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On anti fuzzy ideals of Γ-rings

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Abstract. In this paper, we introduce the concepts of anti fuzzy ideals of a $\Gamma-$ring , a $\Gamma-$residue class ring and anti level subset of a fuzzy set. In particular, we discuss the relation of an anti fuzzy ideals of Γ-rings and the anti level subset of fuzzy sets, and give some characterizations of Artinian and Noetherian Γ-rings in terms of anti fuzzy ideals.

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1. Introduction

The notion of a fuzzy set in a set was introduced by L. A. Zadeh [13] in 1965, and since then, this concept has provided a useful mathematical tool for describing the behavior of systems that are too complex or illdefined to admit precise mathematical analysis by classical methods and tools. N. Nobusawa [9] introduced the notion of a Γ-ring, a concept more general than a ring. W. E. Barnes [1] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. W. E. Barnes, T. S. Ravisankar [12] and J. Luh [7] studied the structure of Γ-rings and obtained various generalizations analogous to corresponding parts in ring theory. W. Liu [6] has studied fuzzy ideals of a ring. Y. B. Jun and C. Y. Lee [5] and M. A. Öztürk et al. [11] applied the concept of fuzzy sets to the theory of Γ-rings, they gave the notion of fuzzy ideals in a Γ-ring and some properties of fuzzy ideals of Γ-rings. Then, S. M. Hong and Y. B. Jun [3] defined the normalized fuzzy ideals and fuzzy maximal ideals in a Γ-ring and Jun [4] further characterized the fuzzy prime ideals of a Γ-ring. Furthermore, M. A. Öztürk et al. [10] characterized the Artinian and Noetherian Γ-rings. R. Biswas [2] introduced the concept of anti fuzzy subgroups of groups in 1990, then many researchers such as T. K. Mukherjee et al. [8] are engaged in extending the concepts. In this article, by use of the method presented by the article [10] and the article [11], we propose the concept of anti fuzzy ideals of

Γ-rings and the anti level subset of a fuzzy set, investigate some related properties of them, study the relation of anti fuzzy ideals of Γ-rings and anti level subsets of fuzzy sets. In particular, we discuss the characterizations of Artinian Γ-rings, and give a condition for a Γ-ring to be Noetherian.

2. Preliminaries

Definition 2.1. Let M and Γ be two additive abelian groups. If for all $x,\ y,\ z \in M$ and all $\alpha,\ \beta \in \Gamma$, the following conditions are satisfied:

- $x\alpha y \in M$,
- $(x+y)\alpha z = x\alpha z + y\alpha z,\ x(\alpha+\beta)y = x\alpha y + x\beta y,\ x\alpha(y+z) = x\alpha y + x\alpha z,$
- $(x\alpha y)\beta z = x\alpha(y\beta z)$.

then M is called a Γ-ring.

Definition 2.2. A subset A of the Γ-ring M is a left (resp. right) ideal of M if A is an additive subgroup of M and the set

$$M\Gamma A = \{x\alpha y |\ x \in M,\ \alpha \in \Gamma,\ y \in A\}\ (A\Gamma M)$$

is contained in A. If A is both a left and a right ideal, then A is a two-side ideal, or simply an ideal of M.

Definition 2.3. Let U be an ideal of a Γ-ring M. If for each $a+U,\ b+U$ in the factor group M/U, and each $\gamma \in \Gamma$, we define $(a+U)\gamma(b+U) = a\gamma b+U$, then M/U is a Γ-ring which is called the Γ-residue class ring of M with respect to U.

Definition 2.4. A Γ-ring M is said to satisfy the left (resp. right) ascending chain condition of left (resp. right) ideals (or to be left (resp. right) Noetherian) if every strictly increasing sequence $U_1 \subset U_2 \subset U_3 \subset ...$ of left (resp. right) ideals of M is of finite length. A Γ-ring M is said to satisfy the left (resp. right) descending chain condition of left (resp. right) ideals (or to be left (resp. right) Artinian) if every strictly decreasing sequence $V_1 \supset V_2 \supset V_3 \supset ...$ of left (resp. right) ideals of M is of finite length. A Γ-ring M is left (resp. right) Noetherian if M satisfies the left (resp. right) ascending chain condition on left (resp. right) ideals. M is said to be Noetherian if M is both left and right Noetherian. A Γ-ring M is left (resp. right) Artinian if M satisfies the left (resp. right) descending chain condition on left (resp. right) ideals. M is said to be Artinian if M is both left and right Artinian.

3. Anti fuzzy ideals

Definition 3.1. A fuzzy set μ in a Γ-ring M is called an anti fuzzy left (resp. right) ideal of M if

- $\mu(x-y) \leq \max\{\mu(x),\ \mu(y)\}$,
- $\mu(x\alpha y) \leq \mu(y) \quad (\mu(x\alpha y) \leq \mu(x))$, for all $x,\ y \in M$ and all $\alpha \in \Gamma$.

A fuzzy set μ in a Γ-ring M is called an anti fuzzy ideal of M if μ is both an anti fuzzy left and an anti fuzzy right ideal of M.

We note that μ is an anti fuzzy ideal of M if and only if

- $\mu(x-y) \leq \max\{\mu(x),\ \mu(y)\}$,
- $\mu(x\alpha y) \leq \min\{\mu(x),\ \mu(y)\}$,

for all $x,\ y \in M$ and all $\alpha \in \Gamma$.

Throughout this paper, all proofs are going to proceed the only left cases, as the right cases are obtained from similar method. In order to facilitate discussion, we denote the zero element of a $\Gamma-$ring M by 0 .

Example 3.2. If G and H are additive abelian groups and $M = Hom(G,\ H)$, $\Gamma = Hom(H,\ G)$, then M is a Γ-ring with the operations pointwise addition and composition of homomorphisms. Define a fuzzy set $\mu : M \to [0,1]$ by $\mu(0) = t_1,\ \mu(f) = t_2,\ 0 \leq t_1 < t_2 \leq 1$, where f is any member of M with $f \neq 0$. Routine calculations give that μ is an anti fuzzy left (resp. right) ideal of M.

Definition 3.3. Let μ be a fuzzy set in a set S. For $t \in [0,1]$, the set

$$\mu_t := \{x \in S | \mu(x) \leq t\}$$

is called an anti level subset of μ.

Theorem 3.4. *Let μ be a fuzzy set of a Γ-ring M. Then μ is an anti fuzzy left (resp. right) ideal of M if and only if μ_t is a left (resp. right) ideal of M for all $t \in Im(\mu)$.*

Proof. Suppose that μ is an anti fuzzy left ideal of M. Let $x,\ y \in \mu_t$, then $\mu(x) \leq t$ and $\mu(y) \leq t$. It follows that

$$\mu(x-y) \leq \max\{\mu(x),\ \mu(y)\} \leq t,$$

i.e., $x - y \in \mu_t$. Now let $x \in\ M,\ \alpha \in \Gamma$ and $y \in \mu_t$. Since μ is an anti fuzzy left ideal of M, $\mu(x\alpha y) \leq \mu(y) \leq t$. Therefore, $x\alpha y \in \mu_t$ and μ_t is a left ideal of M.

Conversely, assume that μ_t is a left ideal of M. Let us prove that

$$\mu(x-y) \leq \max\{\mu(x),\ \mu(y)\} \tag{3.1}$$

and

$$\mu(x\alpha y) \leq \mu(y) \tag{3.2}$$

for all $x,\ y \in M$ and all $\alpha \in \Gamma$.

Firstly, If (3.1) is not true, then

$$\mu(x-y) > \max\{\mu(x),\ \mu(y)\}$$

for some $x,\ y \in M$. For these elements $x,\ y$, there exist $t_i,\ t_j \in Im(\mu)$, say $t_i < t_j$, such that $\mu(x) = t_i,\ \mu(y) = t_j$. Then, on the one hand, $x \in \mu_{t_i},\ y \in \mu_{t_j},\ \mu_{t_i} \subseteq \mu_{t_j}$ so $x - y \in \mu_{t_j}$; on the other hand,

$$\mu(x-y) > \max\{\mu(x),\ \mu(y)\} = t_j,$$

and so $x - y \notin \mu_{t_j}$. This is a contradiction.

Secondly, if (3.2) is not true, then for a fixed $\alpha \in \Gamma$, there exist $x,\ y \in M$ such that $\mu(x\alpha y) > \mu(y)$. Let $s_i,\ s_j \in Im(\mu)$ be such that $s_i < s_j,\ \mu(x) = s_i$, and $\mu(y) = s_j$, Then $\mu(x\alpha y) < \mu(y) = s_j$ and so $x\alpha y \notin \mu_{s_j}$, on the other hand, $x \in \mu_{s_i} \subseteq \mu_{s_j},\ y \in \mu_{s_j},\ x\alpha y \in \mu_{s_j}$, a contradiction. This completes the proof. □

Theorem 3.5. *Let* $S = \{\lambda_n \in (0,\ 1) \mid n \in \mathbf{N}\} \cup \{1\}$ *(where* $\lambda_1 > \lambda_2 > \lambda_3 > ...$ *) and* $\{U_n \mid n \in \mathbf{N}\}$ *be a family of left ideals of a* Γ*-ring* M *such that* $U_1 \supset U_2 \supset U_3 \supset ...$*, then a fuzzy set* μ *in* M *defined by*

$$\mu(x) = \begin{cases} \lambda_1, & x \in U_1, \\ \lambda_{n-1}, & x \in U_{n-1} \setminus U_n,\ n = 2, 3, ..., \\ 1, & x \in M \setminus \cap_{n=1}^{\infty} U_n, \end{cases}$$

is an anti fuzzy left ideal of M*.*

Proof. Using Theorem 3.4, the proof is straightforward. □

Theorem 3.6. *Let* μ *be an anti fuzzy left (resp. right) ideal of a* Γ*-ring* M *and* $\mu_{t_1},\ \mu_{t_2}$ *with* $(t_1 < t_2)$ *are two anti level subsets of* μ*, then* μ_{t_1} *and* μ_{t_2} *are equal if and only if there is no* $x \in M$ *such that* $t_1 < \mu(x) \le t_2$*.*

Proof. Suppose $t_1 < t_2$ and $\mu_{t_1} = \mu_{t_2}$. If there exists $x \in M$ such that $t_1 < \mu(x) \le t_2$, then μ_{t_1} is a proper subset of μ_{t_2}. This is a contradiction.

Conversely, assume that there is no $x \in M$ such that $t_1 < \mu(x) \le t_2$. From $t_1 < t_2$, it follows that $\mu_{t_1} \subseteq \mu_{t_2}$. If $x \in \mu_{t_2}$, i.e., $\mu(x) \le t_2$, and so $\mu(x) \le t_1$ because $\mu(x) \not> t_1$, Hence $x \in \mu_{t_1}$, $\mu_{t_2} \subseteq \mu_{t_1}$, therefore $\mu_{t_1} = \mu_{t_2}$. This completes the proof. □

Theorem 3.7. *Let* M *be a* Γ*-ring. If* μ *is an anti fuzzy left (resp. right) ideal of* M*, and* $Im(\mu) = \{t_0,\ t_1,\ t_2,\ ...,\ t_n\}$*, where* $t_0 < t_1 < t_2 < ... < t_n$*, then* $\mu_{t_i} (i = 1,\ 2,\ ...,\ n)$ *constitutes all the anti level subsets of* μ*.*

Proof. Let $t \in [0, 1]$ and $t \notin Im(\mu)$. If $t > t_n$, then $\mu_{t_n} \subseteq \mu_t$. Since $\mu_{t_n} = M$, it follows that $\mu_t = M$, so $\mu_t = \mu_{t_n}$. Assume that $t_i < t < t_{i+1} (1 \le i \le n-2)$, then there is no $x \in M$ such that $t < \mu(x) \le t_{i+1}$ for $Im(\mu) = \{t_0,\ t_1,\ t_2,\ ...,\ t_n\}$. From Theorem 3.6, we have that $\mu_t = \mu_{t_{i+1}}$. This shows that the anti level set μ_t is in $\{\mu_{t_i} \mid 1 \le i \le n\}$ for any $t \in [0, 1]$ with $t \ge \mu(0)$. □

Theorem 3.8. *Let* A *be a nonempty subset of a* Γ*-ring* M*. If* μ *is a fuzzy set in* M *such that*

$$\mu(x) = \begin{cases} 0, & x \in A, \\ 1, & x \notin A. \end{cases}$$

Then μ *is an anti fuzzy left (resp. right) ideal of* M *if and only if* A *is a left (resp. right) ideal of* M*.*

Proof. Assume that μ is an anti fuzzy left ideal of M. Let $x,\ y \in A$, then $x - y \in A$, $\mu(x) = \mu(y) = \mu(x - y) = 0$. Thus $\mu(x - y) \le \max\{\mu(x),\ \mu(y)\} = 0$. Considering $\mu(x - y) \ge 0$, so we obtain $\mu(x - y) = 0$. This means that $\mu(x - y) \in A$. Therefor A is an additive subgroup of M. Let $x \in M,\ y \in A,\ \alpha \in \Gamma$. Then $\mu(x\alpha y) \le \mu(y) = 0$, considering $\mu(x\alpha y) \ge 0$, hance $\mu(x\alpha y) = 0$. i.e., $x\alpha y \in A$, and A is a left ideal of M.

Conversely, let $x,\ y \in M$ and $\alpha \in \Gamma$, we consider the following four cases:

(1) If $x,\ y \in A$, then $x - y \in A$, $\mu(x) = \mu(y) = 0$ and

$$\mu(x - y) = 0 \le \max\{\mu(x),\ \mu(y)\} = 0,$$

(2) If $x,\ y \notin A$, then $x - y \notin A$, $\mu(x) = \mu(y) = 1$ and

$$\mu(x - y) = 1 \le \max\{\mu(x),\ \mu(y)\} = 1,$$

(3) If $x \in A,\ y \notin A$, then $x - y \notin A,\ \mu(x) = 0,\ \mu(y) = 1$ and

$$\mu(x - y) = 1 \leq \max\{\mu(x),\ \mu(y)\} = 1,$$

(4) If $x \notin A,\ y \in A$, then it is similar to (3).
So $\mu(x - y) \leq \max\{\mu(x),\ \mu(y)\}$. Now if $y \in A$, then $x\alpha y \in A$ for A is a left ideal of M, hence $\mu(x\alpha y) = 0 = \mu(y)$. if $y \notin A$, then $\mu(y) = 1$ and so $\mu(x\alpha y) \leq \mu(y)$. Therefore μ is an anti fuzzy left ideal of M. □

Definition 3.9. Let μ be a fuzzy set in M and f a function defined on M, then the fuzzy set ν in $f(M)$ defined by

$$\nu(y) = \inf_{x \in f^{-1}(y)} \mu(x)$$

for all $y \in f(M)$, is called the image of μ under f. Similarly, if ν is a fuzzy set in $f(M)$, then the fuzzy set $\mu = \nu \circ f$ in M (that is, the fuzzy set defined by $\mu(x) = \nu(f(x))$) for all $x \in M$ is called the preimage of ν under f.

Theorem 3.10. *A $\Gamma-$homomorphic preimage of an anti fuzzy left (resp. right) ideal is an anti fuzzy left (resp. right) ideal.*

Proof. If $\theta : M \to N$ is a $\Gamma-$homomorphism of Γ-rings, ν is an anti fuzzy left ideal of N and μ is the preimage of ν under θ. Then

$$\begin{aligned} \mu(x - y) &= \nu(\theta(x - y)) \\ &= \nu(\theta(x) - \theta(y)) \\ &\leq \max\{\nu(\theta(x)),\ \nu(\theta(y))\} \\ &= \max\{\mu(x),\ \mu(y)\} \end{aligned}$$

and

$$\begin{aligned} \mu(x\alpha y) &= \nu(\theta(x\alpha y)) \\ &= \nu(\theta(x)\alpha\theta(y)) \\ &\leq (\nu(\theta(y)) \\ &= \mu(y) \end{aligned}$$

for all $x,\ y \in M$ and $\alpha \in \Gamma$. This completes the proof. □

Definition 3.11. Let μ be a fuzzy set in a set M. Then μ has inf property if N is a subset of M, there exists $n_0 \in N$ such that

$$\mu(n_0) = \inf_{n \in N} \mu(n).$$

Theorem 3.12. *A $\Gamma -$ homomorphic image of an anti fuzzy left (resp. right) ideal which has the inf property is an anti fuzzy left (resp. right) ideal.*

Proof. Let $\theta : M \to N$ be a $\Gamma -$homomorphism of Γ-rings, μ an anti fuzzy left ideal of M with the inf property and ν the image of μ under θ. Given $\theta(x),\ \theta(y) \in \theta(M)$, let $x_0 \in \theta^{-1}(\theta(x)),\ y_0 \in \theta^{-1}(\theta(y))$ be such that

$$\mu(x_0) = \inf_{n \in \theta^{-1}(\theta(x))} \mu(n), \quad \mu(y_0) = \inf_{n \in \theta^{-1}(\theta(y))} \mu(n)$$

respectively. Then

$$\begin{aligned}
\nu(\theta(x)-\theta(y)) &= \inf_{z\in\theta^{-1}(\theta(x)-\theta(y))}\mu(z)\\
&\leq \mu(x_0-y_0)\\
&\leq \max\{\mu(x_0),\ \mu(y_0)\}\\
&= \max\{\inf_{n\in\theta^{-1}\theta(x)}\mu(n),\ \inf_{n\in\theta^{-1}\theta(y)}\mu(n)\}\\
&= \max\{\nu(\theta(x)),\ \nu(\theta(y))\}
\end{aligned}$$

and for any $\alpha\in\Gamma$, we have,

$$\begin{aligned}
\nu(\theta(x)\alpha\theta(y)) &= \inf_{z\in\theta^{-1}(\theta(x)\alpha\theta(y))}\mu(z)\\
&\leq \mu(x_0\alpha y_0)\\
&\leq \mu(y_0)\\
&= \inf_{n\in\theta^{-1}(\theta(y))}\mu(n)\\
&= \nu(\theta(y)).
\end{aligned}$$

This completes the proof. □

Theorem 3.13. *Assume that U is an ideal of a Γ-ring M, μ is an anti fuzzy left (resp. right) ideal of M, and $\mu(x)$ is monotonous with respect to x, then the fuzzy set $\bar{\mu}$ of M/U defined by*

$$\bar{\mu}(a+U)=\inf_{x\in U}\mu(a+x)$$

is an anti fuzzy left (resp. right) ideal of the $\Gamma-$ residue class ring M/U of M with respect to U.

Proof. Let $a,\ b\in M$ be such that $a+U=b+U$. Then $b-a\in U$, therefor $b=a+y$ for some $y\in U$, and so

$$\bar{\mu}(b+U)=\inf_{x\in U}\mu(b+x)=\inf_{x\in U}\mu(a+y+x)=\inf_{x+y=z\in U}\mu(a+z)=\bar{\mu}(a+U).$$

Hence $\bar{\mu}$ is well-defined. For any $x+U,\ y+U\in M/U$ and $\alpha\in\Gamma$, we have,

$$\begin{aligned}
\bar{\mu}((x+U)-(y+U)) &= \bar{\mu}((x-y)+U))\\
&= \inf_{z\in U}\mu((x-y)+z)\\
&= \inf_{z=u-v\in U}\mu((x-y)+(u-v))\\
&= \inf_{u,v\in U}\mu((x+u)-(y+v))\\
&\leq \inf_{u,v\in U}\max\{\mu(x+u),\ \mu(y+v)\}\\
&= \max\{\inf_{u\in U}\mu(x+u),\ \inf_{v\in U}\mu(y+v)\}\\
&= \max\{\bar{\mu}(x+U),\ \bar{\mu}(y+U)\}.
\end{aligned}$$

and

$$\begin{aligned}\bar{\mu}((x+U)\alpha(y+U)) &= \bar{\mu}((x\alpha y)+U)) \\ &= \inf_{z\in U}\mu((x\alpha y)+z) \\ &\leq \inf_{z\in U}\mu(x\alpha y)+x\alpha z) \quad \text{since} \quad x\alpha z\in U \\ &= \inf_{z\in U}\mu(x\alpha(y+z)) \\ &\leq \inf_{z\in U}\mu(y+z) \\ &= \bar{\mu}(y+U).\end{aligned}$$

This completes the proof. □

Theorem 3.14. *Let U be an ideal of a Γ-ring M. If $\bar{\mu}$ with $\bar{\mu}(a+u)=\mu(a)$(where $a\in M$) is an anti fuzzy left (resp. right) ideal of M/U, then μ is an anti fuzzy left (resp. right) ideal of M.*

Proof. Assume that $\bar{\mu}$ is an anti fuzzy left ideal of M/U, For every $x,\ y\in M$ and $\alpha\in\Gamma$, we have

$$\begin{aligned}\mu(x-y) &= \bar{\mu}((x-y)+U)) \\ &= \bar{\mu}((x+U)-(y+U)) \\ &\leq \max\{\bar{\mu}(x+U),\ \bar{\mu}(y+U)\} \\ &= \max\{\mu(x),\ \mu(y)\}.\end{aligned}$$

$$\begin{aligned}\mu(x\alpha y) &= \bar{\mu}((x\alpha y)+U)) \\ &= \bar{\mu}((x+U)\alpha(y+U)) \\ &\leq \bar{\mu}(y+U) \\ &= \mu(y).\end{aligned}$$

This completes the proof. □

Theorem 3.15. *If every anti fuzzy left ideal of a Γ-ring M has finite number of values, then M is left Noetherian.*

Proof. Suppose that every anti fuzzy left ideal of a Γ-ring M has finite number of values and M is not left Noetherian. Then there exists strictly increasing sequence $U_0\subset U_1\subset U_2\subset ...$ of left ideals of M. Define a fuzzy set μ in M by

$$\mu(x)=\begin{cases}\frac{1}{n+1}, & x\in U_{n+1}\setminus U_n,\ n=0,1,2,... \\ 1, & x\in\cup_{n=0}^{\infty}U_n.\end{cases}$$

Let us prove that μ is an anti fuzzy left ideal, i.e., $\mu(x-y)\leq\max\{\mu(x),\ \mu(y)\}$ and $\mu(x\alpha y)\leq\mu(y)$ for every $x,\ y\in M$ and $\alpha\in\Gamma$.

It is easy to see μ is an anti fuzzy left ideal in the following three case:
case 1: $x,\ y\in\cup_{n=0}^{\infty}U_n$.
case 2: $x\in U_{n+1}\setminus U_n,\ y\in\cup_{n=0}^{\infty}U_n$.
case 3: $y\in U_{n+1}\setminus U_n,\ x\in\cup_{n=0}^{\infty}U_n$.
Then, only case 4: $x,\ y\in U_{n+1}\setminus U_n$ need to consider. In fact, On the one hand, $x-y\in U_{n+1}\setminus U_n$ for some n (n=0, 1, 2 ...), and so either $x\notin U_n$ or $y\notin U_n$. So for definiteness, let $y\in U_{k+1}\setminus U_k$ for $k\leq n$. It follows that

$$\mu(x-y)=\frac{1}{n+1}\leq\frac{1}{k+1}=\mu(y).$$

So $\mu(x-y) \leq \max\{\mu(x),\ \mu(y)\}$; On the other hand, there exists a nonnegative integer n such that $x\alpha y \in U_{n+1} \setminus U_n$, then $y \notin U_n$, and hence $y \in U_{k+1} \setminus U_k$ for $k \leq n$, i.e.,

$$\mu(x\alpha y) = \frac{1}{n+1} \leq \frac{1}{k+1} = \mu(y).$$

Therefore μ is an anti fuzzy left ideal of M and μ has infinite number of different values. This contradiction prove that M is a left Noetherian $\Gamma-$ring M. □

Theorem 3.16. *A Γ-ring M is left Artinian if and only if the set of values of any anti fuzzy left ideal of M is a well ordered subset of [0, 1].*

Proof. Suppose that a Γ-ring M is left Artinian and μ is an anti fuzzy left ideal of M whose set is not a well ordered subset of [0, 1]. Then there exists a strictly decreasing sequence $\{\lambda_n\}$ such that $\mu(x_n) = \lambda_n$. Define

$$U_n = \{x \in M \mid \mu(x) \leq \lambda_n\}.$$

Then $U_1 \supset U_2 \supset U_3 \supset ...$ is a strictly descending chain of left ideals of M, which contradicts that M is left Artinian.

Conversely, assume that the set of values of any anti fuzzy left ideal of M is a well ordered subset of [0, 1] and M is not a left Artinian, then there exists a strictly descending chain

$$U_1 \supset U_2 \supset U_3 \supset ... \tag{3.1}$$

of left ideals of M. Note that $U = \cap U_i$ (i=1, 2, 3 ...) is a left ideal of M. Define a fuzzy set μ in M by

$$\mu(x) = \begin{cases} 1, & x \notin U_i, \\ \frac{1}{k}, & where\ k = \max\{i \in N \mid x \in U_i\}. \end{cases}$$

It can be easily seen that μ is an anti fuzzy left ideal of M. Because the chain (3.1) is not terminating, μ has a strictly descending sequence of values, contradicting that the value set of any anti fuzzy left ideal is well ordered. So M is left Artinian. □

Theorem 3.17. *Let $S = \{\lambda_1,\ \lambda_2,\ ...,\ \lambda_n,\ ...\} \cup \{1\}$, where $\{\lambda_n\}$ is a fixed strictly descending sequence, and $0 < \lambda_n < 1$. Then a Γ-ring M is left Artinian if and only if for each anti fuzzy left ideal μ of M, $Im(\mu) \subset S$ implies that there exists $n_0 \in \mathbf{N}$ such that $Im(\mu) \subset \{\lambda_1,\ \lambda_2,\ ...\ \lambda_{n_0}\} \cup \{1\}$.*

Proof. Assume M is left Artinian, then $Im(\mu)$ is a well ordered subset of [0, 1] by Theorem 3.16 and so the condition is necessary by noticing that a set is well ordered if and only if it does not contain any infinite descending sequence, i.e., there exists $n_0 \in \mathbf{N}$ such that $Im(\mu) \subset \{\lambda_1,\ \lambda_2,\ ...\ \lambda_{n_0}\} \cup \{1\}$.

Conversely, assume that there exists $n_0 \in \mathbf{N}$ such that $Im(\mu) \subset \{\lambda_1,\ \lambda_2,\ ...\ \lambda_{n_0}\} \cup \{1\}$ and M is not left Artinian, then there exists a strictly descending chain of left ideals of M $U_1 \supset U_2 \supset U_3 \supset ...$, define a fuzzy set μ in M by

$$\mu(x) = \begin{cases} \lambda_1, & x \in U_1, \\ \lambda_{n-1}, & x \in U_{n-1} \setminus U_n,\ n = 2,\ 3,\ ..., \\ 1, & x \in M \setminus\ \cap_{n=1}^{\infty} U_n, \end{cases}$$

Then μ is an anti fuzzy left ideal of M by Theorem 3.5. Because the descending chain is not terminating, $Im(\mu)$ has infinite number of values, this contradicts the assumption. Therefore M is left Artinian. □

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