Multi-granulation rough set: from crisp to fuzzy case

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ABSTRACT.   Multi–granulation is an improvement of the classical rough set theory since it uses a family of binary relations instead of of a single indiscernibility relation for the constructing of approximation. In this paper, the multi–granulation rough set approach is further generalized into fuzzy environment. A family of fuzzy T–similarity relations are used to define the optimistic and pessimistic fuzzy rough sets respectively. The basic properties about these fuzzy rough sets are then discussed. Finally, the relationships among single relation based fuzzy rough set, optimistic and pessimistic multi–granulation fuzzy rough sets are addressed.

1. Introduction

Rough set theory [12, 13, 14, 15], proposed by Pawlak, is a powerful tool which can be used to deal with the inconsistency problems by separation of certain and doubtful knowledge extracted from the exemplary decisions. Presently, rough set theory has been demonstrated to be useful in the fields such as knowledge discovery, decision analysis, data mining, pattern recognition and so on.

However, it is well known that the indiscernibility relation is too restrictive for classification analysis in practical applications, especially in the fuzzy environment. Therefore, how to generalize the rough set model to fuzzy case plays an important role in the development of rough set theory. For example, in Ref. [6], the model of rough fuzzy set is proposed by using the indiscernibility relation to approximate a

fuzzy concept. Alternatively, the fuzzy rough model is the approximation of a crisp set or a fuzzy set in a fuzzy approximation space. In Ref. [19], Sun et al. presented the interval–valued fuzzy rough set by combining the interval–valued fuzzy set and rough set. By employing an approximation space constituted from an intuitionistic fuzzy triangular norm, an intuitionistic fuzzy implicator, and an intuitionistic fuzzy T–equivalence relation, Cornelis et al. [4] defined the concept of intuitionistic fuzzy rough sets in which the lower and upper approximations are both intuitionistic fuzzy sets in the universe. Zhou et al. proposed the intuitionistic fuzzy rough approximation by constructive and axiomatic approaches in Ref. [24]. Bhatt [1] presented the fuzzy–rough sets on compact computational domain. Zhao et al. [23] investigated the fuzzy variable precision rough sets by combining fuzzy rough set and variable precision rough set with the goal of making fuzzy rough set a special case. Hu et al. [7] proposed the Gaussian Kernel based fuzzy rough set, it uses the Gaussian Kernel to compute fuzzy T–equivalence relation for objective approximation. The same authors proposed the fuzzy preference–based rough sets in Ref. [8]. Ouyang et al. [11] presented the fuzzy rough model which is based on the fuzzy tolerance relation. More details about recent advancements of fuzzy rough set can be found in the literatures [2, 3, 5, 9, 10, 20, 21, 22].

Obviously, it is not difficult to observe that the above fuzzy rough set approaches are proposed on the basis of one and only one fuzzy binary relation. However, it should be noticed that in Ref. [16, 17, 18], the authors argued that we often need to describe concurrently a target concept through multi binary relations (e.g. equivalence relation, tolerance relation, reflexive relation and neighborhood relation) on the universe according to a user's requirements or targets of problem solving. Therefore, they proposed the concept of Multi–Granulation Rough Set (MGRS) [16, 17, 18] model which is based on a family of indiscernibility relations. In their papers, Qian et al. said that the MGRS are useful in the following cases [18]:

(1) we cannot perform the intersection operations between two different attributes sets;
(2) decision makers may be independent for the same project;
(3) extract decision rules from distributive information systems and groups of intelligent agents through using rough set approaches.

The purpose of this paper is to further generalize the MGRS to fuzzy environment. In our approach, a family of fuzzy T–similarity relations is used for constructing multi–granulation fuzzy rough set. To facilitate our discussion, we first present Pawlak's rough set, Qian's MGRS in Section 2. In Section 3, the optimistic and pessimistic multi–granulation fuzzy rough approximations are defined as the generalization of classical fuzzy rough set and Qian's MGRS. Not only the basic properties about the multi–granulation fuzzy rough set are discussed, but also the relationships among several fuzzy rough sets are presented. Results are summarized in Section 4.

2. Preliminary knowledge on rough sets

In this section, we review some basic concepts such as information system, Pawlak's rough set, multi–granulation rough set and dominance–based rough set.

2.1. **Pawlak's rough set.** Formally, let $U \neq \varnothing$ be the universe of discourse, $\mathbf{R}$ is a family of equivalence relations on U, then the pair $K = (U, \mathbf{R})$ is referred to as a knowledge base.

$\forall R \in \mathbf{R}$, $AP = (U, R)$ is referred to as a approximate space, we use U/R to represent the family of all equivalence classes in terms of R (or classification of U). Such family of equivalence classes is referred to as categories or concepts of R. Therefore, $\forall x \in U$, $[x]_R$ is used to denotes a category (equivalence class) in terms of R which contains x.

Suppose that $A \subseteq \mathbf{R}$, then $\bigcap A$ (intersection of all equivalence relations in A) is also an equivalence relation, and will be denoted by $IND(A)$, it is referred to as an indiscernibility relation over A. $U/IND(A)$ is the family of all equivalence classes in terms of the set of equivalence relations A, each element in $U/IND(A)$ is referred to as an A–basic knowledge, $[x]_A = \{y \in U : (x, y) \in IND(A)\}$ is the equivalence class of x in terms of the set of equivalence relations A.

By the indiscernibility relation $IND(A)$, one can derive the lower and upper approximations of an arbitrary subset X of U. They are defined as

$$\underline{A}(X) = \{x \in U : [x]_A \subseteq X\} \text{ and } \overline{A}(X) = \{x \in U : [x]_A \cap X \neq \varnothing\}$$

respectively. The pair $[\underline{A}(X), \overline{A}(X)]$ is referred to as the Pawlak's rough set of X with respect to the set of attributes A.

2.2. **Multi–granulation rough set.** The multi–granulation rough set is different from Pawlak's rough set model because the former is constructed on the basis of a family of binary relations instead of a single indiscernibility relation.

The concept of multi–granulation rough set theory was firstly proposed by Qian et al. In their appraoch, two different models have been defined. The first one is the optimistic multi–granulation rough set [16, 18], the second one is the pessimistic multi–granulation rough set [17].

2.2.1. *Optimistic MGRS.*

Definition 2.1. [18] Let K be a knowledge base in which $A = \{R_1, R_2, \cdots, R_m\} \subseteq \mathbf{R}$, then $\forall X \subseteq U$, the optimistic multi–granulation lower and upper approximations are denoted by $\sum\limits_{i=1}^{m} R_i \overset{O}{}(X)$ and $\overline{\sum\limits_{i=1}^{m} R_i}^{O}(X)$ respectively where

$$(2.1) \quad \sum_{i=1}^{m} R_i \overset{O}{}(X) = \{x \in U : [x]_{A_1} \subseteq X \vee [x]_{A_2} \subseteq X \vee \cdots \vee [x]_{A_m} \subseteq X\};$$

$$(2.2) \quad \overline{\sum_{i=1}^{m} R_i}^{O}(X) = \sim \sum_{i=1}^{m} R_i \overset{O}{}(\sim X);$$

$[x]_{R_i}$ ($1 \leq i \leq m$) is the equivalence class of x in terms of the equivalence relation R_i, $\sim X$ is the complement of set X.

By the lower and upper approximations $\sum\limits_{i=1}^{m} R_i{}^O (X)$ and $\overline{\sum\limits_{i=1}^{m} R_i}{}^O (X)$, the optimistic multi–granulation boundary region of $\overline{X}$ is

$$(2.3) \qquad BN_{\sum_{i=1}^{m} R_i}^{O}(X) = \overline{\sum_{i=1}^{m} R_i}{}^O (X) - \sum_{i=1}^{m} R_i{}^O (X).$$

Theorem 2.2. *Let K be a knowledge base in which $A = \{R_1, R_2, \cdots, R_m\} \subseteq \mathbf{R}$, then $\forall X \subseteq U$, we have*

$$(2.4) \quad \overline{\sum_{i=1}^{m} R_i}{}^O (X) = \{x \in U : [x]_{R_1} \cap X \neq \varnothing \wedge [x]_{R_2} \cap X \neq \varnothing \wedge \cdots \wedge [x]_{R_m} \cap X \neq \varnothing\}.$$

Proof. By Definition 2.1, we have

$$x \in \overline{\sum_{i=1}^{m} R_i}{}^O (X) \quad \Leftrightarrow \quad x \notin \sum_{i=1}^{m} R_i{}^O (\sim X)$$

$$\Leftrightarrow \quad [x]_{R_1} \nsubseteq (\sim X) \wedge [x]_{E_2} \nsubseteq (\sim X) \wedge \cdots \wedge [x]_{R_m} \nsubseteq (\sim X)$$

$$\Leftrightarrow \quad [x]_{R_1} \cap X \neq \varnothing \wedge [x]_{R_2} \cap X \neq \varnothing \wedge \cdots \wedge [x]_{R_m} \cap X \neq \varnothing$$

$\square$

By Theorem 2.2, we can see that though the optimistic upper approximation is defined by the complement of the lower approximation, it can also be considered as a subset in which each object has a non–empty intersection with the target in terms of each subset of attributes.

2.2.2. *Pessimistic MGRS.*

Definition 2.3. [17] Let K be a knowledge base in which $A = \{R_1, R_2, \cdots, R_m\} \subseteq \mathbf{R}$, then $\forall X \subseteq U$, the pessimistic multi–granulation lower and upper approximations are denoted by $\sum\limits_{i=1}^{m} R_i{}^P (X)$ and $\overline{\sum\limits_{i=1}^{m} R_i}{}^P (X)$ respectively where

$$(2.5) \quad \sum_{i=1}^{m} R_i{}^P (X) \quad = \quad \{x \in U : [x]_{R_1} \subseteq X \wedge [x]_{R_2} \subseteq X \wedge \cdots \wedge [x]_{R_m} \subseteq X\};$$

$$(2.6) \quad \overline{\sum_{i=1}^{m} R_i}{}^P (X) \quad = \quad \sim \sum_{i=1}^{m} R_i{}^P (\sim X).$$

By the lower and upper approximations $\sum\limits_{i=1}^{m} R_i{}^P (X)$ and $\overline{\sum\limits_{i=1}^{m} R_i}{}^P (X)$, the pessimistic multi–granulation boundary region of $\overline{X}$ is

$$(2.7) \qquad BN_{\sum_{i=1}^{m} R_i}^{P}(X) = \overline{\sum_{i=1}^{m} R_i}{}^P (X) - \sum_{i=1}^{m} R_i{}^P (X).$$

Theorem 2.4. *Let K be a knowledge base in which $A = \{R_1, R_2, \cdots, R_m\} \subseteq \mathbf{R}$, then $\forall X \subseteq U$, we have*

$$(2.8) \quad \overline{\sum_{i=1}^{m} R_i}^{P}(X) = \{x \in U : [x]_{R_1} \cap X \neq \varnothing \vee [x]_{R_2} \cap X \neq \varnothing \vee \cdots \vee [x]_{R_m} \cap X \neq \varnothing\}.$$

Proof. By Definition 2.3, we have

$$
\begin{aligned}
x \in \overline{\sum_{i=1}^{m} R_i}^{P}(X) \quad &\Leftrightarrow \quad x \in U - \underline{\sum_{i=1}^{m} R_i}^{P}(\sim X) \\
&\Leftrightarrow \quad [x]_{R_1} \not\subseteq (\sim X) \vee [x]_{R_2} \not\subseteq (\sim X) \vee \cdots \vee [x]_{R_m} \not\subseteq (\sim X) \\
&\Leftrightarrow \quad [x]_{R_1} \cap X \neq \varnothing \vee [x]_{R_2} \cap X \neq \varnothing \vee \cdots \vee [x]_{R_m} \cap X \neq \varnothing
\end{aligned}
$$

$\square$

Similar to the upper approximation of optimistic MGRS, the upper approximation of pessimistic MGRS can also be represented as a subset in which each object has a non–empty intersection with the target in terms of one of the subsets of attributes.

More details about MGRS can be found in Ref. [18].

2.3. **Single granulation fuzzy rough set.** The notion of fuzzy sets provides a convenient tool for representing vague concepts by allowing partial memberships. Presently, the concept of fuzzy rough sets was proposed and studied by many authors. We briefly review some of the basic concepts of the fuzzy rough set.

Firstly, we need the following notions. A triangular norm, or shortly t–norm, is an increasing, associative and commutative mapping $T : [0,1] \times [0,1] \rightarrow [0,1]$ that satisfies the boundary condition $(\forall x \in [0,1], T(x,1) = x)$. The most popular continuous t–norms are:

(1) *min* operator $T_M(x,y) = min\{x,y\}$;
(2) algebraic product: $T_P(x,y) = x \cdot y$;
(3) bold intersection(also called the Łukasiewicz t–norm): $T_L(x,y) = max\{0, x + y - 1\}$.

A triangular conorm, or shortly t–conorm, is an increasing, associative and commutative mapping $S : [0,1] \times [0,1] \rightarrow [0,1]$ that satisfies the boundary condition $(\forall x \in [0,1], S(x,0) = x)$. The well–known continuous t–norms are:

(1) *max* operator $S_M(x,y) = max\{x,y\}$;
(2) probabilistic sum: $S_P(x,y) = x + y - x \cdot y$;
(3) bounded sum: $S_L(x,y) = min\{1, x + y\}$.

A negator N is a decreasing mapping $[0,1] \rightarrow [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$. The negator $N_S(x) = 1 - x$ is referred to as the standard negator. A negator N is involutive of $N(N(x)) = x$ for each $x \in [0,1]$, an involutive negator is continuous and strictly decreasing.

If U is the universe of discourse, then the family of all fuzzy sets on U is denoted by $\mathscr{F}(U)$. Moreover, a fuzzy relation on U is a fuzzy subset $\mathscr{R} \in \mathscr{F}(U \times U)$ where

$$\mathscr{R} : U \times U \rightarrow [0,1].$$

The fuzzy relation $\mathscr{R}$ is referred to as serial iff $\forall x \in U$, there is a $y \in U$ such that $R(x, y) = 1$; the fuzzy relation $\mathscr{R}$ is referred to as reflexive iff $\mathscr{R}(x, x) = 1$ holds for each $x \in U$; the fuzzy relation is referred to as symmetric iff $\mathscr{R}(x, y) = \mathscr{R}(y, x)$ ($\forall x, y \in U$); the fuzzy relation is referred to as T–transitive iff $T(\mathscr{R}(x, y), \mathscr{R}(y, z)) \leq \mathscr{R}(x, z)$. If a fuzzy relation $\mathscr{R}$ is reflexive, symmetric and T–transitive, then it is referred to as a fuzzy T–similarity relation. Moreover, it should be noticed that there is a canonical one-to-one correspondence between fuzzy T–similarity relation and fuzzy T–partition.

Based on the definition of T–similarity relation, we can further generalize Pawlak's knowledge based to fuzzy environment. Let $U \neq \varnothing$ be the universe of discourse, $\mathbf{R}$ is a family of fuzzy T–similarity relations on U, then the pair $\mathscr{K} = (U, \mathbf{R})$ is referred to as a fuzzy T–knowledge base. $\forall \mathscr{R} \in \mathbf{R}$, $FAP = (U, \mathscr{R})$ is referred to as a fuzzy T–approximate space.

Given a fuzzy T–knowledge base $\mathscr{K}$, Suppose that $\mathscr{A} = \{\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_m\} \subseteq \mathbf{R}$ is a family of fuzzy T–similarity relations, then $\mathscr{R}_{\mathscr{A}} = \bigcap\{\mathscr{R}_i : i = 1, 2, \cdots, m\}$ (intersection of all fuzzy T–similarity relations in $\mathscr{A}$) is also a fuzzy T–similarity relation. Therefore, $\forall x, y \in U$,

$$\mathscr{R}_{\mathscr{A}}(x, y) = \bigwedge\{\mathscr{R}_i(x, y) : i = 1, 2, \cdots, m\}.$$

The fuzzy rough approximations in terms of fuzzy T–similarity relation in terms of $\mathscr{R}_{\mathscr{A}}$ is defined as follows.

Definition 2.5. [6] Let U be the universe of discourse, $\mathscr{R}_{\mathscr{A}}$ is a fuzzy T–similarity relation on U, then $\forall F \in \mathscr{F}(U)$, then fuzzy rough lower approximation and fuzzy rough upper approximation of F are denoted by $\underline{\mathscr{R}}(F)$ and $\overline{\mathscr{R}}(F)$ respectively, whose memberships for each $x \in U$ are:

$$(2.9) \qquad \underline{\mathscr{R}_{\mathscr{A}}}(F)(x) = \bigwedge_{y \in U} S\big(1 - \mathscr{R}_{\mathscr{A}}(x, y), F(y)\big),$$

$$(2.10) \qquad \overline{\mathscr{R}_{\mathscr{A}}}(F)(x) = \bigvee_{y \in U} T\big(\mathscr{R}_{\mathscr{A}}(x, y), F(y)\big).$$

Theorem 2.6. [6] *Let U be the universe of discourse, $\mathscr{R}_{\mathscr{A}}$ is a fuzzy T–similarity relation on U, then $\forall F, F' \in \mathscr{F}(U)$, the fuzzy rough approximations have the following properties:*

(1) $\underline{\mathscr{R}_{\mathscr{A}}}(U) = \overline{\mathscr{R}_{\mathscr{A}}}(U) = U$;

(2) $\underline{\mathscr{R}_{\mathscr{A}}}(\varnothing) = \overline{\mathscr{R}_{\mathscr{A}}}(\varnothing) = \varnothing$;

(3) $\underline{\mathscr{R}_{\mathscr{A}}}(F) \subseteq F \subseteq \overline{\mathscr{R}_{\mathscr{A}}}(F)$;

(4) $\underline{\mathscr{R}_{\mathscr{A}}}(F) = \sim \overline{\mathscr{R}_{\mathscr{A}}}(\sim F)$;

(5) $\overline{\mathscr{R}_{\mathscr{A}}}(F) = \sim \underline{\mathscr{R}_{\mathscr{A}}}(\sim F)$;

(6) $\underline{\mathscr{R}_{\mathscr{A}}}(F \cap F') = \underline{\mathscr{R}_{\mathscr{A}}}(F) \cap \underline{\mathscr{R}_{\mathscr{A}}}(F')$;

(7) $\overline{\mathscr{R}_{\mathscr{A}}}(F \cup F') = \overline{\mathscr{R}_{\mathscr{A}}}(F) \cup \overline{\mathscr{R}_{\mathscr{A}}}(F')$;

(8) $\underline{\mathscr{R}_{\mathscr{A}}}(F \cup F') \supseteq \underline{\mathscr{R}_{\mathscr{A}}}(F) \cup \underline{\mathscr{R}_{\mathscr{A}}}(F')$;

(9) $\overline{\mathscr{R}_{\mathscr{A}}}(F \cap F') \subseteq \overline{\mathscr{R}_{\mathscr{A}}}(F) \cap \overline{\mathscr{R}_{\mathscr{A}}}(F')$;

(10) $\underline{\mathscr{R}_{\mathscr{A}}}(\underline{\mathscr{R}_{\mathscr{A}}}(F)) = \underline{\mathscr{R}_{\mathscr{A}}}(F)$;

(11) $\overline{\mathscr{R}_{\mathscr{A}}}(\overline{\mathscr{R}_{\mathscr{A}}}(F)) = \overline{\mathscr{R}_{\mathscr{A}}}(F)$;

(12) $F \subseteq F' \Rightarrow \underline{\underline{\mathscr{R}_{\mathscr{A}}}}(F) \subseteq \underline{\underline{\mathscr{R}_{\mathscr{A}}}}(F')$;

(13) $F \subseteq F' \Rightarrow \overline{\overline{\mathscr{R}_{\mathscr{A}}}}(F) \subseteq \overline{\overline{\mathscr{R}_{\mathscr{A}}}}(F')$.

3. Constructive approach to multi–granulation fuzzy rough set

Following Qian's work, it is natural to introduce the multi–granulation concept into fuzzy environment. This is what will be discussed in this section.

In the following, we will use a family of fuzzy T–similarity relations instead of a single fuzzy T–similarity relation to define the concept of multi–granulation fuzzy rough set. Similar to Qian's approach, the optimistic and pessimistic multi–granulation fuzzy rough sets will be presented respectively.

3.1. **Optimistic multi–granulation fuzzy rough set.**

Definition 3.1. Let U be the universe of discourse, $\{\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_m\} \subseteq \mathbf{R}$ is a family of fuzzy T–similarity relations on U, then $\forall F \in \mathscr{F}(U)$, the optimistic multi–granulation fuzzy rough lower approximation upper approximation of F are denoted by $\sum\limits_{i=1}^{m} \underline{\mathscr{R}_i}^{O}(F)$ and $\overline{\sum\limits_{i=1}^{m} \mathscr{R}_i}^{O}(F)$ respectively, whose memberships for each $x \in U$ are:

$$(3.1) \qquad \sum_{i=1}^{m} \underline{\mathscr{R}_i}^{O}(F)(x) = \bigvee_{i=1}^{m} \bigwedge_{y \in U} S\big(1 - R_i(x,y), F(y)\big),$$

$$(3.2) \qquad \overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F)(x) = \bigwedge_{i=1}^{m} \bigvee_{y \in U} T\big(R_i(x,y), F(y)\big).$$

If $\sum\limits_{i=1}^{m} \underline{\mathscr{R}_i}^{O}(F) = \overline{\sum\limits_{i=1}^{m} \mathscr{R}_i}^{O}(F)$, then F is referred to as optimistic definable under the multi–granulation fuzzy T–environment; otherwise, F is referred to as optimistic undefinable. $\left[\sum\limits_{i=1}^{m} \underline{\mathscr{R}_i}^{O}(F), \overline{\sum\limits_{i=1}^{m} \mathscr{R}_i}^{O}(F)\right]$ is referred to as a pair of optimistic rough approximation of F under the multi–granulation fuzzy T–environment. In the following, let us examine some properties of such fuzzy rough approximation.

Theorem 3.2. *Let U be the universe of discourse, $\{\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_m\} \subseteq \mathscr{F}\mathscr{R}(U)$ is a family of fuzzy relations on U, then $\forall F, F' \in \mathscr{F}(U)$, the optimistic multi–granulation fuzzy rough set has the following properties:*

(1) $\sum\limits_{i=1}^{m} \underline{\mathscr{R}_i}^{O}(F) \subseteq F \subseteq \overline{\sum\limits_{i=1}^{m} \mathscr{R}_i}^{O}(F)$;

(2) $\sum\limits_{i=1}^{m} \underline{\mathscr{R}_i}^{O}(\varnothing) = \overline{\sum\limits_{i=1}^{m} \mathscr{R}_i}^{O}(\varnothing) = \varnothing, \sum\limits_{i=1}^{m} \underline{\mathscr{R}_i}^{O}(U) = \overline{\sum\limits_{i=1}^{m} \mathscr{R}_i}^{O}(U) = U$;

(3) $\sum\limits_{i=1}^{m} \underline{\mathscr{R}_i}^{O}(F) = \bigcup\limits_{i=1}^{m} \underline{\mathscr{R}_i}(F), \overline{\sum\limits_{i=1}^{m} \mathscr{R}_i}^{O}(F) = \bigcap\limits_{i=1}^{m} \overline{\mathscr{R}_i}(F)$;

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(4) $\sum\limits_{i=1}^{m}\underline{\mathscr{R}_i}{}^{O}\left(\sum\limits_{i=1}^{m}\underline{\mathscr{R}_i}{}^{O}(X)\right)=\sum\limits_{i=1}^{m}\underline{\mathscr{R}_i}{}^{O}(X),\ \overline{\sum\limits_{i=1}^{m}\mathscr{R}_i}{}^{O}\left(\overline{\sum\limits_{i=1}^{m}\mathscr{R}_i}{}^{O}(X)\right)=\overline{\sum\limits_{i=1}^{m}\mathscr{R}_i}{}^{O}(X);$

(5) $\sum\limits_{i=1}^{m}\underline{\mathscr{R}_i}{}^{O}(\sim F)=\sim\overline{\sum\limits_{i=1}^{m}\mathscr{R}_i}{}^{O}(F),\ \overline{\sum\limits_{i=1}^{m}\mathscr{R}_i}{}^{O}(\sim F)=\sim\sum\limits_{i=1}^{m}\underline{\mathscr{R}_i}{}^{O}(F);$

(6) $F\subseteq F'\Rightarrow\sum\limits_{i=1}^{m}\underline{\mathscr{R}_i}{}^{O}(F)\subseteq\sum\limits_{i=1}^{m}\underline{\mathscr{R}_i}{}^{O}(F'),\overline{\sum\limits_{i=1}^{m}\mathscr{R}_i}{}^{O}(F)\subseteq\overline{\sum\limits_{i=1}^{m}\mathscr{R}_i}{}^{O}(F').$

Proof.   (1) $\forall x\in U,$

$$\sum\limits_{i=1}^{m}\underline{\mathscr{R}_i}{}^{O}(F)(x)\ =\ \bigvee\limits_{i=1}^{m}\bigwedge\limits_{y\in U}S\big(1-R_i(x,y),F(y)\big)$$

$$\leq\ \bigvee\limits_{i=1}^{m}S\big(1-R_i(x,x),F(x)\big)=S\big(0,F(x)\big)=F(x)$$

$$\overline{\sum\limits_{i=1}^{m}\mathscr{R}_i}{}^{O}(F)(x)\ =\ \bigwedge\limits_{i=1}^{m}\bigvee\limits_{y\in U}T\big(R_i(x,y),F(y)\big)$$

$$\geq\ \bigwedge\limits_{i=1}^{m}T\big(R_i(x,x),F(x)\big)=T\big(1,F(x)\big)=F(x)$$

(2) $\forall x\in U,$

$$\sum\limits_{i=1}^{m}\underline{\mathscr{R}_i}{}^{O}(\varnothing)(x)\ =\ \bigvee\limits_{i=1}^{m}\bigwedge\limits_{y\in U}S\big(1-R_i(x,y),0\big)$$

$$=\ \bigvee\limits_{i=1}^{m}\bigwedge\limits_{y\in U}\big(1-R_i(x,y)\big)=\bigvee\limits_{i=1}^{m}\big(1-R_i(x,x)\big)=0$$

Similarly, it is not difficult to obtain that $\overline{\sum\limits_{i=1}^{m}\mathscr{R}_i}{}^{O}(\varnothing)=\varnothing.$

$\forall x\in U,$

$$\overline{\sum\limits_{i=1}^{m}\mathscr{R}_i}{}^{O}(U)(x)\ =\ \bigwedge\limits_{i=1}^{m}\bigvee\limits_{y\in U}T\big(R_i(x,y),U(y)\big)$$

$$=\ \bigwedge\limits_{i=1}^{m}\bigvee\limits_{y\in U}T\big(R_i(x,y),1\big)=\bigwedge\limits_{i=1}^{m}\bigvee\limits_{y\in U}R_i(x,y)=1$$

Similarly, it is not difficult to obtain that $\sum\limits_{i=1}^{m}\underline{\mathscr{R}_i}{}^{O}(U)=U.$

(3) $\forall x \in U$,

$$\underline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F)(x) = \bigvee_{i=1}^{m} \bigwedge_{y \in U} S\big(1 - R_i(x,y), F(y)\big) = \bigvee_{i=1}^{m} \underline{\mathscr{R}_i}(F)(x)$$

Similarly, it is not difficult to obtain that $\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F) = \bigcap_{i=1}^{m} \overline{\mathscr{R}_i}(F)$.

(4) By the proof of 1, since $\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F) \subseteq F$, then by the result of 4, we have

$$\underline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}\big(\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F)\big) \subseteq \overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F). \quad \text{Therefore, it must be proved that}$$

$$\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}\big(\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F)\big) \supseteq \overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F).$$

By 3, we have $\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F) = \bigcup_{i=1}^{m} \underline{\mathscr{R}_i}(F)$. Therefore,

$$\begin{aligned}
\underline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}\big(\underline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F)\big) &= \bigcup_{i=1}^{m} \underline{\mathscr{R}_i}\big(\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F)\big) = \bigcup_{i=1}^{m} \underline{\mathscr{R}_i}\big(\bigcup_{i=1}^{m} \underline{\mathscr{R}_i}(F)\big) \\
&\supseteq \bigcup_{i=1}^{m}\bigcup_{i=1}^{m} \underline{\mathscr{R}_i}(\underline{\mathscr{R}_i}(F)) = \bigcup_{i=1}^{m} \underline{\mathscr{R}_i}(F) = \overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F)
\end{aligned}$$

Similarly, it is not difficult to obtain that $\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}\big(\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(X)\big) = \overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(X)$.

(5) $\forall x \in U$,

$$\begin{aligned}
\underline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(\sim F)(x) &= \bigvee_{i=1}^{m} \bigwedge_{y \in U} S\big(1 - R_i(x,y), 1 - F(y)\big) \\
&= \bigvee_{i=1}^{m} \bigwedge_{y \in U} \big(1 - T\big(R_i(x,y), F(y)\big)\big) \\
&= 1 - \bigwedge_{i=1}^{m} \bigvee_{y \in U} T\big(R_i(x,y), F(y)\big) \\
&= 1 - \overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F)(x)
\end{aligned}$$

Similarly, it is not difficult to obtain that $\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(\sim F) = \sim \underline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F)$.

(6) $\forall x \in U$, since $F \subseteq F'$, then

$$\sum_{i=1}^{m} \underline{\mathscr{R}_i}^{O} (F)(x) = \bigvee_{i=1}^{m} \bigwedge_{y \in U} S\big(1 - R_i(x,y), F(y)\big)$$

$$\leq \bigvee_{i=1}^{m} \bigwedge_{y \in U} S\big(1 - R_i(x,y), F'(y)\big) = \sum_{i=1}^{m} \underline{\mathscr{R}_i}^{O} (F')(x)$$

Similarly, it is not difficult to obtain that $\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O} (F) \subseteq \overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O} (F')$.

$\square$

In the above theorem, (1) says that the optimistic multi–granulation lower and upper approximations satisfy the contraction and extension respectively; (2) says that the optimistic multi–granulation fuzzy rough set satisfies the normality and conormality; (3) expresses the relationships between multi–granulation and single granulation fuzzy rough sets; (4) and (5) shows the idempotency and complement of multi–granulation fuzzy rough set respectively; (6) shows the monotone of optimistic multi–granulation fuzzy rough approximation w.r.t. the variety of fuzzy target. These results are all multi–granulation generalization of the properties we showed in Theorem 2.6.

Theorem 3.3. *Let U be the universe of discourse, $\{\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_m\} \subseteq \mathscr{FR}(U)$ is a family of fuzzy relations on U, $\forall F_1, F_2, \cdots, F_n \in \mathscr{F}(U)$, the optimistic multi–granulation fuzzy rough set has the following properties:*

(1) $\sum_{i=1}^{m} \underline{\mathscr{R}_i}^{O} (\bigcap_{j=1}^{n} F_j) = \bigcup_{i=1}^{m} (\bigcap_{j=1}^{n} \underline{\mathscr{R}_i}(F_j))$, $\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O} (\bigcup_{j=1}^{n} F_j) = \bigcap_{i=1}^{m} (\bigcup_{j=1}^{n} \overline{\mathscr{R}_i}(F_j))$;

(2) $\sum_{i=1}^{m} \underline{\mathscr{R}_i}^{O} (\bigcap_{j=1}^{n} F_j) = \bigcap_{j=1}^{n} (\sum_{i=1}^{m} \underline{\mathscr{R}_i}^{O} (F_j))$, $\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O} (\bigcup_{j=1}^{n} F_j) = \bigcup_{j=1}^{n} (\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O} (F_j))$;

(3) $\sum_{i=1}^{m} \underline{\mathscr{R}_i}^{O} (\bigcup_{j=1}^{n} F_j) \supseteq \bigcup_{j=1}^{n} (\sum_{i=1}^{m} \underline{\mathscr{R}_i}^{O} (F_j))$, $\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O} (\bigcap_{j=1}^{n} F_j) \subseteq \bigcap_{j=1}^{n} (\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O} (F_j))$.

Proof. (1) $\forall x \in U,$

$$
\begin{aligned}
\underline{\sum_{i=1}^{m}\mathscr{R}_i}\,{}^{O}(\bigcap_{j=1}^{n}F_j)(x) &= \bigvee_{i=1}^{m}\bigwedge_{y\in U} S\big(1-R_i(x,y),(\bigcap_{j=1}^{n}F_j)(y)\big) \\
&= \bigvee_{i=1}^{m}\bigwedge_{y\in U} S\big(1-R_i(x,y),\bigwedge_{j=1}^{n}F_j(y)\big) \\
&= \bigvee_{i=1}^{m}\bigwedge_{y\in U}\bigwedge_{j=1}^{n} S\big(1-R_i(x,y),F_j(y)\big) \\
&= \bigvee_{i=1}^{m}\bigwedge_{j=1}^{n}\bigwedge_{y\in U} S\big(1-R_i(x,y),F_j(y)\big) \\
&= \bigvee_{i=1}^{m}\bigwedge_{j=1}^{n}\underline{\mathscr{R}_i}(F_j)(x) = \bigvee_{i=1}^{m}\big((\bigcap_{j=1}^{n}\underline{\mathscr{R}_i}(F_j))(x)\big) \\
&= (\bigcup_{i=1}^{m}(\bigcap_{j=1}^{n}\underline{\mathscr{R}_i}(F_j)))(x)
\end{aligned}
$$

Similarly, it is not difficult to prove $\overline{\sum_{i=1}^{m}\mathscr{R}_i}\,{}^{O}(\bigcup_{j=1}^{n}F_j) = \bigcap_{i=1}^{m}(\bigcup_{j=1}^{n}\overline{\mathscr{R}_i}(F_j)).$

(2) $\forall x \in U,$

$$
\begin{aligned}
\underline{\sum_{i=1}^{m}\mathscr{R}_i}\,{}^{O}(\bigcap_{j=1}^{n}F_j)(x) &= \bigvee_{i=1}^{m}\bigwedge_{y\in U} S\big(1-R_i(x,y),(\bigcap_{j=1}^{n}F_j)(y)\big) \\
&= \bigvee_{i=1}^{m}\bigwedge_{y\in U} S\big(1-R_i(x,y),\bigwedge_{j=1}^{n}F_j(y)\big) \\
&= \bigvee_{i=1}^{m}\bigwedge_{y\in U}\bigwedge_{j=1}^{n} S\big(1-R_i(x,y),F_j(y)\big) \\
&= \bigwedge_{j=1}^{n}\bigvee_{i=1}^{m}\bigwedge_{y\in U} S\big(1-R_i(x,y),F_j(y)\big) \\
&= \bigwedge_{j=1}^{n}\big(\underline{\sum_{i=1}^{m}\mathscr{R}_i}\,{}^{O}(F_j)(x)\big) = \big(\bigcap_{j=1}^{n}(\underline{\sum_{i=1}^{m}\mathscr{R}_i}\,{}^{O}(F_j))\big)(x)
\end{aligned}
$$

Similarly, it is not difficult to prove $\overline{\sum_{i=1}^{m}\mathscr{R}_i}\,{}^{O}(\bigcup_{j=1}^{n}F_j) = \bigcup_{j=1}^{n}(\overline{\sum_{i=1}^{m}\mathscr{R}_i}\,{}^{O}(F_j)).$

(3) By 3 of Theorem 3.2, since $\sum_{i=1}^{m}\mathscr{R}_i{}^{O}(F) = \bigcup_{i=1}^{m}\underline{\mathscr{R}_i}(F)$, then $\sum_{i=1}^{m}\mathscr{R}_i{}^{O}(\bigcup_{j=1}^{n}F_j) =$ $\bigcup_{i=1}^{m}\underline{\mathscr{R}_i}(\bigcup_{j=1}^{n}F_j)$. Moreover, by (8) of Theorem 2.6, it is not difficult to observe

that $\bigcup_{i=1}^{m}\underline{\mathscr{R}_i}(\bigcup_{j=1}^{n}F_j) \supseteq \bigcup_{i=1}^{m}\bigcup_{j=1}^{n}\underline{\mathscr{R}_i}(F_j) = \bigcup_{j=1}^{n}\bigcup_{i=1}^{m}\underline{\mathscr{R}_i}(F_j) = \bigcup_{j=1}^{n}(\sum_{i=1}^{m}\mathscr{R}_i{}^{O}(F_j))$, it

follows that $\sum_{i=1}^{m}\mathscr{R}_i{}^{O}(\bigcup_{j=1}^{n}F_j) \supseteq \bigcup_{j=1}^{n}(\sum_{i=1}^{m}\mathscr{R}_i{}^{O}(F_j))$.

Similarly, it is not difficult to obtain $\sum_{i=1}^{m}\overline{\mathscr{R}_i}{}^{O}(\bigcap_{j=1}^{n}F_j) \subseteq \bigcap_{j=1}^{n}(\sum_{i=1}^{m}\overline{\mathscr{R}_i}{}^{O}(F_j))$.

$\square$

In the above theorem, not only multi–fuzzy T–relations, but also a family of fuzzy targets are approximated. These formulas express the relationship between the optimistic fuzzy rough approximations of a single set and the optimistic fuzzy rough approximations of multi–sets under the multi–granulation fuzzy T–environment.

3.2. Pessimistic multi–granulation fuzzy rough set.

Definition 3.4. Let U be the universe of discourse, $\{\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_m\} \subseteq \mathbf{R}$ is a family of fuzzy T–similarity relations on U, then $\forall F \in \mathscr{F}(U)$, the pessimistic multi–granulation fuzzy rough lower approximation upper approximation of F are denoted by $\sum_{i=1}^{m}\mathscr{R}_i{}^{P}(F)$ and $\sum_{i=1}^{m}\overline{\mathscr{R}_i}{}^{P}(F)$ respectively, whose memberships for each $x \in U$ are:

(3.3)
$$\sum_{i=1}^{m}\mathscr{R}_i{}^{P}(F)(x) = \bigwedge_{i=1}^{m}\bigwedge_{y\in U}S\big(1 - R_i(x,y), F(y)\big),$$

(3.4)
$$\sum_{i=1}^{m}\overline{\mathscr{R}_i}{}^{P}(F)(x) = \bigvee_{i=1}^{m}\bigvee_{y\in U}T\big(R_i(x,y), F(y)\big).$$

If $\sum_{i=1}^{m}\mathscr{R}_i{}^{P}(F) = \sum_{i=1}^{m}\overline{\mathscr{R}_i}{}^{P}(F)$, then F is referred to as pessimistic definable under the multi–granulation fuzzy T–environment; otherwise, F is referred to as pessimistic undefinable. $\big[\sum_{i=1}^{m}\mathscr{R}_i{}^{P}(F), \sum_{i=1}^{m}\overline{\mathscr{R}_i}{}^{P}(F)\big]$ is referred to as a pair of pessimistic rough approximation of F under the multi–granulation fuzzy T–environment. The following are properties of pessimistic multi–granulation fuzzy rough approximation.

Theorem 3.5. *Let U be the universe of discourse, $\{\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_m\} \subseteq \mathbf{R}$ is a family of fuzzy T–similarity relations on U, then $\forall F \in \mathscr{F}(U)$, the pessimistic multi–granulation fuzzy rough set has the following properties:*

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(1) $\underline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (F) \subseteq F \subseteq \overline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (F)$;

(2) $\underline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (\varnothing) = \overline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (\varnothing) = \varnothing$, $\underline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (U) = \overline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (U) = U$;

(3) $\underline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (X) = \bigcap_{i=1}^{m} \underline{\mathscr{R}_i}(X)$, $\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (X) = \bigcup_{i=1}^{m} \overline{\mathscr{R}_i}(X)$;

(4) $\underline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (\underline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (X)) = \underline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (X)$, $\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (X)) = \overline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (X)$;

(5) $\underline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (\sim X) = \sim \overline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (X)$, $\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (\sim X) = \sim \underline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (X)$;

Proof. The proof of Theorem 3.5 is similar to the proof of Theorem 3.2. $\qquad\square$

Similar to the optimistic case, in Theorem 3.5, (1) says that the pessimistic multi–granulation lower and upper approximations satisfy the contraction and extension respectively; (2) says that the pessimistic multi–granulation fuzzy rough set satisfies the normality and conormality; (3) expresses the relationships between pessimistic multi–granulation and single granulation fuzzy rough sets; (4) and (5) shows the idempotency and complement of pessimistic multi–granulation fuzzy rough set respectively; (6) shows the monotone of pessimistic multi–granulation fuzzy rough approximation w.r.t. the variety of fuzzy target. These results are also multi–granulation generalization of the properties we showed in Theorem 2.6.

Theorem 3.6. *Let U be the universe of discourse, $\{\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_m\} \subseteq \mathscr{FR}(U)$ is a family of fuzzy relations on U, $\forall F_1, F_2, \cdots, F_n \in \mathscr{F}(U)$, the pessimistic multi–granulation fuzzy rough set has the following properties:*

(1) $\underline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (\bigcap_{j=1}^{n} X_j) = \bigcap_{i=1}^{m} (\bigcap_{j=1}^{n} \underline{\mathscr{R}_i}(X_j))$, $\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (\bigcup_{j=1}^{n} X_j) = \bigcup_{i=1}^{m} (\bigcup_{j=1}^{n} \overline{\mathscr{R}_i}(X_j))$;

(2) $\underline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (\bigcap_{j=1}^{n} X_j) = \bigcap_{j=1}^{n} (\underline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (X_j))$, $\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (\bigcup_{j=1}^{n} X_j) = \bigcup_{j=1}^{n} (\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (X_j))$;

(3) $\underline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (\bigcup_{j=1}^{n} X_j) \supseteq \bigcup_{j=1}^{n} (\underline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (X_j))$, $\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (\bigcap_{j=1}^{n} X_j) \subseteq \bigcap_{j=1}^{n} (\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{P} (X_j))$.

Proof. The proof of Theorem 3.6 is similar to the proof of Theorem 3.3. $\qquad\square$

Theorem 3.6 expresses the relationship between the pessimistic fuzzy rough approximations of a single set and the pessimistic fuzzy rough approximations of multi–sets under the multi–granulation fuzzy T–environment.

3.3. **Relationships among several models.**

Theorem 3.7. *Let U be the universe of discourse, $\{\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_m\} \subseteq \mathscr{FR}(U)$ is a family of fuzzy relations on U, $\forall F \in \mathscr{F}(U)$, we have*

(3.5)
$$\underline{\sum_{i=1}^{m} \mathscr{R}_i}^{P}(F) \subseteq \underline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F),$$

(3.6)
$$\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F) \subseteq \overline{\sum_{i=1}^{m} \mathscr{R}_i}^{P}(F).$$

Proof. $\forall x \in U$, we have

$$
\begin{aligned}
\underline{\sum_{i=1}^{m} \mathscr{R}_i}^{P}(F)(x) &= \bigwedge_{i=1}^{m} \bigwedge_{y \in U} S\big(1 - R_i(x,y), F(y)\big) \\
&\leq \bigvee_{i=1}^{m} \bigwedge_{y \in U} S\big(1 - R_i(x,y), F(y)\big) \\
&= \underline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F)(x).
\end{aligned}
$$

Similarly, it is not difficult to prove $\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F) \subseteq \overline{\sum_{i=1}^{m} \mathscr{R}_i}^{P}(F)$. $\qquad\square$

The above theorem express the relationship between optimistic and pessimistic multi–granulation fuzzy rough approximation. Formula (3.5) tells us that the pessimistic multi–granulation fuzzy lower approximation is included into the optimistic multi–granulation fuzzy lower approximation, formula (3.6) tells us that the pessimistic multi–granulation fuzzy upper approximation includes optimistic multi–granulation fuzzy upper approximation.

Theorem 3.8. *Let U be the universe of discourse, $\{\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_m\} \subseteq \mathscr{FR}(U)$ is a family of fuzzy relations on U, $\forall F \in \mathscr{F}(U), \forall i = 1, 2, \cdots, m$, we have*

(3.7)
$$\underline{\mathscr{R}_i}(F) \subseteq \underline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F),$$

(3.8)
$$\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{O}(F) \subseteq \overline{\mathscr{R}_i}(F),$$

(3.9)
$$\underline{\mathscr{R}_i}(F) \supseteq \underline{\sum_{i=1}^{m} \mathscr{R}_i}^{P}(F),$$

(3.10)
$$\overline{\sum_{i=1}^{m} \mathscr{R}_i}^{P}(F) \supseteq \overline{\mathscr{R}_i}(F).$$

Proof. It can be derived directly from Definition 2.5, Definition 3.1 and Definition 3.4. $\qquad\square$

The above theorem expresses the relationships between single granulation fuzzy rough approximation and optimistic(pessimistic) multi–granulation fuzzy rough approximation. Formula (3.7) says that the single granulation fuzzy lower approximation is included into the optimistic multi–granulation fuzzy lower approximation; formula (3.8) says that the single granulation fuzzy upper approximation includes the optimistic multi–granulation fuzzy upper approximation; formula (3.9) says that the single granulation fuzzy lower approximation includes the pessimistic multi–granulation fuzzy lower approximation; formula (3.10) says that the single granulation fuzzy upper approximation is included into the pessimistic multi–granulation fuzzy upper approximation.

4. Conclusions

In this paper, the constructive approach is used to generalize the multi–granulation rough set to fuzzy environment. The models of optimistic and pessimistic fuzzy rough approximations are then defined respectively. These multi–granulation fuzzy rough set are constructed on the basis of a family of fuzzy T–similarity relations instead of a single fuzzy relation.

In our further research, the approach to attribute reductions w.r.t. the proposed fuzzy rough model is an important issue to be addressed.

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