Positive implicative vague ideals in BCK-algebras

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ABSTRACT. The notion of positive implicative vague ideals in BCK-algebras is introduced. A relation between a vague ideal and a positive implicative vague ideal is discussed. Characterizations of a positive implicative vague ideal are considered.

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1. Introduction

Several authors have made a number of generalizations of Zadeh’s fuzzy set theory [10]. Of these, the notion of vague set theory introduced by Gau and Buehrer [3] is of interest to us. Using the vague set in the sense of Gau and Buehrer, Biswas [2] studied vague groups. Jun and Park [5, 9] studied vague ideals and vague deductive systems in subtraction algebras. Lee et al. [7] introduced the notion of vague BCK/BCI-algebras and vague ideals, and investigated their properties. They also provided conditions for a vague set to be a vague ideal. They discussed characterizations of a vague ideal. Ahn et al. [1] introduced the notion of vague quick ideals of BCK/BCI-algebras, and discussed related properties. Lee et al. [6] introduced the notions of vague $d$-subalgebras, vague $d$-ideals, vague $d^s$-ideals and vague $d^*$-ideals. They established relations between vague $d$-subalgebras, vague BCK-ideals, vague $d$-ideals, vague $d^s$-ideals and vague $d^*$-ideals. In this paper, we also use the notion of vague set in the sense of Gau and Buehrer to discuss the vague theory on BCK/BCI-algebras. We introduce the notion of positive implicative vague ideals in BCK-algebras, and then we investigate their properties. We investigate a relation between a vague ideal and a positive implicative vague ideal. We establish characterizations of a positive implicative vague ideal.
2. Preliminaries

We review some definitions and properties that will be useful in our results. By a *BCI-algebra* we mean an algebra \((X, *, 0)\) of type \((2,0)\) satisfying the following conditions:

\[(a1) \ (\forall x, y, z \in X) \ ((x * y) * (x * z)) * (z * y) = 0),\]
\[(a2) \ (\forall x, y \in X) \ ((x * (x * y)) * y = 0),\]
\[(a3) \ (\forall x \in X) \ (x * x = 0),\]
\[(a4) \ (\forall x, y \in X) \ (x * y = 0, y * x = 0 \Rightarrow x = y).\]

A BCI-algebra \(X\) satisfying the additional condition:

\[(a5) \ (\forall x \in X) \ (0 * x = 0)\]

is called a *BCK-algebra*. In any BCK/BCI-algebra \(X\) one can define a partial order “\(\leq\)” by putting \(x \leq y\) if and only if \(x * y = 0\).

A BCK/BCI-algebra \(X\) has the following properties:

\[(b1) \ (\forall x \in X) \ (x * 0 = x),\]
\[(b2) \ (\forall x, y, z \in X) \ ((x * y) * z = (x * z) * y),\]
\[(b3) \ (\forall x, y, z \in X) \ (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x).\]

In particular, if \(X\) is a BCK-algebra then the following property hold:

\[(b5) \ (\forall x, y \in X) \ ((x * y) * x = 0).\]

A subset \(S\) of a BCK/BCI-algebra \(X\) is called a *subalgebra* of \(X\) if \(x * y \in S\) whenever \(x, y \in S\). A subset \(A\) of a BCK-algebra \(X\) is called an *ideal* of \(X\) if it satisfies:

\[(c1) \ 0 \in A,\]
\[(c2) \ (\forall x \in A) \ (\forall y \in X) \ (y * x \in A \Rightarrow y \in A).\]

Note that every ideal \(A\) of a BCK/BCI-algebra \(X\) satisfies:

\[(2.1) \ (\forall x \in A) \ (\forall y \in X) \ (y \leq x \Rightarrow y \in A).\]

A subset \(A\) of a BCK-algebra \(X\) is called a *positive implicative ideal* of \(X\) if it satisfies \((c1)\) and

\[(c3) \ (\forall x, y, z \in A) \ ((x * y) * z \in A, y * z \in A \Rightarrow x * z \in A).\]

Note that any positive implicative ideal is an ideal, but the converse is not true in general.

**Lemma 2.1.** [8] Let \(X\) be a BCK-algebra. Then an ideal \(A\) of \(X\) is positive implicative if and only if it satisfies:

\[(2.2) \ (\forall x, y \in X) \ ((x * y) * y \in A \Rightarrow x * y \in A).\]

We refer the reader to the books [4] and [8] for further information regarding BCK/BCI-algebras.

**Definition 2.2.** [2] A vague set \(A\) in the universe of discourse \(U\) is characterized by two membership functions given by:

1. A true membership function \(t_A : U \rightarrow [0, 1]\),
and

(2) A false membership function

\[ f_A : U \rightarrow [0, 1], \]

where \( t_A(u) \) is a lower bound on the grade of membership of \( u \) derived from the “evidence for \( u \)”, \( f_A(u) \) is a lower bound on the negation of \( u \) derived from the “evidence against \( u \)”, and

\[ t_A(u) + f_A(u) \leq 1. \]

Thus the grade of membership of \( u \) in the vague set \( A \) is bounded by a subinterval \([t_A(u), 1 - f_A(u)]\) of \([0, 1]\). This indicates that if the actual grade of membership of \( u \) is \( \mu(u) \), then

\[ t_A(u) \leq \mu(u) \leq 1 - f_A(u). \]

The vague set \( A \) is written as

\[ A = \{(u, [t_A(u), f_A(u)]) \mid u \in U\}, \]

where the interval \([t_A(u), 1 - f_A(u)]\) is called the vague value of \( u \) in \( A \), denoted by \( V_A(u) \).

Recall that if \( I_1 = [a_1, b_1] \) and \( I_2 = [a_2, b_2] \) are two subintervals of \([0, 1]\), we can define a relation between \( I_1 \) and \( I_2 \) by \( I_1 \succeq I_2 \) if and only if \( a_1 \geq a_2 \) and \( b_1 \geq b_2 \). For \( \alpha, \beta \in [0, 1] \) we now define \((\alpha, \beta)\)-cut and \( \alpha \)-cut of a vague set.

**Definition 2.3.** [2] Let \( A \) be a vague set of a universe \( X \) with the true-membership function \( t_A \) and the false-membership function \( f_A \). The \((\alpha, \beta)\)-cut of the vague set \( A \) is a crisp subset \( A_{(\alpha,\beta)} \) of the set \( X \) given by

\[ A_{(\alpha,\beta)} = \{x \in X \mid V_A(x) \succeq [\alpha, \beta]\}. \]

Clearly \( A_{(0,0)} = X \). The \((\alpha, \beta)\)-cuts of the vague set \( A \) are also called vague-cuts of \( A \).

**Definition 2.4.** [2] The \( \alpha \)-cut of the vague set \( A \) is a crisp subset \( A_\alpha \) of the set \( X \) given by \( A_\alpha = A_{(\alpha,\alpha)} \).

Note that \( A_0 = X \), and if \( \alpha \geq \beta \) then \( A_\alpha \subseteq A_\beta \) and \( A_{(\alpha,\beta)} = A_\alpha \).

Equivalently, we can define the \( \alpha \)-cut as

\[ A_\alpha = \{x \in X \mid t_A(x) \geq \alpha\}. \]

### 3. Positive implicative vague ideals

For our discussion, we shall use the following notations on interval arithmetic:

Let \( I[0, 1] \) denote the family of all closed subintervals of \([0, 1]\). We define the term “imax” to mean the maximum of two intervals as

\[ \text{imax}(I_1, I_2) := [\max(a_1, a_2), \max(b_1, b_2)], \]

where \( I_1 = [a_1, b_1], I_2 = [a_2, b_2] \in I[0, 1] \). Similarly we define “imin”. The concepts of “imax” and “imin” could be extended to define “isup” and “iinf” of infinite number of elements of \( I[0, 1] \).
It is obvious that $L = \{I[0, 1], \text{sup}, \text{inf}, \geq\}$ is a lattice with universal bounds $[0, 0]$ and $[1, 1]$ (see [1]).

In what follows let $X$ denote a BCK-algebra unless otherwise specified.

**Definition 3.1.** [7] A vague set $A$ of $X$ is called a *vague BCK/BCI-algebra* of $X$ if the following condition is true:

\[(\forall x, y \in X) \ (V_A(x \ast y) \geq \text{imin}\{V_A(x), V_A(y)\}),\]

that is,

\[
t_A(x \ast y) \geq \min\{t_A(x), t_A(y)\},
1 - f_A(x \ast y) \geq \min\{1 - f_A(x), 1 - f_A(y)\}\]

for all $x, y \in X$.

**Definition 3.2.** [7] A vague set $A$ of $X$ is called a *vague ideal* of $X$ if the following conditions are true:

\[
\begin{align*}
&\ (d1) \ (\forall x \in X) \ (V_A(0) \geq V_A(x)), \\
&\ (d2) \ (\forall x, y \in X) \ (V_A(x) \geq \text{imin}\{V_A(x \ast y), V_A(y)\}),
\end{align*}
\]

that is,

\[
\begin{align*}
&\ t_A(0) \geq t_A(x), \ 1 - f_A(0) \geq 1 - f_A(x), \\
&\ t_A(x) \geq \min\{t_A(x \ast y), t_A(y)\}, \\
&\ 1 - f_A(x) \geq \min\{1 - f_A(x \ast y), 1 - f_A(y)\}
\end{align*}
\]

for all $x, y \in X$.

**Proposition 3.3.** [7] Every vague ideal $A$ of $X$ satisfies:

\[(\forall x, y \in X) \ (x \leq y \Rightarrow V_A(x) \geq V_A(y)).\]

**Proposition 3.4.** [7] Every vague ideal $A$ of $X$ satisfies:

\[
(\forall x, y, z \in X) \ (V_A(x \ast z) \geq \text{imin}\{V_A((x \ast y) \ast z), V_A(y)\}).
\]

**Proposition 3.5.** For a vague ideal $A$ of $X$, the following conditions are equivalent:

\[
\begin{align*}
&\ (1) \ (\forall x, y \in X) \ (V_A(x \ast y) \geq V_A((x \ast y) \ast y)), \\
&\ (2) \ (\forall x, y, z \in X) \ (V_A((x \ast z) \ast (y \ast z)) \geq V_A((x \ast y) \ast z)).
\end{align*}
\]

**Proof.** Assume that $A$ satisfies the condition (1). Since

\[
((x \ast (y \ast z)) \ast z) \ast z = ((x \ast (y \ast z)) \ast (y \ast z)) \ast z \leq (x \ast y) \ast z
\]

for all $x, y, z \in X$, we have $t_A((x \ast y) \ast z) \leq t_A(((x \ast (y \ast z)) \ast z) \ast z)$ and

\[1 - f_A((x \ast y) \ast z) \leq 1 - f_A(((x \ast (y \ast z)) \ast z) \ast z)
\]

for all $x, y, z \in X$. It follows from (b2) and (1) that

\[
t_A((x \ast z) \ast (y \ast z)) = t_A(((x \ast (y \ast z)) \ast z) \ast z)
\]

\[
\geq t_A(((x \ast (y \ast z)) \ast z) \ast z) \geq t_A((x \ast y) \ast z)
\]

100
\[ \begin{array}{c|ccc} * & 0 & a & b \\ \hline 0 & 0 & 0 & 0 \\ a & a & 0 & 0 \\ b & b & b & 0 \end{array} \]

Table 1. *-operation for \( X \)

\[ 1 - f_A((x * z) * (y * z)) = 1 - f_A((x * (y * z)) * z) \]
\[ \geq 1 - f_A(((x * (y * z)) * z) * z) \geq 1 - f_A((x * y) * z). \]

Therefore \( A \) satisfies the second condition. Conversely, if we put \( z = y \) in (2) and use (a3) and (b1), then we obtain the condition (1). This completes the proof. \( \Box \)

**Definition 3.6.** A vague set \( A \) of \( X \) is called a positive implicative vague ideal of \( X \) if it satisfies (d1) and (d3)

\[ (\forall x, y, z \in X) \left( V_A(x * z) \geq \text{imin}\{V_A((x * y) * z), V_A(y * z)\} \right). \]

Note that (d3) is equivalent to the following assertion:

\[ t_A(x * z) \geq \text{min}\{t_A((x * y) * z), t_A(y * z)\}, \]
\[ 1 - f_A(x * z) \geq \text{min}\{1 - f_A((x * y) * z), 1 - f_A(y * z)\} \]

for all \( x, y, z \in X \).

**Example 3.7.** Let \( X = \{0, a, b\} \) be a BCK-algebra in which the *-operation is given by Table 1. Let \( A \) be a vague set in \( X \) defined by

\[ A = \{\langle 0, [0.7, 0.2]\rangle, \langle a, [0.5, 0.3]\rangle, \langle b, [0.4, 0.4]\rangle\}. \]

It is routine to verify that \( A \) is a positive implicative vague ideal of \( X \).

**Theorem 3.8.** Every positive implicative vague ideal is a vague ideal.

**Proof.** Let \( A \) be a positive implicative vague ideal of \( X \). If we take \( z = 0 \) in (d3) and use (b1), then we obtain (d2). Hence \( A \) is a vague ideal of \( X \). \( \Box \)

The following example shows that the converse of Theorem 3.8 may not be true.

**Example 3.9.** Consider a BCK-algebra \( X = \{0, a, b, c\} \) with Cayley table which is given by Table 2. Let \( A \) be a vague set in \( X \) defined by

\[ A = \{\langle 0, [0.8, 0.1]\rangle, \langle a, [0.7, 0.2]\rangle, \langle b, [0.7, 0.2]\rangle, \langle c, [0.5, 0.4]\rangle\}. \]

It is routine to verify that \( A \) is a vague ideal of \( X \). But it is not a positive implicative vague ideal of \( X \) since

\[ V_A(b * a) \not\geq \text{imin}\{V_A((b * a) * a), V_A(a * a)\}. \]

It is natural to ask what is the condition under which a vague ideal is a positive implicative vague ideal? We now answer this question.

**Theorem 3.10.** For a vague ideal \( A \) of \( X \), the following are equivalent:
(1) $A$ is a positive implicative vague ideal of $X$.
(2) $A$ satisfies the condition (1) in Proposition 3.5.

Proof. Assume that $A$ is a positive implicative vague ideal of $X$. If we put $z = y$ in (d3) and use (a3) and (d1), then
\[
V_A(x*y) \geq \text{imin} \{ V_A((x*y)*y), V_A(y*y) \} = \text{imin} \{ V_A((x*y)*y), V_A(0) \} = V_A((x*y)*y)
\]
for all $x,y \in X$.

Conversely, suppose that $A$ satisfies the condition (1) of Proposition 3.5. Note that $((x*z) * z) * (y*z) \leq (x*y) * z$ for all $x,y,z \in X$. Using Proposition 3.5(1), (d2) and Proposition 3.3, we have

\[
V_A(x*z) \geq V_A((x*z)*z) \geq \text{imin} \{ V_A(((x*z)*z) * (y*z)), V_A(y*z) \} \geq \text{imin} \{ V_A((x*y)*z), V_A(y*z) \},
\]

and so $A$ is a positive implicative vague ideal of $X$. \qed

Theorem 3.11. For a vague ideal $A$ of $X$, the following are equivalent:

(1) $A$ is a positive implicative vague ideal of $X$.
(2) $A$ satisfies the condition (2) in Proposition 3.5.

Proof. Assume that $A$ is a positive implicative vague ideal of $X$. Combining Theorems 3.8 and 3.10 and Proposition 3.3, $A$ satisfies the condition (2) in Proposition 3.5.

Conversely, suppose that (2) is valid. For any $x,y,z \in X$, we have

\[
V_A(x*z) \geq \text{imin} \{ V_A((x*z)*(y*z)), V_A(y*z) \} \geq \text{imin} \{ V_A((x*y)*z), V_A(y*z) \}.
\]

Therefore $A$ is a positive implicative vague ideal of $X$. \qed

Theorem 3.12. For a vague set $A$ in $X$, the following are equivalent:

(1) $A$ is a positive implicative vague ideal of $X$.
(2) $A$ satisfies conditions (d1) and
(3.8) $(\forall x,y,z \in X) (V_A(x*y) \geq \text{imin} \{ V_A(((x*y)*y)*z), V_A(z) \}).$

<table>
<thead>
<tr>
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Table 2. Cayley table for $X$
Proof. Assume that \( A \) is a positive implicative vague ideal of \( X \). Then \( A \) is a vague ideal of \( X \) by Theorem 3.8 and so \( A \) satisfies the condition (d1). Using (d2), (b2), (a3), (b1) and Theorem 3.11 we have

\[
V_A(x \star y) \geq \text{imin} \{V_A((x \star y) \star z), V_A(z)\} \\
= \text{imin} \{V_A(((x \star z) \star y) \star (y \star y)), V_A(z)\} \\
\geq \text{imin} \{V_A(((x \star z) \star y) \star y), V_A(z)\} \\
= \text{imin} \{V_A(((x \star y) \star y) \star z), V_A(z)\}.
\]

Therefore (3.8) is valid.

Conversely, let \( A \) satisfy conditions (d1) and (3.8). For any \( x, y \in X \), we have

\[
V_A(x) = V_A(x \star 0) \geq \text{imin} \{V_A(((x \star 0) \star 0) \star y), V_A(y)\} \\
= \text{imin} \{V_A(x \star y), V_A(y)\}.
\]

Hence \( A \) is a vague ideal of \( X \). If we put \( z = 0 \) in (3.8), then

\[
V_A(x \star y) \geq \text{imin} \{V_A(((x \star y) \star 0) \star 0), V_A(0)\} = V_A((x \star y) \star y)
\]

for all \( x, y \in X \). It follows from Theorem 3.10 that \( A \) is a positive implicative vague ideal of \( X \).

Combining the above results, we have characterizations of a positive implicative vague ideal.

**Theorem 3.13.** For a vague set \( A \) in \( X \), the following are equivalent:

1. \( A \) is a positive implicative vague ideal of \( X \).
2. \( A \) is a vague ideal of \( X \) satisfying the condition (1) in Proposition 3.5.
3. \( A \) is a vague ideal of \( X \) satisfying the condition (2) in Proposition 3.5.
4. \( A \) satisfies the conditions (d1) and (3.8).

**Lemma 3.14.** For a vague set \( A \) in \( X \), the following are equivalent:

1. \( A \) is a vague ideal of \( X \).
2. \( A \) satisfies the following implication:

\[
(\forall x, y, z \in X) \ (x \star y) \star z = 0 \Rightarrow V_A(x) \geq \text{imin} \{V_A(y), V_A(z)\}.
\]

**Theorem 3.15.** For a vague set \( A \) in \( X \), the following are equivalent:

1. \( A \) is a positive implicative vague ideal of \( X \).
2. \( A \) satisfies the following implication:

\[
(((x \star y) \star a) \star b = 0 \Rightarrow V_A(x \star y) \geq \text{imin} \{V_A(a), V_A(b)\}.
\]

for all \( x, y, a, b \in X \).

Proof. Assume that \( A \) is a positive implicative vague ideal of \( X \). Then \( A \) is a vague ideal of \( X \) by Theorem 3.8. Let \( x, y, a, b \in X \) be such that \(((x \star y) \star a) \star b = 0 \). It follows from Theorem 3.10 and Lemma 3.14 that

\[
V_A(x \star y) \geq V_A((x \star y) \star y) \geq \text{imin} \{V_A(a), V_A(b)\}.
\]

Therefore \( A \) satisfies (3.10).
Conversely, suppose that \( A \) satisfies the condition (3.10) and let \( x, a, b \in X \) be such that \((x + a) \ast b = 0\). Then \(((x \ast 0) \ast 0) \ast a \ast b = 0\), and so \(V_A(x) = V_A(x \ast 0) \succeq \text{imin} \{V_A(a), V_A(b)\}\).

Using Lemma 3.14 we know that \( A \) is a vague ideal of \( X \). Note that 
\[((x \ast y) \ast (x \ast y)) \ast 0 = 0\]
for all \( x, y \in X \). It follows from (3.10) that
\[V_A(x \ast y) \succeq \text{imin} \{V_A((x \ast y) \ast y), V_A(0)\} = V_A((x \ast y) \ast y)\]
and so \( A \) is a positive implicative vague ideal of \( X \) by Theorem 3.10.

**Theorem 3.16.** For a vague set \( A \) in \( X \), the following are equivalent:

1. \( A \) is a positive implicative vague ideal of \( X \).
2. \( A \) satisfies the following implication:

\[
((x \ast y) \ast z) \ast a \ast b = 0 \Rightarrow V_A((x \ast z) \ast (y \ast z)) \succeq \text{imin} \{V_A(a), V_A(b)\}
\]
for all \( x, y, z, a, b \in X \).

**Proof.** Suppose that \( A \) is a positive implicative vague ideal of \( X \). Then \( A \) is a vague ideal of \( X \) by Theorem 3.8. Let \( x, y, z, a, b \in X \) be such that \(((x \ast y) \ast z) \ast a \ast b = 0\). Then \(V_A((x \ast z) \ast (y \ast z)) \succeq \text{imin} \{V_A(a), V_A(b)\}\) by Theorem 3.11 and Lemma 3.14. Thus (3.11) is valid.

Conversely, if we put \( z = y \) in (3.11), then
\[V_A(x \ast y) = V_A((x \ast y) \ast (y \ast y)) \succeq \text{imin} \{V_A(a), V_A(b)\}\]
whenever \(((x \ast y) \ast y) \ast a \ast b = 0\) for all \( x, y, a, b \in X \). It follows from Theorem 3.15 that \( A \) is a positive implicative vague ideal of \( X \).

By the mathematical induction, the above two theorems have more general forms.

**Theorem 3.17.** For a vague set \( A \) in \( X \), the following are equivalent:

1. \( A \) is a positive implicative vague ideal of \( X \).
2. \( A \) satisfies the following inequality:

\[
V_A(x \ast y) \succeq \text{imin} \{V_A(a_1), \cdots, V_A(a_n)\}
\]
whenever \((\cdots ((x \ast y) \ast a_1) \ast \cdots) \ast a_n = 0\) for all \( x, a_1, \cdots, a_n \in X \).

**Theorem 3.18.** For a vague set \( A \) in \( X \), the following are equivalent:

1. \( A \) is a positive implicative vague ideal of \( X \).
2. \( A \) satisfies the following inequality:

\[
V_A((x \ast z) \ast (y \ast z)) \succeq \text{imin} \{V_A(a_1), \cdots, V_A(a_n)\}
\]
whenever \((\cdots ((x \ast y) \ast z) \ast a_1) \ast \cdots) \ast a_n = 0\) for all \( x, y, z, a_1, \cdots, a_n \in X \).

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