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Applications of fuzzy soft sets in ring theory

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Abstract. This paper aims to extend the notion of ring to the algebraic structures of fuzzy soft sets. Firstly, we give some new notions such as the product, extended product, restricted product, sum, extended sum and restricted sum of two fuzzy soft sets. Then, we define a new binary relation on fuzzy soft sets using binary relations on the universe and parameter sets. Also, we introduce the concept of fuzzy soft rings (ideals) and study some of their properties and structural characteristics. Finally, we define fuzzy soft function and fuzzy soft ring homomorphism, and then give theorem of homomorphic image and homomorphic pre-image of a fuzzy soft set under a fuzzy soft function.

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1. Introduction

To solve complicated problems in economics, engineering and environment, we cannot succesfully use classical mathematic methods because of various uncertainties typical for those problems. There are several well-known theories to describe uncertainty. For instance fuzzy sets theory [31], rough sets theory [24] and other mathematical tools. But all of these theories have their inherit difficulties as pointed out by Molodtsov [21]. To overcome these difficulties, Molodtsov introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties affecting existing methods.

At present, works on the soft set theory are progressing rapidly. Maji et al. [19] studied several operations on the theory of soft sets. Ali et al. [3] gave some new notions on soft sets. Some researches have studied algebraic properties of soft sets. Initially, Aktaş and Çağman [2] compared soft sets to the related concepts of fuzzy sets and rough sets. They also defined the notion of soft groups and derived their basic properties using Molodtsov's definition of the soft sets. Feng et al. [9] introduced the notions of soft semirings, soft ideals and idealistic soft semirings, and then investigated several related properties. Jun et al. [16] introduced the notions of soft p-ideals and p-idealistic soft BCI-algebras, and then gave characterizations of p-ideals in BCI-algebras. Liu et al. [17] described some classes of soft rings and gave the first, second and third fuzzy isomorphism theorems for soft rings. Acar et al. [1] defined soft rings, and introduced basic notions of soft rings. Çelik et al. [7] defined some new binary relation on soft sets, and also they investigated some new properties of soft rings. Yamak et al. [29] introduced the notion of soft hypergroupoids. Shabir and Ahmad [26] introduced the notions of soft ternary semigroups, soft ideals, soft quasi-ideals, soft bi-ideals and characterized some classes of the ternary semigroups. Aslam and Qurashi [5] extended the concept of soft group, and discussed some of their properties. They also defined normal soft group, cyclic soft group, abelian soft group, product of soft group, coset of a soft subgroup of a soft group. Jun et al. [15] introduced the notions of intersectional soft BCK/BCI-algebras and intersectional soft BCK/BCI-ideals in BCK/BCI-algebras based on subalgebras, and investigated of their properties. Wu and Zhan [28] introduce the notions of soft h-hemiregular hemirings, soft h-intra-hemiregular hemirings and h-semisimple hemirings, and gave some properties of them.

Some researches have studied algebraic properties of fuzzy soft sets. Firstly, Maji et al. [18] defined fuzzy soft set, and established some results on them. Jin-liang et al. [13] defined the operations on fuzzy soft groups, and proved some results on them. Aygünoğlu and Aygün [6] gave the concept of fuzzy soft group, and defined fuzzy soft function and fuzzy soft homomorphism. Majumdar and Samanta [20] defined generalised fuzzy soft sets, and studied some of their properties. They also showed applications of generalised fuzzy soft sets. Jun et al. [14] introduced the notion of fuzzy soft BCK/BCI algebras, and investigated some properties of them. Jiang et al. [12] proposed the notion of the interval-valued intuitionistic fuzzy soft set theory. İnan and Öztürk [10] introduced the concepts of fuzzy soft ring and $(\in, \in \vee q)$-fuzzy soft subring. They also studied some of their basic properties. Yin et al. [30] discussed the operation properties and algebraic structure of intuitionistic fuzzy soft sets. They also derived the lattice structures of intuitionistic fuzzy soft sets. Roy and Samanta [25] constructed a topology on a fuzzy soft set. Also they introduced the concepts of fuzzy soft base and fuzzy soft subbase. Srinivasan and Palaniappan [27] defined some operators on intuitionistic fuzzy sets of root type, and established their properties. Lastly, Dinda et al. [8] studied relations on generalised intuitionistic fuzzy soft sets, and gave a few algebraic propetties of them.

In this paper, we introduce the notion of fuzzy soft ring which is a generalization of soft rings introduced by Acar et al. [1].

From the beginning the majority of studies on fuzzy soft sets for algebraic structures such as, groups, rings, semigroups and BCK/BCI-algebras have concentrated on usual binary operations (see [6], [10], [13], [14], [18], [20]). However, this seems to restrict application for algebraic sets. To solve this problem, we define a new binary relation on fuzzy soft sets. From this point of view, in Section 2.3., we summarize some basic concepts of fuzzy soft sets which will be used throughout the paper. Also, we define product, extended product, restricted product, sum, extended sum and restricted sum of two fuzzy soft sets. In Section 3., we recall the concept of fuzzy soft rings introduced in [10], and discuss its further properties. Also, we derive some properties of fuzzy soft ideals. Moreover, we define fuzzy soft function and fuzzy soft ring homomorphism, and then give theorem of homomorphic image and homomorphic pre-image of a fuzzy soft set under a fuzzy soft function.

2. Preliminaries

In this section, we will give some known definitions and notations regarding fuzzy subsets, fuzzy subring (ideal), soft sets, soft ring (ideal) and fuzzy soft sets.

2. 1. Fuzzy Subsets on Rings

The definitions and notions, in this part, may be found in references [11], [22], [23], [31], [32]. Troghout this paper, let R be a ring.

Let $(R, +, \cdot)$ be a ring with zero element 0_R and A, B be two non-empty subsets of R. Then the set of all products $\{a_1b_1 + a_2b_2 + ... + a_nb_n \mid n \in \mathbb{N}, a_i \in A, b_i \in B\}$ is denoted by $A \cdot B$, the set of all sums $\{a + b \mid a \in A, b \in B\}$ is denoted by $A + B$ and the set of all mines $\{-a \mid a \in A\}$ is denoted by $-A$.

A non-empty subset I of a ring R is called a *subring* if and only if $a - b \in I$ and $ab \in I$ for all $a, b \in I$. A non-empty subset I of a ring R is called an *ideal* if and only if $a - b \in I$ and $ra, ar \in I$ for all $a, b \in I$ and $r \in R$.

Let R, S be two rings. A mapping $f : R \to S$ is called a *homomorphism* if $f(x+y) = f(x)+f(y)$ and $f(x\cdot y) = f(x)\cdot f(y)$ for all $x, y \in R$. A ring homomorphism $f : R \to S$ is called a *monomorphism* (resp. *epimorphism*, *isomorphism*), if it is an injective (resp. surjective, bijective) mapping (see [11]).

Fuzzy subset was introduced firstly by Zadeh [31]. A fuzzy subset μ of R is defined as a map from R to $[0, 1]$. The family of all fuzzy subsets of R is denoted by $\mathcal{F}(R)$. The following are most popular operations on fuzzy subsets: For all $\mu, \nu \in \mathcal{F}(R)$, $\omega \in \mathcal{F}(S)$ and $x \in R, y \in S$;

$$\begin{aligned}
(\mu \vee \nu)(x) &= \mu(x) \vee \nu(x); \\
(\mu \wedge \nu)(x) &= \mu(x) \wedge \nu(x); \\
(\mu \times \omega)(x, y) &= \mu(x) \wedge \omega(y); \\
(\mu + \nu)(x) &= \bigvee_{x=a+b} (\mu(a) \wedge \nu(b)); \\
(\mu \cdot \nu)(x) &= \bigvee_{x=\sum_{i=1}^{p} b_i . c_i} \{ \bigwedge_{1\le i\le p} (\mu(b_i) \wedge \nu(c_i)) \mid b_i, c_i \in R, \quad p \in \mathbb{N}\}; \\
-\mu(x) &= \mu(-x); \\
f(\mu)(y) &= \begin{cases} \bigvee_{f(a)=y} \mu(a) & \text{if} \quad f^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases} \\
f^{-1}(\omega)(x) &= \omega(f(x))
\end{aligned}$$

$\mu \le \nu$ if and only if $\mu(x) \le \nu(x)$ for all $x \in R$. For $T \subseteq R$, $\chi_T \in \mathcal{F}(R)$ is called *characteristic function* of T, and defined by $\chi_T(x) = 1$ if $x \in T$ and $\chi_T(x) = 0$ otherwise for all $x \in R$.

For any $\mu \in \mathcal{F}(R)$, the set $\{x \mid x \in R, \mu(x) \ge \alpha\}$ is called α-*level subset* of μ and denoted by μ_α. The mapping given by $Q : R \times R \to [0, 1]$ is called a *fuzzy relation* over R.

Definition 2.1 ([22])**.** Let $\mu \in \mathcal{F}(R)$. Then μ is called a *fuzzy subring* of R if

(i) $\mu(a - b) \geq \mu(a) \wedge \mu(b)$,
(ii) $\mu(ab) \geq \mu(a) \wedge \mu(b)$

for all $a, b \in R$.

If condition (ii) is replaced by (iii) $\mu(ab) \geq \mu(a) \vee \mu(b)$, then μ is called a *fuzzy ideal* of R .

Theorem 2.2 ([22])**.** *Let μ and ν be fuzzy ideals over R. Then $\mu + \nu$ and $\mu \cdot \nu$ are fuzzy ideals over R.*

Theorem 2.3 ([22])**.** *Let $f : X \to Y$ be a mapping. Let μ and ν be fuzzy subrings of X and Y, respectively. Then $f(\mu)$ and $f^{-1}(\nu)$ are fuzzy subrings of Y and X, respectively.*

Theorem 2.4 ([32])**.** *μ is a fuzzy subring over R if and only if $\mu_\alpha (\neq \emptyset)$ is a subring of R for all $\alpha \in [0, 1]$.*

Theorem 2.5 ([23])**.** *μ is a subring (ideal) of R if and only if χ_μ is a fuzzy subring (ideal) of R.*

2. 2. Soft Sets

Molodtsov [21] defined the notion of a soft set in the following way: Let R be an inital universe and E be a set of parameters. Let $P(R)$ denotes the power set of R and A be a non-empty subset of E. Then a pair (F, A) is called a *soft set* over R, where F is a mapping given by $F : A \to P(R)$. In other words, a soft set over R is a parameterized family of subsets of the universe R. For $x \in A$, $F(x)$ may be considered as the set of x-approximate elements of the soft set (F, A).

Definition 2.6 ([2])**.** Let $\mu \in \mathcal{F}(R)$ and μ_α be α-level set of μ. Then the soft set $(F_\mu, [0, 1])$, defined by $F_\mu(\alpha) = \mu_\alpha$ for all $\alpha \in [0, 1]$, is called *level soft set* of μ over R.

Definition 2.7 ([1], [7])**.** Let (F, A) be a soft set over R. Then (F, A) is called a *soft ring (ideal)* over R if $F(a)$ is a subring (ideal) of R for all $a \in A$.

Theorem 2.8. *If μ is a fuzzy subring (ideal) of R, then $(F_\mu, [0, 1])$ is a soft ring (ideal) over R.*

Proof. By using Theorem 2.4, the proof can be achived. □

2. 3. Fuzzy Soft Sets and Operations

In this part, we introduce the concept of fuzzy soft sets, and give some new notions regarding of them.

Definition 2.9 ([18])**.** Let R be a common universe, E be a set of parameters and $A \subseteq E$. Then a pair (F, A) is called a *fuzzy soft set* over R, where F is a mapping given by $F : A \to \mathcal{F}(R)$.

Example 2.10.

(i) Let (F, A) be a soft set over R. Then $(\overline{F}, A)$, defined by $\overline{F}(a) = \chi_{F(a)}$ for all $a \in A$, is a fuzzy soft set over R.

(ii) Let Q be a fuzzy relation over R. Let the mapping $F : R \to \mathcal{F}(R)$ be defined $F(r_1)(r_2) = Q(r_1, r_2)$ for all $r_1, r_2 \in R$. Then (F, R) is a fuzzy soft set over R.

(ii) Let $f : R_1 \to R_2$ be a mapping. Let the mapping $F : R_1 \to \mathcal{F}(R_2)$ be defined

$$F(r_1)(r_2) = \begin{cases} 1 & \text{if} \quad f(r_1) = r_2 \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

for all $r_1 \in R_1, r_2 \in R_2$. Then (F, R_1) is a fuzzy soft set over R_2.

Definition 2.11 ([18])**.** For two fuzzy soft sets (F, A) and (G, B) over a common universe R , we say that (F, A) is a *fuzzy soft subset* of (G, B) if

(i) $A \subseteq B$

(ii) $F(a) \le G(a)$ for all $a \in A$. In this case, we write $(F, A)\widetilde{\subseteq}(G, B)$.

Definition 2.12. Let R be a common universe, E be a set of parameters and $A \subset E$.

(a) (F, A) is called a *relative null fuzzy soft* set (with respect to the parameter set A), denoted by $\widetilde{\varnothing}_A$, if $F(a) = 0_R$ for all $a \in A$.

(b) (G, A) is called a *relative whole fuzzy soft* set (with respect to the parameter set A), denoted by $\widetilde{\Omega}_A$, if $G(a) = R$ for all $a \in A$.

The relative whole fuzzy soft set with respect to the set of parameters E is called the *absolute fuzzy soft set* over R and simply denoted by $\widetilde{\Omega}_E$. In a similar way, the relative null fuzzy soft set with respect to the E is called the *null fuzzy soft set* over R and is denoted by $\widetilde{\varnothing}_E$.

We shall denote by $\widetilde{\varnothing}_\varnothing$ the unique fuzzy soft set over R with an empty parameter set, which is called the *empty fuzzy soft set* over R.
Note that $\widetilde{\varnothing}_\varnothing$ and $\widetilde{\varnothing}_A$ are different fuzzy soft sets over R and $\widetilde{\varnothing}_\varnothing \subset \widetilde{\varnothing}_A \subset (F, A) \subset \widetilde{\Omega}_A \subset \widetilde{\Omega}_E$ over R.

Definition 2.13 ([18])**.** Let (F, A) and (G, B) be two fuzzy soft sets over a common universe R. Then,

(1) The $\wedge$-*intersection* of two fuzzy soft sets (F, A) and (G, B) is defined as the fuzzy soft set $(H, C) = (F, A)\widetilde{\wedge}(G, B)$ over R, where $C = A \times B$ and $H(a, b) = F(a) \wedge G(b)$ for all $(a, b) \in A \times B$.

(2) The $\vee$-*union* of two fuzzy soft sets (F, A) and (G, B) is defined as the fuzzy soft set $(H, C) = (F, A)\widetilde{\vee}(G, B)$ over R, where $C = A \times B$ and $H(a, b) = F(a) \vee G(b)$ for all $(a, b) \in A \times B$.

Definition 2.14 ([4])**.** Let (F, A) and (G, B) be two fuzzy soft sets over a common universe R. Then,

(1) The *extended union* of two fuzzy soft sets (F, A) and (G, B) is defined as the fuzzy soft set $(H, C) = (F, A)\widetilde{\cup}(G, B)$ over R, where $C = A \cup B$ and

$$H(c) = \begin{cases} F(c) & \text{if} \quad c \in A\backslash B \\ G(c) & \text{if} \quad c \in B\backslash A \\ F(c) \vee G(c) & \text{if} \quad c \in A \cap B \end{cases}$$

for all $c \in C$.

(2) The *restricted union* of two fuzzy soft sets (F, A) and (G, B) is defined as the fuzzy soft set $(H, C) = (F, A)\widetilde{\cup}_{\Re}(G, B)$ over R, where $C = A \cap B$ and $H(c) = F(c) \vee G(c)$ for all $c \in C$.

(3) The *extenden intersection* of two fuzzy soft sets (F, A) and (G, B) is defined as the fuzzy soft set $(H, C) = (F, A)\widetilde{\cap}_{\varepsilon}(G, B)$ over R, where $C = A \cup B$ and

$$H(c) = \begin{cases} F(c) & \text{if} \quad c \in A\backslash B \\ G(c) & \text{if} \quad c \in B\backslash A \\ F(c) \wedge G(c) & \text{if} \quad c \in A \cap B \end{cases}$$

for all $c \in C$.

(4) The *restricted intersection* of two fuzzy soft sets (F, A) and (G, B) is defined as the fuzzy soft set $(H, C) = (F, A)\widetilde{\cap}(G, B)$ over R, where $C = A \cap B$ and $H(c) = F(c) \wedge G(c)$ for all $c \in C$.

(5) The *cartesian product* of two fuzzy soft sets (F, A) and (G, B) is defined as the fuzzy soft set $(H, C) = (F, A)\widetilde{\times}(G, B)$ over R, where $C = A \times B$ and $H(a, b) = F(a) \times G(b)$ for all $(a, b) \in A \times B$.

In addition to the above, we can give following definition.

Definition 2.15. Let (F, A) and (G, B) be two fuzzy soft sets over a common universe R. Then,

(1) The *product* of two fuzzy soft sets (F, A) and (G, B) is defined as the fuzzy soft set $(H, C) = (F, A)\widetilde{\odot}(G, B)$ over R, where $C = A \times B$ and $H(a, b) = F(a) \cdot G(b)$ for all $(a, b) \in A \times B$.

(2) The *extended product* of two fuzzy soft sets (F, A) and (G, B) is defined as the fuzzy soft set $(H, C) = (F, A)\widetilde{\odot}_{\cup}(G, B)$ over R, where $C = A \cup B$ and

$$H(c) = \begin{cases} F(c) & \text{if} \quad c \in A\backslash B \\ G(c) & \text{if} \quad c \in B\backslash A \\ F(c) \cdot G(c) & \text{if} \quad c \in A \cap B \end{cases}$$

for all $c \in C$.

(3) The *restricted product* of two fuzzy soft sets (F, A) and (G, B) is defined as the fuzzy soft set $(H, C) = (F, A)\widetilde{\odot}_{\cap}(G, B)$ over R, where $C = A \cap B$ and $H(c) = F(c) \cdot G(c)$ for all $c \in C$.

(4) The *sum* of two fuzzy soft sets (F, A) and (G, B) is defined as the fuzzy soft set $(H, C) = (F, A)\widetilde{\oplus}(G, B)$ over R, where $C = A \times B$ and $H(a, b) = F(a) + G(b)$ for all $(a, b) \in A \times B$.

(5) The *extended sum* of two fuzzy soft sets (F, A) and (G, B) is defined as the fuzzy soft set $(H, C) = (F, A)\widetilde{\oplus}_{\cup}(G, B)$ over R, where $C = A \cup B$ and

$$H(c) = \begin{cases} F(c) & \text{if} \quad c \in A \backslash B \\ G(c) & \text{if} \quad c \in B \backslash A \\ F(c) + G(c) & \text{if} \quad c \in A \cap B \end{cases}$$

for all $c \in C$.

(6) The *restricted sum* of two fuzzy soft sets (F, A) and (G, B) is defined as the fuzzy soft set $(H, C) = (F, A)\widetilde{\oplus}_{\cap}(G, B)$ over R, where $C = A \cap B$ and $H(c) = F(c) + G(c)$ for all $c \in C$.

Now, we define a binary operation on fuzzy soft sets in the following way:
Suppose that $\oplus$ is a binary operation on $P(E)$, and $\otimes$ is a binary operation on $\mathcal{F}(R)$. Then for any two fuzzy soft set (F, A) and (G, B) over R, $(F, A) \oplus_{\otimes} (G, B)$ is defined as the fuzzy soft set (H, C), where $C = A \oplus B$ and

$$H(c) = \begin{cases} F(c) & \text{if} \quad c \in A \backslash B \\ G(c) & \text{if} \quad c \in B \backslash A \\ F(c) \otimes G(c) & \text{if} \quad c \in A \cap B \\ 0_R & \text{otherwise} \end{cases}$$

for all $c \in C$. Here we describe a general binary operation on fuzzy soft sets. Some binary operations given in Definition 2.14 and Definition 2.15 can be obtained a special case of above binary operation by choosing $\oplus \in \{\cup, \cap\}$ and $\otimes \in \{\vee, \wedge, +, \cdot\}$.

3. Fuzzy soft rings

In this section, we introduce the concept of fuzzy soft rings (ideals), and give some new fundamental properties of them.

Definition 3.1 ([10])**.** Let (F, A) be a fuzzy soft set over R. Then,

(1) (F, A) is said to be a *fuzzy soft ring* over R if $F(a)$ is a fuzzy subring of R for all $a \in A$.

(2) (F, A) is said to be a *fuzzy soft ideal* over R if $F(a)$ is a fuzzy ideal of R for all $a \in A$.

Example 3.2.

(1) Let μ be fuzzy subring of R. Then the fuzzy soft set $(F, [0, 1])$, defined by $F(\alpha) = \chi_{\mu_\alpha}$ for all $\alpha \in [0, 1]$, is a fuzzy soft ring over R.

(2) Let $(F, \mathbb{N})$ be a fuzzy soft set over R defined by

$$F(n)(r) = \begin{cases} 1 & \text{if} \quad n.r = 0_R \\ \alpha & \text{if} \quad \text{otherwise} \end{cases}$$

for all $n \in \mathbb{N}$, $r \in R$, $\alpha \in [0, 1]$. Then $(F, \mathbb{N})$ is a fuzzy soft ideal over R.

Theorem 3.3. *(F, A) is a soft ring (ideal) over R if and only if $(\overline{F}, A)$ is a fuzzy soft ring (ideal) over R.*

Proof. By using Theorem 2.5, the proof can be achived. □

Theorem 3.4. *Let (F, A) and (G, B) be two fuzzy soft rings (ideals) over R. Then,*

(1) *$(F,A)\widetilde{\cap}_{\varepsilon}(G,B)$ is a fuzzy soft ring (ideal) over R.*
(2) *$(F,A)\widetilde{\cap}(G,B)$ is a fuzzy soft ring (ideal) over R.*
(3) *If $F(c) \le G(c)$ or $G(c) \le F(c)$ for all $c \in A \cup B$, then $(F,A)\widetilde{\cup}(G,B)$ is a fuzzy soft ring (ideal) over R.*
(4) *If $F(c) \le G(c)$ or $G(c) \le F(c)$ for all $c \in A \cap B$, then $(F,A)\widetilde{\cup}_{\Re}(G,B)$ is a fuzzy soft ring (ideal) over R.*
(5) *If $F(a) \le G(b)$ or $G(b) \le F(a)$ for all $(a,b) \in A \times B$, then $(F,A)\widetilde{\vee}(G,B)$ is a fuzzy soft ring (ideal) over R.*

Proof.

(1) Let $(F,A)\widetilde{\cap}_{\varepsilon}(G,B) = (H, A \cup B)$. Clearly if $c \in A\backslash B$ or $c \in B\backslash A$ for all $c \in A \cup B$, then $H(c)$ is a fuzzy subring (ideal) of R. If $c \in A \cap B$, then $H(c) = F(c) \wedge G(c)$ is a fuzzy subring (ideal) of R since the intersection of two fuzzy subring (ideal) is a fuzzy subring (ideal) over R. Hence, $(F,A)\widetilde{\cap}_{\varepsilon}(G,B)$ is a fuzzy soft ring (ideal) over R.
(2) It is similar to (1).
(3) Let $(F,A)\widetilde{\cup}(G,B) = (H, A \cup B)$. Clearly if $c \in A\backslash B$ or $c \in B\backslash A$ for all $c \in A \cup B$, then $H(c)$ is a fuzzy subring (ideal) of R. If $F(c) \le G(c)$ or $G(c) \le F(c)$, then $H(c) = F(c) \vee G(c)$ is a fuzzy subring of R for all $c \in A \cap B$. Hence, $(F,A)\widetilde{\cup}_{\Re}(G,B)$ is a fuzzy soft ring (ideal) over R.
(4) It is similar to (3).
(5) It is straightforward.

□

Theorem 3.5. *Let (F,A) and (G,B) be two fuzzy soft rings (ideals) over R_1 and R_2, respectively. Then $(F,A)\widetilde{\times}(G,B)$ is a fuzzy soft ring (ideal) over $R_1 \times R_2$.*

Proof. Let $(F,A)\widetilde{\times}(G,B) = (H, A \times B)$. Then we have $H(a,b) = F(a) \times G(b)$ for all $(a,b) \in A \times B$. Since (F,A) and (G,B) are fuzzy soft rings over R_1 and R_2, then $F(a)$ and $G(b)$ are fuzzy subrings of R_1 and R_2 for all $(a,b) \in A \times B$, respectively. Also $(F(a) \times G(b))(x,y) = F(a)(x) \wedge G(b)(y)$ for all $x \in R_1$ and $y \in R_2$. Hence, we obtain $H(a,b) = F(a) \times G(b)$ is a fuzzy subring of $R_1 \times R_2$ for all $(a,b) \in A \times B$. Consequently, $(F,A)\widetilde{\times}(G,B)$ is a fuzzy soft ring over $R_1 \times R_2$.

For ideals, the proof is similar. □

Theorem 3.6. *Let (F,A) and (G,B) be two fuzzy soft ideals over R. Then,*

(1) *$(F,A)\widetilde{\odot}(G,B)$ is a fuzzy soft ideal over R.*
(2) *$(F,A)\widetilde{\odot}_{\cup}(G,B)$ is a fuzzy soft ideal over R.*
(3) *$(F,A)\widetilde{\odot}_{\cap}(G,B)$ is a fuzzy soft ideal over R.*
(4) *$(F,A)\widetilde{\oplus}(G,B)$ is a fuzzy soft ideal over R.*
(5) *$(F,A)\widetilde{\oplus}_{\cup}(G,B)$ is a fuzzy soft ideal over R*
(6) *$(F,A)\widetilde{\oplus}_{\cap}(G,B)$ is a fuzzy soft ideal over R.*

Proof.

(1) Let $(F,A)\widetilde{\odot}(G,B) = (H, A \times B)$. Since (F,A) and (G,B) are fuzzy soft ideals over R, then $F(a)$ and $G(b)$ are fuzzy ideals over R for all $a \in A$, $b \in B$. Also $F(a) \cdot G(b)$ is a fuzzy ideal over R for all $(a,b) \in A \times B$ since

the multiplication of two fuzzy ideals is a fuzzy ideal. Hence, $(F, A)\widetilde{\odot}(G, B)$ is a fuzzy soft ideal over R.

(2) Let $(F, A)\widetilde{\odot}_{\cup}(G, B) = (H, A \cup B)$. If $c \in A\backslash B$ or $c \in B\backslash A$ for all $c \in A \cup B$, then $H(c)$ is a fuzzy ideal of R. If $c \in A \cap B$, then $H(c) = F(c) \cdot G(c)$ is a fuzzy ideal of R since the multiplication of two fuzzy ideals is a fuzzy ideal. Hence, $(F, A)\widetilde{\odot}_{\cup}(G, B)$ is a fuzzy soft ideal over R.

(3) It is similar to (2).

(4) Let $(F, A)\widetilde{\oplus}(G, B) = (H, A \times B)$. Since (F, A) and (G, B) are fuzzy soft ideals over R, then $F(a)$ and $G(b)$ are fuzzy ideals over R for all $a \in A$, $b \in B$. Also $F(a) + G(b)$ is a fuzzy ideal over R for all $(a, b) \in A \times B$ since the addition of two fuzzy ideals is a fuzzy ideal. Hence $(F, A)\widetilde{\oplus}(G, B)$ is a fuzzy soft ideal over R.

(5) Let $(F, A)\widetilde{\oplus}_{\cup}(G, B) = (H, A \cup B)$. If $c \in A\backslash B$ or $c \in B\backslash A$ for all $c \in A \cup B$, then $H(c)$ is a fuzzy ideal of R. If $c \in A \cap B$, then $H(c) = F(c) + G(c)$ is a fuzzy ideal of R since the addition of two fuzzy ideal is a fuzzy ideal. Hence, $(F, A)\widetilde{\oplus}_{\cup}(G, B)$ is a fuzzy soft ideal over R.

(6) It is similar to (5).

□

Example 3.7. Let $R = \mathbb{Z}$, $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$. Consider the fuzzy soft ideals (F, A) and (G, B) over R defined as follows:

A	1	2	3
F	$\chi_{\mathbb{Z}}$	$\chi_{4\mathbb{Z}}$	$\chi_{3\mathbb{Z}}$

B	2	3	4
G	$\chi_{6\mathbb{Z}}$	$\chi_{5\mathbb{Z}}$	$\chi_{2\mathbb{Z}}$

If $(F, A)\widetilde{\odot}(G, B) = (H_1, A \cup B)$, $(F, A)\widetilde{\odot}_{\cup}(G, B) = (H_2, A \cup B)$, $(F, A)\widetilde{\odot}_{\cap}(G, B) = (H_3, A \cap B)$, $(F, A)\widetilde{\oplus}(G, B) = (H_4, A \cup B)$, $(F, A)\widetilde{\oplus}_{\cup}(G, B) = (H_5, A \cup B)$, $(F, A)\widetilde{\oplus}_{\cap}(G, B) = (H_6, A \cap B)$,

then we have following product and sum tables, respectively:

$A \times B$	$(1, 2)$	$(1, 3)$	$(1, 4)$	$(2, 2)$	$(2, 3)$	$(2, 4)$	$(3, 2)$	$(3, 3)$	$(3, 4)$
H_1	$\chi_{6\mathbb{Z}}$	$\chi_{5\mathbb{Z}}$	$\chi_{2\mathbb{Z}}$	$\chi_{12\mathbb{Z}}$	$\chi_{20\mathbb{Z}}$	$\chi_{4\mathbb{Z}}$	$\chi_{6\mathbb{Z}}$	$\chi_{15\mathbb{Z}}$	$\chi_{6\mathbb{Z}}$

$A \cup B$	1	2	3	4
H_2	$\chi_{\mathbb{Z}}$	$\chi_{12\mathbb{Z}}$	$\chi_{15\mathbb{Z}}$	$\chi_{2\mathbb{Z}}$

$A \cap B$	2	3
H_3	$\chi_{12\mathbb{Z}}$	$\chi_{15\mathbb{Z}}$

$A \times B$	$(1, 2)$	$(1, 3)$	$(1, 4)$	$(2, 2)$	$(2, 3)$	$(2, 4)$	$(3, 2)$	$(3, 3)$	$(3, 4)$
H_4	$\chi_{\mathbb{Z}}$	$\chi_{\mathbb{Z}}$	$\chi_{\mathbb{Z}}$	$\chi_{2\mathbb{Z}}$	$\chi_{\mathbb{Z}}$	$\chi_{2\mathbb{Z}}$	$\chi_{3\mathbb{Z}}$	$\chi_{\mathbb{Z}}$	$\chi_{\mathbb{Z}}$

$A \cup B$	1	2	3	4
H_5	$\chi_{\mathbb{Z}}$	$\chi_{2\mathbb{Z}}$	$\chi_{\mathbb{Z}}$	$\chi_{2\mathbb{Z}}$

$A \cap B$	2	3
H_6	$\chi_{2\mathbb{Z}}$	$\chi_{\mathbb{Z}}$

It is easy to see that the product, extended product, restricted product, sum, extended sum and restricted sum of two fuzzy soft ideals (F, A) and (G, B) is a fuzzy soft ideal over R.

Definition 3.8. Let (F, A) and (G, B) be two fuzzy soft sets over R_1 and R_2, respectively. Let $\phi : R_1 \to R_2$, $\psi : A \to B$ be two functions. Then we say that the pair (ϕ, ψ) is a *fuzzy soft function* from (F, A) to (G, B) denoted by $(\phi, \psi) : (F, A) \to (G, B)$ if $\phi(F(x)) = G(\psi(x))$ for all $x \in A$. If ϕ and ψ are injective (resp. surjective, bijective), then (ϕ, ψ) is said to be injective (resp. surjective, bijective).

In this definition, If ϕ is a ring homomorphism from R_1 to R_2, then (ϕ, ψ) is said to be a *fuzzy soft ring homomorphism*, and that (F, A) is *fuzzy soft homomorphic* to (G, B). The latter is denoted by $(F, A) \sim (G, B)$. If ϕ is an isomorphism from R_1 to R_2 and ψ is a bijective mapping from A onto B, then we say that (ϕ, ψ) is a *fuzzy soft ring isomorphism* and that (F, A) is *fuzzy soft isomorphic* to (G, B). The latter is denoted by $(F, A) \simeq (G, B)$.

Definition 3.9. Let (F, A) and (G, B) be two fuzzy soft sets over R_1 and R_2, respectively. Let (ϕ, ψ) be a fuzzy soft function from (F, A) to (G, B). Then,

(1) The image of (F, A) under the fuzzy soft function (ϕ, ψ) is defined as the fuzzy soft set $(\phi, \psi)(F, A) = (\phi(F), B)$ over R_2, where

$$\phi(F)(y) = \begin{cases} \bigvee_{\psi(x)=y} \phi(F(x)) & \text{if} \quad y \in Im\psi \\ 0_{R_2} & \text{otherwise} \end{cases}$$

for all $y \in B$.

(2) The pre-image of (G, B) under the fuzzy soft function (ϕ, ψ) is defined as the fuzzy soft set $(\phi, \psi)^{-1}(G, B) = (\phi^{-1}(G), A)$ over R_1, where

$$\phi^{-1}(G)(x) = \phi^{-1}(G(\psi(x)))$$

for all $x \in A$.

Lemma 3.10. *Let (F, A), (G, B) and (H, C) be fuzzy soft sets over R_1, R_2 and R_3, respectively. Let $(\phi, \psi) : (F, A) \to (G, B)$ and $(\varphi, \gamma) : (G, B) \to (H, C)$ be two fuzzy soft functions. Then $(\varphi \circ \phi, \gamma \circ \psi) : (F, A) \to (H, C)$ is a fuzzy soft function.*

Proof. Let $(\phi, \psi) : (F, A) \to (G, B)$ and $(\varphi, \gamma) : (G, B) \to (H, C)$ be two fuzzy soft functions. Then we can write $\phi(F(x)) = G(\psi(x))$ and $\varphi(G(y)) = H(\gamma(y))$ for all $x \in A, y \in B$. Since $\varphi \circ \phi : R_1 \to R_3$ and $\gamma \circ \psi : A \to C$, then $\varphi \circ \phi(F(x)) = \varphi(\phi(F(x))) = \varphi(G(\psi(x))) = H(\gamma(\psi(x))) = H(\gamma \circ \psi(x))$ for all $x \in A$. Hence, $(\varphi \circ \phi, \gamma \circ \psi) : (F, A) \to (H, C)$ is a fuzzy soft function. □

Theorem 3.11. *Let (F, A), (G, B) and (H, C) be fuzzy soft sets over the rings R_1, R_2 and R_3, respectively. Let $(\phi, \psi) : (F, A) \to (G, B)$ and $(\varphi, \gamma) : (G, B) \to (H, C)$ be two fuzzy soft ring homomorphism. Then $(\varphi \circ \phi, \gamma \circ \psi) : (F, A) \to (H, C)$ is a fuzzy soft ring homomorphism.*

Proof. It is straightforward. □

Theorem 3.12. *Let (F, A) and (G, B) be two fuzzy soft rings over R_1 and R_2, respectively. Let (ϕ, ψ) be a fuzzy soft ring homomorphism from (F, A) to (G, B). Then,*

(1) *If $\phi : R_1 \to R_2$ is a epimorphism of rings and ψ is a bijective mapping, then $(\phi(F), B)$ is a fuzzy soft ring over R_2.*
(2) *If $\phi : R_2 \to R_1$ is a epimorphism of rings, then $(\phi^{-1}(G), A)$ is a fuzzy soft ring over R_1.*

Proof.

(1) Let $y \in B$. Since ψ is bijective, then there exist a unique $x \in A$ such that $\psi(x) = y$. Since $F(x)$ is a fuzzy subring of R_1 and ϕ is a epimorphism, then $\phi(F(x))$ is a fuzzy subring of R_2. Also, $\phi(F)(y) = \phi(F(x))$ is a fuzzy subring over R_2. Hence, $(\phi(F), B)$ is a fuzzy soft ring over R_2.
(2) Since $\psi(x) \in B$ for all $x \in A$ and (G, B) is a fuzzy soft ring over R_2, then $G(\psi(x))$ is a fuzzy subring of R_2 for all $x \in A$. Also, its homomorphic inverse image $\phi^{-1}(G(\psi(x)))$ is also a fuzzy subring of R_1 for all $x \in A$. Hence, $(\phi^{-1}(G), A)$ is a fuzzy soft ring over R_1.

□

4. Conclusions

In this paper, we studied algebraic properties of fuzzy soft sets in ring structure. We established some new notions for fuzzy soft sets such as product, extended product, restricted product, sum, extended sum and restricted sum. Also, we introduced the notion of fuzzy soft rings (ideals), and gave related examples. Moreover, we investigated some new properties of fuzzy soft rings (ideals). To extend this work, one could study the properties of fuzzy soft sets in other algebraic structures such as modules and fields. In the future, we will consider some applications of fuzzy soft rings to a decision making problem.

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