wp(G)$ is defined by $f_1(p_x) = p_x$, then $h : \wp(G) \times \wp(G) \to \wp(G) \times \wp(G)$ defined by $h(p_x, q_y) = (f_1, I)\ (p_x, q_y) = (p_x, q_y^{-1})$ is continuous. Now $M(p_x, q_y) = M_I(h(p_x, q_y)) = M_I \circ h$ and hence M is continuous. □

Theorem 3.11. *Let (G, δ) be a strong intuitionistic fuzzy topological group. Every subgroup of G is a strong intuitionistic fuzzy topological subgroup of G with its subspace topology.*

Proof. Let H be a subgroup of G. We have to prove that $(H, \delta|H)$ is also a strong intuitionistic fuzzy topological group. By hypothesis, $M_G : \wp(G) \times \wp(G) \to \wp(G)$ defined by $M_G(p_x, q_y) = p_x q_y$ for every $(p_x, q_y) \in \wp(G) \times \wp(G)$ is continuous. We have to prove that $M_H : \wp(H) \times \wp(H) \to \wp(H)$ defined by $M_H(p_x, q_y) = p_x q_y$, for every $(p_x, q_y) \in \wp(H) \times \wp(H)$ is continuous . Let $p_x, q_y \in \wp(H)$ i.e., $p_x = (\mu_{p_x}, \nu_{p_x}) \in \wp(H)$ and $q_y = (\mu_{q_y}, \nu_{q_y}) \in \wp(H)$ are the intuitionistic fuzzy singleton and let $V_A \in \tau_{\delta|H}$ be a basic intuitionistic fuzzy open set in the subspace $(H, \tau_{\delta|H})$ such that $\mu_{p_x q_y} \le \mu_A$ and $\nu_{p_x q_y} \ge \nu_A$. So $A = (\mu_A, \nu_A) \in \delta|H$, there exists an intuitionistic fuzzy open set $B = (\mu_B, \nu_B) \in \delta$ such that $\mu_{B\,|H} = \mu_A$ and $\nu_{B\,|H} = \nu_A$. Let $p_x^{'}, q_y^{'} \in \wp(G)$ such that $p_x^{'}|H = p_x$ and $q_y^{'}|H = q_y$ and so $V_B \in \tau_\delta$ with $\mu_{p_x^{'} q_y^{'}} \le \mu_B$ and $\nu_{p_x^{'} q_y^{'}} \ge \nu_B$ and hence $p_x^{'} q_y^{'} \in V_B$. By the continuity of M_G, there exists $V_{B_1}, V_{B_2} \in \tau_\delta$ such that $M_G(V_{B_1} \times V_{B_2}) \subseteq V_B$ and $p_x^{'} \in V_{B_1}$ and $q_y^{'} \in V_{B_2}$ i.e., $\mu_{p_x^{'}} \le \mu_{B_1}$ and $\mu_{q_y^{'}} \le \mu_{B_2}$ implies $\min(\mu_{p_x^{'}}, \mu_{q_y^{'}}) \le \min(\mu_{B_1}, \mu_{B_2})$ and $\nu_{p_x^{'}} \ge \nu_{B_1}$ and $\nu_{q_y} \ge \nu_{B_2}$ implies $\max(\nu_{p_x}, \nu_{q_y}) \ge \max(\nu_{B_1}, \nu_{B_2})$ hence $(p_x^{'}, q_y^{'}) \in B_1 \times B_2$. $\mu_{p_x} \le \mu_{B_1|H}$, $\mu_{q_y} \le \mu_{B_2||H}$ and $\nu_{p_x} \ge \nu_{B_1|H}$ and $\nu_{q_y} \ge \nu_{B_2|H}$ implies $p_x \in V_{B_1|H}$ and $q_y \in V_{B_2|H}$ and hence $(p_x, q_y) \in (B_1|H \times B_2|H)$. To prove that $M_H(V_{B_1|H} \times V_{B_2|H}) \subseteq V_{B\,|H}$. Let $(r, s) \in \wp(H)$ i.e., $r = (\mu_r, \nu_r) \in \wp(H)$ and $s = (\mu_s, \nu_s) \in \wp(H)$ be a intuitionistic fuzzy singleton defined on x and y such that $r \in V_{B_1|H}$ and $s \in V_{B_2|H}$. Now extend r and s to $r^{'} \in \wp(G)$ and $s^{'} \in \wp(G)$ where $r^{'}|H = r$ and $s^{'}|H = s$ and hence $M_G(r^{'}, s^{'}) \subseteq V_B$. Hence $r^{'} \in V_{B_1}$ and $s^{'} \in V_{B_2}$. Now $(r^{'}, s^{'}) \in B_1 \times B_2$ implies $r \in V_{B_1|H}$ and $s \in V_{B_2|H}$ and hence $M_H(V_{B_1|H} \times V_{B_2|H}) \subseteq V_{B\,|H}$. Hence M_H is continuous. Clearly $I_H = I_G|p(H)$ is continuous. Hence the theorem holds. □

Theorem 3.12. *Let $(G_1, *_1, \delta_1)$ and $(G_2, *_2, \delta_2)$ be strong intuitionistic fuzzy topological groups. Then $(G_1 \times G_2, *, \delta_1 \times \delta_2)$ is a strong intuitionistic fuzzy topological group.*

Proof. We note that $*((x_1,x_2),\ (y_1,y_2)) = (x_1 *_1 y_1, x_2 *_2 y_2)$. Clearly $(G_1 \times G_2,\ \delta_1 \times \delta_2)$ is intuitionistic fuzzy Hausdorff in the product topology. By Theorem 7.3 of Foster and Theorem 3.7, i is continuous.

Now to prove that $(G_1 \times G_2,\ *, \delta_1 \times \delta_2)$ is a strong intuitionistic fuzzy topological group, it is enough to prove that $M : \wp(G_1{\times}G_2) \times \wp(G_1{\times}G_2) \to \wp(G_1{\times}G_2)$ defined by $M(p_x, q_y) = p_x q_y$ is continuous, where p_x and q_y is intuitionistic fuzzy singletons defined on (x_1, x_2) and (y_1, y_2) respectively. To prove that M is continuous, let $((\mu_{p_x}, \nu_{p_x}),\ (\mu_{q_y}, \nu_{q_y})) \in \wp(G_1 \times G_2) \times \wp(G_1 \times G_2)$ and $V_A \in \tau_{\delta_1 \times \delta_2}$ with $A = (\mu_A, \nu_A) \in \delta_1 \times \delta_2$ such that $M(p_x, q_y) = p_x q_y \in V_A$. So

$$\mu_{pq}(x_1 *_1 y_1, x_2 *_2 y_2) \le \mu_A(x_1 *_1 y_1, x_2 *_2 y_2)$$

and

$$\nu_{pq}(x_1 *_1 y_1, x_2 *_2 y_2) \ge \nu_A(x_1 *_1 y_1, x_2 *_2 y_2).$$

Since $A = (\mu_A, \nu_A) \in \delta_1 \times \delta_2$, $\mu_A = \vee(\mu_{A_\alpha} \times \mu_{B_\alpha})$ and $\nu_A = \wedge(\nu_{A_\alpha} \times \nu_{B_\alpha})$. Hence

$$\mu_{p_x q_y}(x_1 *_1 y_1, x_2 *_2 y_2) \le \min(\vee\mu_{A_\alpha}(x_1 *_1 y_1),\ \vee\mu_{B_\alpha}(x_2 *_2 y_2))$$

and $\nu_{p_x q_y}(x_1 *_1 y_1, x_2 *_2 y_2) \ge \max(\wedge\nu_{A_\alpha}(x_1 *_1 y_1),\ \wedge\nu_{B_\alpha}(x_2 *_2 y_2))$. Let $\vee A_\alpha = A_1$ and $\vee B_\alpha = A_2$. We note that $A_i \in \delta_i$. Hence

$$\mu_{p_x q_y}(x_1 *_1 y_1, x_2 *_2 y_2) \le\ \min(\mu_{A_1}(x_1 *_1 y_1),\ \mu_{A_2}(x_2 *_2 y_2))$$

and $\nu_{p_x q_y}(x_1 *_1 y_1, x_2 *_2 y_2) \ge\ \max(\nu_{A_1}(x_1 *_1 y_1),\ \nu_{A_2}(x_2 *_2 y_2))$. Now we define intuitionistic fuzzy singletons $p_1 = (\mu_{p_1}, \nu_{p_1})$,$p_2 = (\mu_{p_2}, \nu_{p_2})$ on x_1, x_2 with intuitionistic fuzzy value $\mu_p(x_1, x_2)$ and $\nu_p(x_1, x_2)$. We also define intuitionistic fuzzy singletons $q_1 = (\mu_{q_1}, \nu_{q_1})$, $q_2 = (\mu_{q_2}, \nu_{q_2})$ on y_1, y_2 with intuitionistic fuzzy value $\mu_q(x_1, x_2)$ and $\nu_q(x_1, x_2)$. Here we note that $p_1, q_1 \in \wp(G_1)$ and $p_2, q_2 \in \wp(G_2)$. Clearly $\mu_{p_1 q_1}(x_1 *_1 y_1) = \mu_{pq}(x_1 *_1 y_1, x_2 *_2 y_2) \le \mu_{A_1}(x_1 *_1 y_1)$ and $\nu_{p_1 q_1}(x_1 *_1 y_1) = \nu_{pq}(x_1 *_1 y_1, x_2 *_2 y_2) \ge \nu_{A_1}(x_1 *_1 y_1)$ and $\mu_{p_1 q_1} \le \mu_{A_1}$ and $\nu_{p_1 q_1} \ge \nu_{A_1}$ and so hence $p_1 q_1 \in V_{A_1}$. So we have $(p_i, q_i) \in \wp(G_i) \times \wp(G_i)$ and $p_i q_i \in V_{A_i} \in \tau_{\delta_i}$. Let $M : \wp(G_i) \times \wp(G_i) \to \wp(G_i)$ be defined by $M_i(a_i, b_i) = a_i *_i b_i$ where $a_i, b_i \in \wp(G_i)$. Since $(G_i, *_i, \delta_i)$ are strong intuitionistic fuzzy topological groups, M_i are continuous. Hence by continuity of M_i, there exists $V_{B_i} \times V_{C_i}$ such that $(p_i, q_i) \in V_{B_i} \times V_{C_i}$ and $M_i(V_{B_i} \times V_{C_i}) \subseteq V_{A_i}$. Hence $p_i \le B_i$ and $q_i \le C_i$. Hence $p = p_1 \times p_2 \le B_1 \times B_2$ and $q = q_1 \times q_2 \le C_1 \times C_2$. Let $B = B_1 \times B_2$ and $C = C_1 \times C_2$. Clearly $(p, q) \in V_B \times V_C$. To prove M is continuous, it is enough to prove that $M(V_B \times V_C) \subseteq V_A$. Let $r \in V_B$ and $s \in V_C$, where r and s are intuitionistic fuzzy points defined on (z_1, z_2) and (t_1, t_2). We have to prove that $rs \in V_A$. Now

$$\begin{aligned}
\mu_{rs}(z_1 *_1 t_1, z_2 *_2 t_2) &= \min\{\mu_r(z_1, z_2),\ \mu_s(t_1, t_2)\} \le \min\{\mu_B(z_1, z_2),\ \ \mu_C(t_1, t_2)\} \\
&= \min\{\mu_{B_1} \times \mu_{B_2}(z_1, z_2),\ \mu_{C_1} \times \mu_{C_2}(t_1, t_2)\} \\
&= \min\{\mu_{B_1} \times \mu_{C_1}(z_1, t_1),\ \mu_{B_2} \times \mu_{C_2}(z_2, t_2)\} \\
&\le \min\{\mu_{A_1}(z_1 *_1 t_1),\ \mu_{A_2}(z_2 *_2 t_2)\} \\
&= \mu_{A_1} \times \mu_{A_2}(z_1 *_1 t_1,\ z_2 *_2 t_2)
\end{aligned}$$

and

$$\begin{aligned}\nu_{rs}(z_1 *_1 t_1, z_2 *_2 t_2) &= \max\{\nu_r(z_1,z_2),\ \nu_s(t_1,t_2)\} \geq \max\{\nu_B(z_1,z_2),\ \ \nu_C(t_1,t_2)\}\\ &= \max\{\nu_{B_1}\times\nu_{B_2}(z_1,z_2),\ \ \nu_{C_1}\times\nu_{C_2}(t_1,t_2)\}\\ &= \max\{\nu_{B_1}\times\nu_{C_1}(z_1,t_1),\ \nu_{B_2}\times\nu_{C_2}(z_2,t_2)\}\\ &\geq \max\{\nu_A(z_1 *_1 t_1),\ \nu_{A_2}(z_2 *_2 t_2)\}\\ &= \nu_{A_1}\times\nu_{A_2}(z_1 *_1 t_1,\ z_2 *_2 t_2)\end{aligned}$$

Hence $rs \in V_{A_1\times A_2} \subseteq V_A$ and so the theorem is true. □

Theorem 3.13. *Let $f : (G,\delta) \to (G^{'},\sigma)$ be an injective intuitionistic fuzzy continuous intuitionistic fuzzy open homomorphism. Then the image of a strong intuitionistic fuzzy topological subgroup H of (G,δ) is again a strong intuitionistic fuzzy topological subgroup of $(G^{'},\sigma)$.*

Proof. Let $f : (G,\delta) \to (G^{'},\sigma)$ be an injective intuitionistic fuzzy continuous intuitionistic fuzzy open homomorphism. We have to prove that $(f(H),\ \sigma|f(H))$ is a strong intuitionistic fuzzy topological subgroup of $(G^{'},\sigma)$. By Theorem 3.10, it suffices to prove that $M_{I_{f(H)}} : \wp(f(H))\times\wp(f(H)) \to \wp(f(H))$ defined by $M_{I_{f(H)}}(q_1,q_2) = q_1q_2^{-1}$, for every $(q_1,q_2) \in \wp(f(H))\times\wp(f(H))$, is continuous. Let intuitionistic fuzzy singletons $(q_1,q_2) \in \wp(f(H))\times\wp(f(H))$ i.e., $((\mu_{q_1},\nu_{q_1}),(\mu_{q_2},\nu_{q_2})) \in \wp(f(H))\times\wp(f(H))$ whose supports are y_1 and y_2 respectively. Let $A = (\mu_A,\nu_A)$ be an intuitionistic fuzzy open set in $G^{'}|f(H)$ and $V_A \in \tau_{\sigma|f(H)}$ be a basic intuitionistic fuzzy open set in $f(H)$ with $\mu_{q_1q_2^{-1}} \leq \mu_A$ and $\nu_{q_1q_2^{-1}} \geq \nu_A$ and hence $q_1q_2^{-1} \in V_A$. By definition, there exists $B = (\mu_B,\nu_B) \in \sigma$ such that $A = B|f(H)$ i.e., $\mu_A = \mu_B|f(H)$ and $\nu_A = \nu_B|f(H)$. Since f is an intuitionistic fuzzy continuous, $f^{-1}(B) \in \delta$. Now we define intuitionistic fuzzy singletons $p_1 = (\mu_{p_1},\nu_{p_1})$, $p_2 = (\mu_{p_2},\nu_{p_2})$ on x_1, x_2 respectively where $x_1,x_2 \in H$ with $f(x_1) = y_1$ and $f(x_2) = y_2$. An intuitionistic fuzzy singleton $p_1 = (\mu_{p_1},\nu_{p_1})$ has an intuitionistic fuzzy values $\mu_{q_1(y_1)}$ and $\nu_{q_1(y_1)}$. An intuitionistic fuzzy singleton $p_2 = (\mu_{p_2},\nu_{p_2})$ has an intuitionistic fuzzy values $\mu_{q_2(y_2)}$ and $\nu_{q_2(y_2)}$. Since H is a subgroup and f is intuitionistic fuzzy homomorphism,

$$\begin{aligned}\mu_{p_1p_2^{-1}}(x_1x_2^{-1}) &= \min\,(\mu_{p_1}(x_1),\mu_{p_2^{-1}}(x_2^{-1})) = \min(\mu_{q_1}(y_1),\ \mu_{q_2^{-1}}(y_2^{-1}))\\ &= \mu_{q_1q_2^{-1}}(y_1y_2^{-1}) \leq \mu_A(y_1y_2^{-1})\\ &= \mu_{B\,|H}(y_1y_2^{-1}) = \mu_{f^{-1}(B)\ |\ H}(x_1x_2^{-1}),\end{aligned}$$

$$\begin{aligned}\nu_{p_1p_2^{-1}}(x_1x_2^{-1}) &= \max\,(\nu_{p_1}(x_1),\nu_{p_2^{-1}}(x_2^{-1})) = \max(\nu_{q_1}(y_1),\ \nu_{q_2^{-1}}(y_2^{-1}))\\ &= \nu_{q_1q_2^{-1}}(y_1y_2^{-1}) \geq \nu_A(y_1y_2^{-1}) = \nu_{B\,|H}(y_1y_2^{-1})\\ &= \nu_{f^{-1}(B)\ |\ H}(y_1y_2^{-1}).\end{aligned}$$

Now $p_1p_2^{-1} \in V_{f^{-1}(B)\ |\ H}$. Since H is a strong intuitionistic fuzzy topological group, there exists $B_1,\ B_2 \in \delta|H$ such that $(p_1,p_2) \in V_{B_1}\times V_{B_2}$ and $M_{I_H}(V_{B_1}\times V_{B_2}) \subseteq V_{f^{-1}(B)\ |\ H}$. Clearly there exists $A_1,A_2 \in \delta$ such that $A_1|H = B_1$, $A_2|H = B_2$. Since f is intuitionistic fuzzy open, $f(A_1)$, $f(A_2) \in \sigma$ with $f(A_1)|f(H) = f(B_1)$, $f(A_2)|f(H) = f(B_2)$. $f(A_2)|f(H) = f(B_2)$. Since $p_1 \in B_1$ implies $\mu_{p_1} \leq \mu_{B_1} =$

$\mu_{A|H}$ and $\nu_{p_1} \geq \nu_{B_1} = \nu_{A\,|H}$. Here $\mu_{q_1}(y_1) = \mu_{p_1}(x_1) \leq \mu_{B_1}(x_1) = \mu_{f(B_1)}(y_1)$ and $\nu_{q_1}(y_1) = \nu_{p_1}(x_1) \geq \nu_{B_1}(x_1) = \nu_{f(B_1)}(y_1)$. Clearly $(q_1, q_2) \in V_{f(B_1)} \times V_{f(B_2)}$. Now we claim that $M_{I_{f(H)}}(V_{f(B_1)} \times V_{f(B_2)}) \subseteq V_A$. Let $(q_1^{'}, q_2^{'}) \in V_{f(B_1)} \times V_{f(B_2)}$. Since f is an intuitionistic fuzzy injective, there exists $(p_1^{'}, p_2^{'}) \in V_{B_1} \times V_{B_2}$ with $f(p_1^{'}) = q_1^{'}$, $f(p_2^{'}) = q_2^{'}$. Since $M_{I_H}(V_{B_1} \times V_{B_2}) \subseteq V_{f^{-1}(B)\,|\,H}$ and $\mu_{p_1^{'}p_2^{'}{}^{-1}} \leq \mu_{f^{-1}(B)\,|\,H}$ and $\nu_{p_1^{'}p_2^{'}{}^{-1}} \geq \nu_{f^{-1}(B)\,|\,H}$. So

$$\begin{aligned}\mu_{q_1^{'}q_2^{'}{}^{-1}}(y_1y_2^{-1}) &= \mu_{p_1^{'}p_2^{'}{}^{-1}}(x_1x_2^{-1}) \leq \mu_{f^{-1}(B)\,|H}(x_1x_2^{-1})\\ &= \mu_B(f(x_1)f(x_2)^{-1}) = \mu_A(y_1y_2^{-1})\end{aligned}$$

and

$$\begin{aligned}\nu_{q_1^{'}q_2^{'}{}^{-1}}(y_1y_2^{-1}) &= \nu_{p_1^{'}p_2^{'}{}^{-1}}(x_1x_2^{-1}) \geq \nu_{f^{-1}(B)\,|\,H}(x_1x_2^{-1})\\ &= \nu_B(f(x_1)f(x_2)^{-1}) = \nu_A(y_1y_2^{-1})\end{aligned}$$

Hence the theorem is true. □

Theorem 3.14. *Let $f : (G, \delta) \to (G^{'}, \sigma)$ be a intuitionistic fuzzy continuous intuitionistic fuzzy open homomorphism such that every intuitionistic fuzzy open set of (G, δ) is f- invariant. Then the image of a strong intuitionistic fuzzy topological subgroup H of (G, δ) is again a strong intuitionistic fuzzy topological subgroup of $(G^{'}, \sigma)$.*

Proof. Let $f : (G, \delta) \to (G^{'}, \sigma)$ be a intuitionistic fuzzy continuous intuitionistic fuzzy open homomorphism such that every intuitionistic fuzzy open set of (G, δ) is f- invariant. We have to prove that $(f(H),\ \sigma|f(H))$ is a strong intuitionistic fuzzy topological subgroup of $(G^{'}, \sigma)$. By theorem 3.10, it suffices to prove that $M_{I_{f(H)}} : \wp(f(H)) \times \wp(f(H)) \to \wp(f(H))$ defined by $M_{I_{f(H)}}(q_1, q_2) = q_1q_2^{-1}$, for every $(q_1, q_2) \in \wp(f(H)) \times \wp(f(H))$, is continuous. Let intuitionistic fuzzy singletons $(q_1, q_2) \in \wp(f(H)) \times \wp(f(H))$ i.e., $((\mu_{q_1}, \nu_{q_1}), (\mu_{q_2}, \nu_{q_2})) \in \wp(f(H)) \times \wp(f(H))$ whose supports are y_1 and y_2 respectively. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy open set in $G^{'}|f(H)$ and $V_A \in \tau_{\sigma|f(H)}$ be a basic intuitionistic fuzzy open set in $f(H)$ with $\mu_{q_1q_2^{-1}} \leq \mu_A$ and $\nu_{q_1q_2^{-1}} \geq \nu_A$ and hence $q_1q_2^{-1} \in V_A$. By definition, there exists $B = (\mu_B, \nu_B) \in \sigma$ such that $A = B|f(H)$ i.e., $\mu_A = \mu_B|f(H)$ and $\nu_A = \nu_B|f(H)$. Since f is an intuitionistic fuzzy continuous, $f^{-1}(B) \in \delta$. Now we define intuitionistic fuzzy singletons $p_1 = (\mu_{p_1}, \nu_{p_1})$, $p_2 = (\mu_{p_2}, \nu_{p_2})$ on x_1,x_2 respectively where $x_1, x_2 \in H$with $f(x_1) = y_1$ and $f(x_2) = y_2$. An intuitionistic fuzzy singleton $p_1 = (\mu_{p_1}, \nu_{p_1})$ has intuitionistic fuzzy values $\mu_{q_1(y_1)}$ and $\nu_{q_1(y_1)}$. An intuitionistic fuzzy singleton $p_2 = (\mu_{p_2}, \nu_{p_2})$ has intuitionistic fuzzy values $\mu_{q_2(y_2)}$ and $\nu_{q_2(y_2)}$. Since H is a subgroup and f is intuitionistic fuzzy homomorphism,

$$\begin{aligned}\mu_{p_1p_2^{-1}}(x_1{\cdot}x_2^{-1}) &= \min\ (\mu_{p_1}(x_1), \mu_{p_2^{-1}}(x_2^{-1})) = \min(\mu_{q_1}(y_1),\ \mu_{q_2^{-1}}(y_2^{-1}))\\ &= \mu_{q_1q_2^{-1}}(y_1y_2^{-1}) \leq \mu_A(y_1y_2^{-1}) = \mu_{B\,|H}(y_1y_2^{-1}) = \mu_{f^{-1}(B)\,|\,H}(x_1x_2^{-1}),\end{aligned}$$

$$\begin{aligned}\nu_{p_1p_2^{-1}}(x_1{\cdot}x_2^{-1}) &= \max\ (\nu_{p_1}(x_1), \nu_{p_2^{-1}}(x_2^{-1})) = \max(\nu_{q_1}(y_1),\ \nu_{q_2^{-1}}(y_2^{-1}))\\ &= \nu_{q_1q_2^{-1}}(y_1y_2^{-1}) \geq \nu_A(y_1y_2^{-1}) = \nu_{B\,|H}(y_1y_2^{-1}) = \nu_{f^{-1}(B)\,|\,H}(y_1y_2^{-1}),\end{aligned}$$

$p_1p_2^{-1} \in V_{f^{-1}(B)\,|\,H}$. Since H is a strong intuitionistic fuzzy topological group, there exists $B_1,\ B_2 \in \delta|H$ such that $(p_1,p_2) \in V_{B_1} \times V_{B_2}$ and $M_{I_H}(V_{B_1} \times V_{B_2}) \subseteq V_{f^{-1}(B)\,|H}$. Clearly there exists $A_1, A_2 \in \delta$ such that $A_1|H = B_1,\ A_2|H = B_2$. Since f is intuitionistic fuzzy open, $f(A_1),\ f(A_2) \in \sigma$ with $f(A_1)|f(H) = f(B_1)$, $f(A_2)|f(H) = f(B_2)$. Since $p_1 \in B_1$ implies $\mu_{p_1} \leq \mu_{B_1} = \mu_{A|H}$ and $\nu_{p_1} \geq \nu_{B_1} = \nu_{A\,|H}$. Here $\mu_{q_1}(y_1) = \mu_{p_1}(x_1) \leq \mu_{B_1}(x_1) = \mu_{f(B_1)}(y_1)$ and $\nu_{q_1}(y_1) = \nu_{p_1}(x_1) \geq \nu_{B_1}(x_1) = \nu_{f(B_1)}(y_1)$. Clearly $(q_1,q_2) \in V_{f(B_1)} \times V_{f(B_2)}$. Now we claim that $M_{I_{f(H)}}(V_{f(B_1)} \times V_{f(B_2)}) \subseteq V_A$. Since B_1 and B_2 are $f-$ invariant, then $f(B_1) = B_1$ and $f(B_2) = B_2$. Since $M_{I_H}(V_{B_1} \times V_{B_2}) \subseteq V_{f^{-1}(B)\,|H}$ and $f-$ invariant, $M_{I_{f(H)}}(V_{f(B_1)} \times V_{f(B_2)}) \subseteq V_A$ and hence the theorem holds. □

The following theorem is easy to prove.

Theorem 3.15. *Let $f : (G,\delta) \to G'$ be an intuitionistic fuzzy homomorphism on a fuzzy topological space (G,δ). Let $\sigma = \{f(\mu)|\mu \in \delta\}$. Then the image of a strong intuitionistic fuzzy topological subgroup H of (G,δ) is again a strong intuitionistic fuzzy topological subgroup of (G',σ).*

Theorem 3.16. *Let $f : (G,\delta) \to (G',\sigma)$ be an injective intuitionistic fuzzy continuous intuitionistic fuzzy open homomorphism. Then the inverse image of a strong intuitionistic fuzzy topological subgroup H of (G',σ) is again a strong intuitionistic fuzzy topological subgroup of (G,δ).*

Proof. Let $f : (G,\delta) \to (G',\sigma)$ be an injective intuitionistic fuzzy continuous intuitionistic fuzzy open homomorphism. We have to prove that $(f^{-1}(H),\ \delta|f^{-1}(H))$ is a strong intuitionistic fuzzy topological subgroup of (G,δ). By theorem 3.10, it suffices to prove that $M_{I_{f^{-1}(H)}} : \wp(f^{-1}(H)) \times \wp(f^{-1}(H)) \to \wp(f^{-1}(H))$ defined by $M_{I_{f^{-1}(H)}}(p_1,p_2) = p_1p_2^{-1}$, for every $(p_1,p_2) \in \wp(f^{-1}(H)) \times \wp(f^{-1}(H))$, is continuous. Let intuitionistic fuzzy singletons $(p_1,p_2) \in \wp(f^{-1}(H)) \times \wp(f^{-1}(H))$ i.e., $((\mu_{p_1},\nu_{p_1}),(\mu_{p_2},\nu_{p_2})) \in \wp(f^{-1}(H)) \times \wp(f^{-1}(H))$ whose supports are x_1 and x_2 respectively. Let $A = (\mu_A,\nu_A)$ be an intuitionistic fuzzy open set in G and $V_A \in \tau_{\delta|f^{-1}(H)}$ be a basic intuitionistic fuzzy open set in $f^{-1}(H)$ with $\mu_{p_1p_2^{-1}} \leq \mu_A$ and $\nu_{p_1p_2^{-1}} \geq \nu_A$ and hence $p_1p_2^{-1} \in V_A$. By definition, there exists $B = (\mu_B,\nu_B) \in \delta$ such that $A = B|f^{-1}(H)$ i.e., $\mu_A = \mu_B|f^{-1}(H)$ and $\nu_A = \nu_B|f^{-1}(H)$. Since f is intuitionistic fuzzy open, $f(B) \in \sigma$. Now we define intuitionistic fuzzy singletons $q_1 = (\mu_{q_1},\nu_{q_1}),\ q_2 = (\mu_{q_2},\nu_{q_2})$ on $y_1,\ y_2$ respectively where $y_1,y_2 \in H$ with $f(x_1) = y_1$ and $f(x_2) = y_2$. An intuitionistic fuzzy singleton $q_1 = (\mu_{q_1},\nu_{q_1})$ has intuitionistic fuzzy values $\mu_{p_1(x_1)}$ and $\nu_{p_1(x_1)}$. An intuitionistic fuzzy singleton $q_2 = (\mu_{q_2},\nu_{q_2})$ has intuitionistic fuzzy values $\mu_{p_2(x_2)}$ and $\nu_{p_2(x_2)}$. Since H is a subgroup and f is intuitionistic fuzzy homomorphism,

$$\begin{aligned}\mu_{q_1q_2^{-1}}(y_1y_2^{-1}) &= \min\,(\mu_{q_1}(y_1),\,\mu_{q_2^{-1}}(y_2^{-1})) = \min(\mu_{p_1}(x_1),\ \mu_{p_2^{-1}}(x_2^{-1}))\\ &= \mu_{p_1p_2^{-1}}(x_1x_2^{-1}) \leq \mu_A(x_1x_2^{-1}) = \mu_{f^{-1}(B\,|\,H)}(x_1x_2^{-1}) \leq \mu_{f(B)\,|\,H}(y_1y_2^{-1})\end{aligned}$$

and

$$\begin{aligned}\nu_{q_1q_2^{-1}}(y_1y_2^{-1}) &= \max\,(\nu_{q_1}(y_1),\,\nu_{q_2^{-1}}(y_2^{-1})) \geq \max(\nu_{p_1}(x_1),\,\nu_{p_2^{-1}}(x_2^{-1}))\\ &= \nu_{p_1p_2^{-1}}(x_1x_2^{-1}) \geq \nu_A(x_1x_2^{-1}) = \nu_{f^{-1}(B\,|\,H)}(x_1x_2^{-1}) \geq \nu_{f(B)\,|\,H}(y_1y_2^{-1}).\end{aligned}$$

So $q_1q_2^{-1} \in V_{f(B)\mid H}$. Since H is a strong intuitionistic fuzzy topological group, there exists $B_1,\ B_2 \in \sigma|H$ such that $(q_1, q_2) \in V_{B_1} \times V_{B_2}$ and $M_{I_H}(V_{B_1} \times V_{B_2}) \subseteq V_{f(B)\mid H}$. Clearly there exists $A_1, A_2 \in \sigma$ such that $A_1|H = B_1,\ A_2|H = B_2$. Since f is intuitionistic fuzzy continuous, $f^{-1}(A_1),\ f^{-1}(A_2) \in \delta$ with $f^{-1}(A_1)|f^{-1}(H) = f^{-1}(B_1),\ f^{-1}(A_2)|f^{-1}(H) = f^{-1}(B_2)$. Clearly $(p_1, p_2) \in V_{f^{-1}(B_1)} \times V_{f^{-1}(B_2)}$. Now we claim that $M_{I_{f^{-1}(H)}}(V_{f^{-1}(B_1)} \times V_{f^{-1}(B_2)}) \subseteq V_A$. Let $(p_1^{'}, p_2^{'}) \in V_{f^{-1}(B_1)} \times V_{f^{-1}(B_2)}$. Hence there exists $(q_1^{'}, q_2^{'}) \in V_{B_1} \times V_{B_2}$ with $f(p_1^{'}) = q_1^{'},\ f(p_2^{'}) = q_2^{'}$. Since $M_{I_H}(V_{B_1} \times V_{B_2}) \subseteq V_{f(B)\mid H}$, $\mu_{q_1^{'}q_2^{'-1}} \leq \mu_{f(B)\mid H}$ and $\nu_{q_1^{'}q_2^{'-1}} \geq \nu_{f(B)\mid H}$. Now

$$\begin{aligned}\mu_{p_1^{'}p_2^{'-1}}(x_1x_2^{'-1}) &= \mu_{p_1^{'}p_2^{'-1}}(x_1x_2^{'-1}) \leq \mu_{f(B)\mid H}(y_1y_2^{'-1})\\ &= \mu_B(x_1x_2^{'-1}) = \mu_A(x_1x_2^{'-1})\end{aligned}$$

and

$$\begin{aligned}\nu_{p_1^{'}p_2^{'-1}}(x_1x_2^{'-1}) &= \nu_{p_1^{'}p_2^{'-1}}(x_1x_2^{'-1}) \leq \nu_{f(B)\mid H}(y_1y_2^{'-1})\\ &= \nu_B(f(y_1)f(y_2^{'})^{-1}) = \nu_B(x_1x_2^{'-1}) = \nu_A(x_1x_2^{'-1})\end{aligned}$$

by intuitionistic fuzzy injectiveness of f and hence the theorem holds. □

The proof of the following theorems is easy.

Theorem 3.17. *Let $f : (G, \delta) \to (G^{'}, \sigma)$ be an intuitionistic fuzzy continuous intuitionistic fuzzy open homomorphism such that every fuzzy open set of (G, δ) is f-invariant. Then the inverse image of a strong intuitionistic fuzzy topological subgroup H of $(G^{'}, \sigma)$ is again a strong intuitionistic fuzzy topological subgroup of $(G^{'}, \delta)$.*

Theorem 3.18. *Let $f : G \to G^{'}$ be a homomorphism. Let $(G^{'}, \sigma)$ be an intuitionistic fuzzy topological space. Let $\delta = \{f^{-1}(\nu)|\nu \in \sigma\}$. Then the inverse image of a strong intuitionistic fuzzy topological subgroup H of $(G^{'}, \sigma)$ is again a strong fuzzy intuitionistic topological subgroup of $(G^{'}, \delta)$.*

Note 3.19. If (G, δ) is an intuitionistic fuzzy topological space, then (G, δ_1) is a fuzzy topological space, where $\delta_1 = \{(\mu_A,\ 1 - \mu_A)|A \in \delta\}$.

Theorem 3.20. *If (G, δ) is a strong intuitionistic fuzzy topological group, then (G, δ_1) is a strong fuzzy topological group.*

Proof. Let (G, δ) be a strong intuitionistic fuzzy topological group. So $M : (\wp(G) \times \wp(G), \tau_\delta \times \tau_\delta) \to (\wp(G), \tau_\delta)$ and $I : (\wp(G), \tau_\delta) \to (\wp(G), \tau_\delta)$ are continuous. To prove that $M_f : (f\wp(G) \times f\wp(G), \tau_{\delta_1} \times \tau_{\delta_1}) \to (f\wp(G), \tau_{\delta_1})$ is continuous, let p,q be fuzzy singletons such that $(p, q) \in f\wp(G) \times f\wp(G)$ and let $A = (\mu_A,\ 1 - \mu_A)$ be a fuzzy open set in δ_1 and $pq \in V_A$. Now we have to prove that there exist $A_1, A_2 \in \delta_1$ such that $p \in V_{A_1}$ and $q \in V_{A_2}$ and $M_f(V_{A_1} \times V_{A_2}) = V_{A_1}V_{A_2} \subseteq V_A$. Clearly $(p, q) \in (\wp(G), \tau_\delta) \times (\wp(G), \tau_\delta)$ such that $pq \in (\wp(G), \tau_\delta)$. Since $A \in \delta_1$, there exists $A^{'} = (\mu_{A'}, \nu_{A'}) \in \delta$ such that $\mu_{A'} = \mu_A$. Since $pq \in V_A$ we have $\mu_{pq} \leq \mu_A$. Now $\nu_{pq} = 1 - \mu_{pq} \geq 1 - \mu_A = 1 - \mu_{A'} \geq \nu_{A'}$. Hence $pq \in A^{'} = (\mu_{A'}, \nu_{A'})$. Since M is continuous and $pq \in V_{A'} \in \tau_\delta$ we have $V_{A_1^{'}}, V_{A_2^{'}} \in \tau_\delta$ such that $p \in V_{A_1^{'}}$, $q \in V_{A_2^{'}}$ and $V_{A_1^{'}}V_{A_2^{'}} \subseteq V_{A'}$. Let $A_1 = (\mu_{A_1^{'}},\ 1 - \mu_{A_1^{'}}) \in \delta_1$ and $A_2 = (\mu_{A_2^{'}},\ 1 - \mu_{A_2^{'}}) \in \delta_1$. Clearly

$p \in A_1$ and $q \in A_2$. Hence $p \in V_{A_1}$ and $q \in V_{A_2}$. To prove that $V_{A_1}V_{A_2} \subseteq V_A$, let $p_1 \in V_{A_1}$ and $p_2 \in V_{A_2}$. By the choice of p_1 and p_2, $p_1 \in V_{A_1'}$ and $p_2 \in V_{A_2'}$. So $p_1p_2 \in V_{A'}$. By definition of $A^{'}$, $p_1p_2 \in V_A$ and hence $p \in V_{A_1}$ and $q \in V_{A_2}$, $M_f(V_{A_1} \times V_{A_2}) \subseteq V_A$.

Now to prove $I : (f\wp(G), \tau_\delta) \to (f\wp(G), \tau_\delta)$ is continuous, let p be fuzzy singleton such that $p \in f\wp(G)$ and let $A = (\mu_A, 1 - \mu_A)$ be a fuzzy open set in δ_1 and $p^{-1} \in V_A$. Now we have to prove that there exist $A_1 \in \delta_1$ such that $p \in V_{A_1}$ and $I(V_{A_1}) \subseteq V_A$. Clearly $(p^{-1}) \in (\wp(G), \tau_\delta)$. Since $A \in \delta_1$, there exists $A^{'} = (\mu_{A'}, \nu_{A'}) \in \delta$ such that $\mu_{A'} = \mu_A$. Since $p^{-1} \in V_A$ we have $\mu_{p^{-1}} \leq \mu_A$. Now $\nu_{p^{-1}} = 1 - \mu_{p^{-1}} \geq 1 - \mu_A = 1 - \mu_{A'} \geq \nu_{A'}$. Hence $p^{-1} \in A^{'} = (\mu_{A'}, \nu_{A'})$. Since I is continuous and $p^{-1} \in V_{A'} \in \tau_\delta$ we have $V_{A_1'} \in \tau_\delta$ such that $p \in V_{A_1'}$ and $I(V_{A_1'}) \subseteq V_{A'}$. Let $A_1 = (\mu_{A_1'}, 1 - \mu_{A_1'}) \in \delta_1$. Clearly $p \in A_1$. Hence $p \in V_{A_1}$. To prove that $I(V_{A_1}) \subseteq V_A$, let $p_1 \in V_{A_1}$. By the choice of p_1, $p_1 \in V_{A_1'}$. So $p_1^{-1} \in V_{A'}$. By definition of $A^{'}$, $p_1^{-1} \in V_A$ and hence $p \in V_{A_1}$, $I(V_{A_1}) \subseteq V_A$. Hence the theorem is true. □

4. Conclusions

In this paper, the definitions, lemma and theorems of the strong intuitionistic fuzzy topological groups are introduced and discussed.

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