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Cubic q-ideals of BCI-algebras

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Abstract. The notion of cubic q-ideals in BCI-algebras is introduced. Relationship between a cubic ideal and a cubic q-ideal is discussed. Conditions for a cubic ideal to be a cubic q-ideal are provided. Characterizations of a cubic q-ideal are established. The cubic extension property for a cubic q-ideal is considered.

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1. Introduction

The study of BCK/BCI-algebras was initiated by K. Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of BCK/BCI-algebras, in particular, emphasis seems to have been put on the ideal theory of BCK/BCI-algebras. Fuzzy sets, which were introduced by Zadeh [7], deal with possibilistic uncertainty, connected with imprecision of states, perceptions and preferences. Based on the (interval-valued) fuzzy sets, Jun et al. [3] introduced the notion of cubic subalgebras/ideals in BCK/BCI-algebras, and then they investigated several properties. They discussed relationship between a cubic subalgebra and a cubic ideal. Also, they provided characterizations of a cubic subalgebra/ideal, and considered a method to make a new cubic subalgebra from old one. Jun et al. [4] introduced the notion of cubic ∘-subalgebras and closed cubic ideals in BCK/BCI-algebras, and then they investigated several properties. They provided relations between a cubic ideal and a cubic ∘-subalgebra in a BCK-algebra, and the relation between a closed cubic ideal and a cubic subalgebra in a BCI-algebra. They also investigated a condition for a cubic set in a BCK-algebra with condition (S) to be a cubic ideal. Finally, they dealt with a characterization of cubic ideal in a BCK/BCI-algebra.

In this paper, we introduce the notion of cubic q-ideals in BCI-algebras. We discuss relationship between a cubic ideal and a cubic q-ideal, and provide conditions for a cubic ideal to be a cubic q-ideal. We establish characterizations of a cubic q-ideal, and consider the cubic extension property for a cubic q-ideal.

2. Preliminaries

In this section we include some elementary aspects that are necessary for this paper.

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCI-algebra* if it satisfies the following axioms:

(I) $(\forall x,y,z\in X)\ (((x*y)*(x*z))*(z*y)=0)$,
(II) $(\forall x,y\in X)\ ((x*(x*y))*y=0)$,
(III) $(\forall x\in X)\ (x*x=0)$,
(IV) $(\forall x,y\in X)\ (x*y=0, y*x=0\ \Rightarrow\ x=y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x\in X)\ (0*x=0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following conditions:

(a1) $(\forall x\in X)\ (x*0=x)$,
(a2) $(\forall x,y,z\in X)\ (x*y=0\ \Rightarrow\ (x*z)*(y*z)=0,\ (z*y)*(z*x)=0)$,
(a3) $(\forall x,y,z\in X)\ ((x*y)*z=(x*z)*y)$,
(a4) $(\forall x,y,z\in X)\ (((x*z)*(y*z))*(x*y)=0)$.

We can define a partial ordering $\leq$ by $x\leq y$ if and only if $x*y=0$. A BCK-algebra X is said to be with *condition* (S) if, for all $x,y\in X$, the set $\{z\in X\mid z*x\leq y\}$ has a greatest element, written $x\circ y$. A BCI-algebra X is said to be *associative* if $(x*y)*z=x*(y*z)$ for all $x,y,z\in X$. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x*y\in S$ for all $x,y\in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies the following conditions:

(b1) $0\in I$,
(b2) $(\forall x,y\in X)\ (x*y\in I,\ y\in I\ \Rightarrow\ x\in I)$.

A subset I of a BCI-algebra X is called a *q-ideal* of X (see [5]) if it satisfies (b1) and

(b3) $(\forall x,y,z\in X)\ (x*(y*z)\in I,\ y\in I\ \Rightarrow\ x*z\in I)$.

Let I be a closed unit interval, i.e., $I=[0,1]$. By an *interval number* we mean a closed subinterval $\overline{a}=[a^-,a^+]$ of I, where $0\leq a^-\leq a^+\leq 1$. Denote by $D[0,1]$ the set of all interval numbers. Let us define what is known as *refined minimum* (briefly, rmin) of two elements in $D[0,1]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $D[0,1]$. Consider two interval numbers $\overline{a}_1:=\left[a_1^-,a_1^+\right]$ and $\overline{a}_2:=\left[a_2^-,a_2^+\right]$. Then

$$\operatorname{rmin}\{\overline{a}_1,\overline{a}_2\}=\left[\min\left\{a_1^-,a_2^-\right\},\min\left\{a_1^+,a_2^+\right\}\right],$$
$$\overline{a}_1\succeq\overline{a}_2 \text{ if and only if } a_1^-\geq a_2^- \text{ and } a_1^+\geq a_2^+,$$

and similarly we may have $\bar{a}_1 \preceq \bar{a}_2$ and $\bar{a}_1 = \bar{a}_2$. To say $\bar{a}_1 \succ \bar{a}_2$ (resp. $\bar{a}_1 \prec \bar{a}_2$) we mean $\bar{a}_1 \succeq \bar{a}_2$ and $\bar{a}_1 \neq \bar{a}_2$ (resp. $\bar{a}_1 \preceq \bar{a}_2$ and $\bar{a}_1 \neq \bar{a}_2$). Let $\bar{a}_i \in D[0,1]$ where $i \in \Lambda$. We define

$$\operatorname*{rinf}_{i\in\Lambda} \bar{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \operatorname*{rsup}_{i\in\Lambda} \bar{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

An *interval-valued fuzzy set* (briefly, *IVF set*) A defined on X is given by

$$A = \left\{\left(x, \left[\mu_A^-(x), \mu_A^+(x)\right]\right)\right\}, \forall x \in X \ \text{ (briefly, denoted by } A = \left[\mu_A^-, \mu_A^+\right]),$$

where μ_A^- and μ_A^+ are two fuzzy sets in X such that $\mu_A^-(x) \leq \mu_A^+(x)$ for all $x \in X$.

Let $\bar{\mu}_A(x) = \left[\mu_A^-(x), \mu_A^+(x)\right], \forall x \in X$. If $\mu_A^-(x) = \mu_A^+(x) = c$ (say) where $0 \leq c \leq 1$, then we have $\bar{\mu}_A(x) = [c, c]$ which we also assume, for the sake of convenience, to belong to $D[0,1]$. Thus $\bar{\mu}_A(x) \in D[0,1]$, $\forall x \in X$, and therefore the IVF set A is given by

$$A = \{(x, \bar{\mu}_A(x))\}, \forall x \in X, \ \text{ where } \bar{\mu}_A : X \to D[0,1].$$

We refer the reader to the books [1, 6] and the paper [2] for further information regarding BCK/BCI-algebras.

3. Cubic q-ideals

Definition 3.1. [3] Let X be a nonempty set. A *cubic set* $\mathscr{A}$ in a set X is a structure

$$\mathscr{A} = \{\langle x, A(x), \lambda(x)\rangle : x \in X\}$$

which is briefly denoted by $\mathscr{A} = \langle A, \lambda\rangle$ where $A = [\mu_A^-, \mu_A^+]$ is an IVF set in X and λ is a fuzzy set in X.

Definition 3.2. [3] A cubic set $\mathscr{A} = \langle A, \lambda\rangle$ in X is called a *cubic subalgebra* of a BCK/BCI-algebra X if it satisfies: for all $x, y \in X$,

(a) $\bar{\mu}_A(x * y) \succeq \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}$.
(b) $\lambda(x * y) \leq \max\{\lambda(x), \lambda(y)\}$.

Definition 3.3. [3] A cubic set $\mathscr{A} = \langle A, \lambda\rangle$ in a BCK/BCI-algebra X is called a *cubic ideal* of X if it satisfies: for all $x, y \in X$,

(a) $\bar{\mu}_A(0) \succeq \bar{\mu}_A(x)$.
(b) $\lambda(0) \leq \lambda(x)$.
(c) $\bar{\mu}_A(x) \succeq \text{rmin}\{\bar{\mu}_A(x * y), \bar{\mu}_A(y)\}$.
(d) $\lambda(x) \leq \max\{\lambda(x * y), \lambda(y)\}$.

Definition 3.4. A cubic set $\mathscr{A} = \langle A, \lambda\rangle$ in a BCI-algebra X is called a *cubic q-ideal* of X if it satisfies conditions (a) and (b) in Definition 3.3 and for all $x, y \in X$,

(a) $\bar{\mu}_A(x * z) \succeq \text{rmin}\{\bar{\mu}_A(x * (y * z)), \bar{\mu}_A(y)\}$.
(b) $\lambda(x * z) \leq \max\{\lambda(x * (y * z)), \lambda(y)\}$.

Example 3.5. Consider a BCI-algebra $X = \{0, a, b, c, d, e\}$ in which the $*$-operation is given by the Table 1. We define $A = [\mu_A^-, \mu_A^+]$ and λ by

$$A = \begin{pmatrix} 0 & a & b & c & d & e \\ [0.4, 0.8] & [0.4, 0.8] & [0.1, 0.3] & [0.1, 0.3] & [0.1, 0.3] & [0.1, 0.3] \end{pmatrix}$$

TABLE 1. $*$-operation

$*$	0	a	b	c	d	e
0	0	0	0	c	c	c
a	a	0	a	d	c	d
b	b	b	0	e	e	c
c	c	c	c	0	0	0
d	d	c	d	a	0	a
e	e	e	c	b	b	0

TABLE 2. $*$-operation

$*$	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	2	2	1	0	3
3	3	3	2	1	0

and

$$\lambda = \begin{pmatrix} 0 & a & b & c & d & e \\ 0.2 & 0.2 & 0.6 & 0.6 & 0.6 & 0.6 \end{pmatrix}.$$

Then $\mathscr{A} = \langle A, \lambda \rangle$ is a cubic q-ideal of X.

Note that every cubic q-ideal of a BCI-algebra X is a cubic ideal of X by taking $z = 0$ in Definition 3.4 and using (a1). But, the converse is not true as seen in the following example.

Example 3.6. Let $X = \{0, a, 1, 2, 3\}$ be a BCI-algebra with the $*$-operation given by Table 2. We define $A = [\mu_A^-, \mu_A^+]$ and λ by

$$A = \begin{pmatrix} 0 & a & 1 & 2 & 3 \\ [0.5, 0.9] & [0.3, 0.7] & [0.2, 0.6] & [0.2, 0.6] & [0.2, 0.6] \end{pmatrix}$$

and

$$\lambda = \begin{pmatrix} 0 & a & 1 & 2 & 3 \\ 0.2 & 0.2 & 0.6 & 0.4 & 0.6 \end{pmatrix},$$

respectively. Then $\mathscr{A} = \langle A, \lambda \rangle$ is a cubic ideal of X (see [3]). But, $\mathscr{A} = \langle A, \lambda \rangle$ is not a cubic q-ideal of X since

$$\bar{\mu}_A(3 * 1) = [0.2, 0.6] < [0.5, 0.9] = \text{rmin}\{\bar{\mu}_A(3 * (0 * 1)), \bar{\mu}_A(0)\}$$

and/or $\lambda(3 * 1) = 0.4 > 0.2 = \max\{\lambda(3 * (0 * 1)), \lambda(0)\}$.

We provide a condition for a cubic ideal to be a cubic q-ideal.

Theorem 3.7. *In an associative BCI-algebra, every cubic ideal is a cubic q-ideal.*

Proof. Let $\mathscr{A} = \langle A, \lambda\rangle$ be a cubic ideal of an associative BCI-algebra X. For any $x, y, z \in X$, we have

$$\begin{aligned}\bar{\mu}_A(x * z) &\succeq \operatorname{rmin}\{\bar{\mu}_A((x * z) * y), \bar{\mu}_A(y)\}\\ &= \operatorname{rmin}\{\bar{\mu}_A((x * y) * z), \bar{\mu}_A(y)\}\\ &= \operatorname{rmin}\{\bar{\mu}_A(x * (y * z)), \bar{\mu}_A(y)\}\end{aligned}$$

$$\begin{aligned}\lambda(x * z) &\leq \max\{\lambda((x * z) * y), \lambda(y)\}\\ &= \max\{\lambda((x * y) * z), \lambda(y)\}\\ &= \max\{\lambda(x * (y * z)), \lambda(y)\}\end{aligned}$$

Hence $\mathscr{A} = \langle A, \lambda\rangle$ is a cubic q-ideal of X. □

Corollary 3.8. *Let X be a BCI-algebra which satisfies any one of the following assertions:*

(1) $(\forall x \in X)\ (0 * x = x)$.
(2) $(\forall x, y \in X)\ (x * y = y * x)$.

Then every cubic ideal is a cubic q-ideal.

Corollary 3.9. *Let X be a quasi-associative BCI-algebra, that is, X is a BCI-algebra which satisfies the following inequality:*

$$(\forall x, y, z \in X)\ \big((x * y) * z \leq x * (y * z)\big).$$

If X satisfies one of the following conditions:

(1) $(\forall x \in X)\ (0 * (0 * x) = x)$,
(2) $(\forall x, y \in X)\ (0 * (y * x) = x * y)$,
(3) $(\forall x, y \in X)\ (x * y = 0 \ \Rightarrow\ x = y)$,
(4) $(\forall x, y, z \in X)\ (x * z = y * z \ \Rightarrow\ x = y)$,
(5) $(\forall x, y, z \in X)\ (z * x = z * y \ \Rightarrow\ x = y)$,
(6) $(\forall x, y, z \in X)\ ((y * x) * (z * x) = y * z)$,
(7) $(\forall x, y, z \in X)\ ((x * y) * (x * z) = z * y)$,
(8) $(\forall x, y, z \in X)\ ((x * y) * (x * z) = 0 * (y * z))$,
(9) $(\forall x, y, z, u \in X)\ ((x * y) * (z * u) = (x * z) * (y * u))$,
(10) $X = \{0\} \cup \{x \in X \mid 0 * x \neq 0\}$,

then every cubic ideal is a cubic q-ideal.

Theorem 3.10. *Let $\mathscr{A} = \langle A, \lambda\rangle$ be a cubic ideal of a BCI-algebra X in which the following inequalities are valid:*

$$(\forall x, y \in X)\ (\bar{\mu}_A(x * y) \succeq \bar{\mu}_A(x),\ \ \lambda(x * y) \leq \lambda(x)). \tag{3.1}$$

Then $\mathscr{A} = \langle A, \lambda\rangle$ is a cubic q-ideal of X.

Proof. Let $x, y, z \in X$. Using (c) and (d) in Definition 3.3, (a3) and (3.1), we have

$$\begin{aligned}\bar{\mu}_A(x * z) &\succeq \operatorname{rmin}\big\{\bar{\mu}_A\big((x * z) * (y * z)\big), \bar{\mu}_A(y * z)\big\}\\ &= \operatorname{rmin}\big\{\bar{\mu}_A\big((x * (y * z)) * z\big), \bar{\mu}_A(y * z)\big\}\\ &\succeq \operatorname{rmin}\big\{\bar{\mu}_A\big(x * (y * z)\big), \bar{\mu}_A(y)\big\}\end{aligned}$$

$$\begin{aligned}\lambda(x*z) &\leq \max\{\lambda\big((x*z)*(y*z)\big), \lambda(y*z)\}\\ &= \max\{\lambda\big((x*(y*z))*z\big), \lambda(y*z)\}\\ &\leq \max\{\lambda\big(x*(y*z)\big), \lambda(y)\}.\end{aligned}$$

Therefore $\mathscr{A} = \langle A, \lambda\rangle$ is a cubic q-ideal of X. □

Proposition 3.11. *Every cubic q-ideal $\mathscr{A} = \langle A, \lambda\rangle$ of a BCI-algebra X satisfies the following inequalities:*

(1) $\bar{\mu}_A(x*y) \succeq \bar{\mu}_A\big(x*(0*y)\big)$ *and* $\lambda(x*y) \leq \lambda\big(x*(0*y)\big)$,
(2) $\bar{\mu}_A(0*x) \succeq \bar{\mu}_A\big(0*(0*x)\big)$ *and* $\lambda(0*x) \leq \lambda\big(0*(0*x)\big)$

for all $x, y \in X$.

Proof. Straightforward. □

Let $\mathscr{A} = \langle A, \lambda\rangle$ be a cubic set in X. For any $r \in [0,1]$ and $[s,t] \in D[0,1]$, we define $U(\mathscr{A}; [s,t], r)$ as follows:

$$U(\mathscr{A}; [s,t], r) = \{x \in X \mid \bar{\mu}_A(x) \succeq [s,t],\ \lambda(x) \leq r\},$$

and we say it is a *cubic level set* of $\mathscr{A} = \langle A, \lambda\rangle$.

Theorem 3.12. *For a cubic set $\mathscr{A} = \langle A, \lambda\rangle$ in a BCI-algebra X, the following are equivalent:*

(1) *$\mathscr{A} = \langle A, \lambda\rangle$ is a cubic q-ideal of X.*
(2) *Every nonempty cubic level set of $\mathscr{A} = \langle A, \lambda\rangle$ is a q-ideal of X.*

Proof. Assume that $\mathscr{A} = \langle A, \lambda\rangle$ is a cubic q-ideal of X. Let $x, y \in X$, $r \in [0,1]$ and $[s,t] \in D[0,1]$. If $x \in U(\mathscr{A}; [s,t], r)$, then $\bar{\mu}_A(0) \succeq \bar{\mu}_A(x) \succeq [s,t]$ and $\lambda(0) \leq \lambda(x) \leq r$. Thus $0 \in U(\mathscr{A}; [s,t], r)$. Let $x, y, z \in X$ be such that $x*(y*z) \in U(\mathscr{A}; [s,t], r)$ and $y \in U(\mathscr{A}; [s,t], r)$. Then $\bar{\mu}_A(x*(y*z)) \succeq [s,t]$, $\lambda(x*(y*z)) \leq r$, $\bar{\mu}_A(y) \succeq [s,t]$ and $\lambda(y) \leq r$. It follows that

$$\bar{\mu}_A(x*z) \succeq \text{rmin}\{\bar{\mu}_A(x*(y*z)), \bar{\mu}_A(y)\} \succeq \text{rmin}\{[s,t],[s,t]\} = [s,t]$$

and $\lambda(x*z) \leq \max\{\lambda(x*(y*z)), \lambda(y)\} \leq \max\{r, r\} = r$ so that $x*z \in U(\mathscr{A}; [s,t], r)$. Hence $U(\mathscr{A}; [s,t], r)$ is a q-ideal of X. Conversely, suppose that any nonempty cubic level set of $\mathscr{A} = \langle A, \lambda\rangle$ is a q-ideal of X, that is, $U(\mathscr{A}; [s,t], r) \neq \varnothing$ and it is a q-ideal of X for all $r \in [0,1]$ and $[s,t] \in D[0,1]$. Assume that $\bar{\mu}_A(0) \prec \bar{\mu}_A(a)$, that is, $[\mu_A^-(0), \mu_A^+(0)] \prec [\mu_A^-(a), \mu_A^+(a)]$, or $\lambda(0) > \lambda(b)$ for some $a, b \in X$. If we take $s_a = \frac{1}{2}(\mu_A^-(0) + \mu_A^-(a))$, $t_a = \frac{1}{2}(\mu_A^+(0) + \mu_A^+(a))$ and $r_b = \frac{1}{2}(\lambda(0) + \lambda(b))$, then $\bar{\mu}_A(0) = [\mu_A^-(0), \mu_A^+(0)] \prec [s_a, t_a] \prec [\mu_A^-(a), \mu_A^+(a)] = \bar{\mu}_A(a)$ or $\lambda(0) > r_b > \lambda(b)$. Hence $0 \notin U(\mathscr{A}; [s_a, t_a], r_b)$. This is a contradiction, and so $\bar{\mu}_A(0) \succeq \bar{\mu}_A(x)$ and $\lambda(0) \leq \lambda(x)$ for all $x \in X$. Now suppose there exist $a, b, c \in X$ such that

$$\bar{\mu}_A(a*c) \prec \text{rmin}\{\bar{\mu}_A(a*(b*c)), \bar{\mu}_A(b)\}$$

or $\lambda(a*c) > \max\{\lambda(a*(b*c)), \lambda(b)\}$. Let $\bar{\mu}_A(a*c) = [(a*c)^-, (a*c)^+]$, $\bar{\mu}_A(b) = [b^-, b^+]$ and $\bar{\mu}_A(a*(b*c)) = [(a*(b*c))^-, (a*(b*c))^+]$. Take

$$s_0 = \tfrac{1}{2}((a*c)^- + \min\{(a*(b*c))^-, b^-\}),$$

$t_0 = \frac{1}{2}((a*c)^+ + \min\{(a*(b*c))^+, b^+\})$ and $r_0 = \frac{1}{2}(\lambda(a) + \max\{\lambda(a*(b*c)), \lambda(b)\})$. Then $(a*c)^- \prec s_0 \prec \min\{(a*(b*c))^-, b^-\}$ and $(a*c)^+ \prec t_0 \prec \min\{(a*(b*c))^+, b^+\}$, which imply that

$$\begin{aligned}\bar{\mu}_A(a*c) &= [(a*c)^-, (a*c)^+] \prec [s_0, t_0] \\ &\prec [\min\{(a*(b*c))^-, b^-\}, \min\{(a*(b*c))^+, b^+\}] \\ &= \text{rmin}\{\bar{\mu}_A(a*(b*c)), \bar{\mu}_A(b)\}.\end{aligned}$$

Also, $\lambda(a*c) > r_0 > \max\{\lambda(a*(b*c)), \lambda(b)\}$. Thus $a*(b*c) \in U(\mathscr{A}; [s_0, t_0], r_0)$ and $b \in U(\mathscr{A}; [s_0, t_0], r_0)$, but $a*c \notin U(\mathscr{A}; [s_0, t_0], r_0)$. This is a contradiction, and therefore $\bar{\mu}_A(x*z) \succeq \text{rmin}\{\bar{\mu}_A(x*(y*z)), \bar{\mu}_A(y)\}$ and $\lambda(x*z) \le \max\{\lambda(x*(y*z)), \lambda(y)\}$ for all $x, y, z \in X$. Hence $\mathscr{A} = \langle A, \lambda \rangle$ is a cubic q-ideal of X. □

Theorem 3.13. *If $\mathscr{A} = \langle A, \lambda \rangle$ is a cubic q-ideal of a BCI-algebra X, then the set*

$$I := \{x \in X \mid \bar{\mu}_A(x) = \bar{\mu}_A(0),\ \lambda(x) = \lambda(0)\}$$

is a q-ideal of X.

Proof. Obviously, $0 \in I$. Let $x, y, z \in X$ be such that $x*(y*z) \in I$ and $y \in I$. Then $\bar{\mu}_A\big(x*(y*z)\big) = \bar{\mu}_A(0) = \bar{\mu}_A(y)$ and $\lambda\big(x*(y*z)\big) = \lambda(0) = \lambda(y)$, and so

$$\bar{\mu}_A(x*z) \succeq \text{rmin}\big\{\bar{\mu}_A\big(x*(y*z)\big), \bar{\mu}_A(y)\big\} = \bar{\mu}_A(0)$$

and $\lambda(x*z) \le \max\big\{\lambda\big(x*(y*z)\big), \lambda(y)\big\} = \lambda(0)$. It follows from (a) and (b) in Definition 3.3 that $\bar{\mu}_A(x*z) = \bar{\mu}_A(0)$ and $\lambda(x*z) = \lambda(0)$ so that $x*z \in I$. Therefore I is a q-ideal of X. □

Lemma 3.14. *A cubic q-ideal $\mathscr{A} = \langle A, \lambda \rangle$ in a BCI-algebra X satisfies the following implication:*

(3.2) $$(\forall x, y \in X)\ \big(x \le y\ \Rightarrow\ \bar{\mu}_A(x) \succeq \bar{\mu}_A(y),\ \lambda(x) \le \lambda(y)\big).$$

Proof. If $x \le y$, then $x*y = 0$ and hence

$$\begin{aligned}\bar{\mu}_A(x) = \bar{\mu}_A(x*0) &\succeq \text{rmin}\{\bar{\mu}_A(x*(y*0)), \bar{\mu}_A(y)\} \\ &= \text{rmin}\{\bar{\mu}_A(x*y), \bar{\mu}_A(y)\} = \text{rmin}\{\bar{\mu}_A(0), \bar{\mu}_A(y)\} = \bar{\mu}_A(y)\end{aligned}$$

and

$$\begin{aligned}\lambda(x) = \lambda(x*0) &\le \max\{\lambda(x*(y*0)), \lambda(y)\} \\ &= \max\{\lambda(x*y), \lambda(y)\} = \max\{\lambda(0), \lambda(y)\} = \lambda(y).\end{aligned}$$

This completes the proof. □

Theorem 3.15. *If $\mathscr{A} = \langle A, \lambda \rangle$ is a cubic ideal of a BCI-algebra X, then the following assertions are equivalent:*

(1) *$\mathscr{A} = \langle A, \lambda \rangle$ is a cubic q-ideal of X.*
(2) $(\forall x, y \in X)\ \big(\bar{\mu}_A(x*y) \succeq \bar{\mu}_A(x*(0*y)),\ \lambda(x*y) \le \lambda(x*(0*y))\big)$.
(3) $(\forall x, y, z \in X)\ \big(\bar{\mu}_A((x*y)*z) \succeq \bar{\mu}_A(x*(y*z)),\ \lambda((x*y)*z) \le \lambda(x*(y*z))\big)$.

Proof. (1) ⇒ (2) follows from Proposition 3.11(1). Assume that (2) is valid. Note that

$$\begin{aligned}((x*y)*(0*z))*(x*(y*z)) &= ((x*y)*(x*(y*z)))*(0*z)\\ &\leq ((y*z)*y)*(0*z)=(0*z)*(0*z)=0\end{aligned}$$

for all $x, y, z \in X$. It follows from Lemma 3.14 that

$$\bar{\mu}_A(((x*y)*(0*z))*(x*(y*z))) \succeq \bar{\mu}_A(0)$$

and $\lambda(((x*y)*(0*z))*(x*(y*z))) \leq \lambda(0)$ so from (a) and (b) in Definition 3.3 that

$$\bar{\mu}_A(((x*y)*(0*z))*(x*(y*z))) = \bar{\mu}_A(0)$$

and $\lambda(((x*y)*(0*z))*(x*(y*z))) = \lambda(0)$. Using (2) and Definition 3.3, we have

$$\begin{aligned}\bar{\mu}_A((x*y)*z) &\succeq \bar{\mu}_A((x*y)*(0*z))\\ &\succeq \text{rmin}\{\bar{\mu}_A(((x*y)*(0*z))*(x*(y*z))), \bar{\mu}_A(x*(y*z))\}\\ &= \text{rmin}\{\bar{\mu}_A(0), \bar{\mu}_A(x*(y*z))\}\\ &= \bar{\mu}_A(x*(y*z))\end{aligned}$$

and

$$\begin{aligned}\lambda((x*y)*z) &\leq \lambda((x*y)*(0*z))\\ &\leq \max\{\lambda(((x*y)*(0*z))*(x*(y*z))), \lambda(x*(y*z))\}\\ &= \max\{\lambda(0), \lambda(x*(y*z))\}\\ &= \lambda(x*(y*z)).\end{aligned}$$

Therefore (3) is valid. Now suppose that (3) holds. Then

$$\begin{aligned}\bar{\mu}_A(x*z) &\succeq \text{rmin}\{\bar{\mu}_A((x*z)*y), \bar{\mu}_A(y)\}\\ &= \text{rmin}\{\bar{\mu}_A((x*y)*z), \bar{\mu}_A(y)\}\\ &\succeq \text{rmin}\{\bar{\mu}_A(x*(y*z)), \bar{\mu}_A(y)\}\end{aligned}$$

and

$$\begin{aligned}\lambda(x*z) &\leq \max\{\lambda((x*z)*y), \lambda(y)\}\\ &= \max\{\lambda((x*y)*z), \lambda(y)\}\\ &\leq \max\{\lambda(x*(y*z)), \lambda(y)\}\end{aligned}$$

for all $x, y, z \in X$. Hence $\mathscr{A} = \langle A, \lambda\rangle$ is a cubic q-ideal of X. □

Theorem 3.16. *For a cubic ideal $\mathscr{A} = \langle A, \lambda\rangle$ of a BCI-algebra X, the following are equivalent:*

(1) $\mathscr{A} = \langle A, \lambda\rangle$ *is a cubic q-ideal of X.*
(2) $\bar{\mu}_A((x*z)*y) \succeq \bar{\mu}_A((x*z)*(0*y))$ *and* $\lambda((x*z)*y) \leq \lambda((x*z)*(0*y))$ *for all $x, y, z \in X$.*
(3) $\bar{\mu}_A(x*y) \succeq \text{rmin}\{\bar{\mu}_A((x*z)*(0*y)), \bar{\mu}_A(z)\}$ *and* $\lambda(x*y) \leq \max\{\lambda((x*z)*(0*y)), \lambda(z)\}$ *for all $x, y, z \in X$.*

Proof. (1) ⇒ (2). It is straightforward by Theorem 3.15.

(2) ⇒ (3). For any $x, y, z \in X$, we have

$$\begin{aligned}\bar{\mu}_A(x * y) &\succeq \mathrm{rmin}\{\bar{\mu}_A((x * y) * z), \bar{\mu}_A(z)\} \\ &= \mathrm{rmin}\{\bar{\mu}_A((x * z) * y), \bar{\mu}_A(z)\} \\ &\succeq \mathrm{rmin}\{\bar{\mu}_A((x * z) * (0 * y)), \bar{\mu}_A(z)\}\end{aligned}$$

and

$$\begin{aligned}\lambda(x * y) &\leq \max\{\lambda((x * y) * z), \lambda(z)\} \\ &= \max\{\lambda((x * z) * y), \lambda(z)\} \\ &\leq \max\{\lambda((x * z) * (0 * y)), \lambda(z)\}.\end{aligned}$$

Hence (3) is valid.

(3) ⇒ (1). Assume that (3) is true. If we take $z = 0$ in (3), then

$$\begin{aligned}\bar{\mu}_A(x * y) &\succeq \mathrm{rmin}\{\bar{\mu}_A((x * 0) * (0 * y)), \bar{\mu}_A(0)\} \\ &= \mathrm{rmin}\{\bar{\mu}_A(x * (0 * y)), \bar{\mu}_A(0)\} \\ &= \bar{\mu}_A(x * (0 * y))\end{aligned}$$

and

$$\begin{aligned}\lambda(x * y) &\leq \max\{\lambda((x * 0) * (0 * y)), \lambda(0)\} \\ &= \max\{\lambda(x * (0 * y)), \lambda(0)\} \\ &= \lambda(x * (0 * y))\end{aligned}$$

for all $x, y \in X$. It follows from Theorem 3.15 that $\mathscr{A} = \langle A, \lambda \rangle$ is a cubic q-ideal of X. □

Theorem 3.17. (Cubic extension property for a cubic q-ideal) *Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \kappa \rangle$ be cubic ideals of a BCI-algebra X such that $\mathscr{A} \lessapprox \mathscr{B}$ and $\bar{\mu}_A(0) = \bar{\mu}_B(0)$ and $\lambda(0) = \kappa(0)$. If $\mathscr{A} = \langle A, \lambda \rangle$ is a cubic q-ideal of X, then so is $\mathscr{B} = \langle B, \kappa \rangle$.*

Proof. Let $x, y \in X$. If we take $a = x * (0 * y)$, then

$$(x * a) * (0 * y) = (x * (0 * y)) * a = 0.$$

Using Theorem 3.15, we have

$$\bar{\mu}_A((x * a) * y) \succeq \bar{\mu}_A((x * a) * (0 * y)) = \bar{\mu}_A(0) = \bar{\mu}_B(0)$$

and

$$\lambda((x * a) * y) \leq \lambda((x * a) * (0 * y)) = \lambda(0) = \kappa(0).$$

Thus $\bar{\mu}_B((x * a) * y) \succeq \bar{\mu}_A((x * a) * y) \succeq \bar{\mu}_B(0) \succeq \bar{\mu}_B(a)$ and

$$\kappa((x * a) * y) \leq \lambda((x * a) * y) \leq \kappa(0) \leq \kappa(a).$$

Since $\mathscr{B} = \langle B, \kappa \rangle$ is a cubic ideal, it follows that

$$\bar{\mu}_B(x * y) \succeq \mathrm{rmin}\{\bar{\mu}_B((x * y) * a), \bar{\mu}_B(a)\} = \bar{\mu}_B(a) = \bar{\mu}_B(x * (0 * y))$$

and $\kappa(x * y) \leq \max\{\kappa((x * y) * a), \kappa(a)\} = \kappa(a) = \kappa(x * (0 * y))$. Using Theorem 3.15, we conclude that $\mathscr{B} = \langle B, \kappa \rangle$ is a cubic q-ideal of X. □

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References

[1] Y. Huang, BCI-algebra, Science Press, Beijing, 2006.
[2] K. Iséki and S. Tanaka, An introduction to the theory of BCK-algebras, Math. Japonica 23 (1978) 1–26.
[3] Y. B. Jun, C. S. Kim and M. S. Kang, Cubic subalgebras and ideals of BCK/BCI-algebras, Far East. J. Math. Sci. (FJMS) 44 (2010) 239–250.
[4] Y. B. Jun and K. J. Lee, Closed cubic ideals and cubic $\circ$-subalgebras in BCK/BCI-algebras, Appl. Math. Sci. (in press).
[5] Y. L. Liu, J. Meng, X. H. Zhang and Z. C. Yue, q-ideals and a-ideals in BCI-algebras, Southeast Asian Bull. Math. 24 (2000) 243–253.
[6] J. Meng and Y. B. Jun, BCK-algebras, Kyungmoon Sa Co. Seoul, 1994.
[7] L. A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965) 338–353.

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