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Type-2 fuzzy differential inclusion for solving type-2 fuzzy differential equation

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Abstract. In this article, we have solved the type-2 fuzzy differential equation. The nonlinear type-2 triangular fuzzy numbers(NT2TFNs) are taken, and their different arithmetic properties are manifested. We have taken the type-2 fuzzy initial and type-2 fuzzy boundary value problem. Type-2 fuzzy differential inclusion concepts solve the differential equations. Numerical examples with solution graphs have also been provided to illustrate the outcomes of the proposed theory.

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1. Introduction

Information, by its very nature, contains uncertainty. Today, accessing and processing sometimes hazy, imperfect, loud, partial, or occasionally conflicting information is a significant component of real world managerial scenarios. fuzzy logic is just an example of the robust tools in the field of computer intelligence that was made possible by developing soft computing techniques. Fuzzy differential equation (FDE) is an important tool for modelling uncertain systems [1]. It is possible to think of fuzzy differential equations (FDEs) as an instance of differential equations that have been fuzzy logic-generalized. When specific coefficients, parameters, or boundary conditions are thought of as belonging to a category of fuzzy sets, the resulting differential equation is referred to as a fuzzy differential equation. A fuzzy differential equation is simplified to a differential equation whose uncertain parameters, coefficients, and conditions are viewed as the degranulation of the precise ends. The outcome of granular precision in the setting of FDEs is of the sort of possibility distribution. As a result, a fuzzy differential equation can also be referred to as a probabilistic differential equation. After introducing the fuzzy set by Zadeh [2], Chang and Zadeh [3] defined the fuzzy derivative, which later preceded the literature on fuzzy differential equations. Following their path, the calculus of fuzzy valued functions gained the research community's attention resulting innovation of different definitions of fuzzy derivatives, including the Hukuhara derivative by Puri and Ralescu [4].

Despite being presented in various ways, it has been demonstrated that all of these fuzzy derivatives are analogous, given that the lower and higher level cuts of the relevant fuzzy function are continuous. After that, Kaleva [5, 6] and Seikkala [7] developed the FDE theory. Hukuhara and Seikkala derivatives are the two most well-known fuzzy derivatives among those given. Hukuhara and Seikkala derivative definitions differ in that Seikkala derivative is based on derivatives of the lower and upper-level cuts of the relevant fuzzy function. In contrast, Hukuhara derivative is defined essentially based on what is known as the Hukuhara difference (H-difference). Investigations on the existence and distinctness of the solution for FDEs under the H-derivative and Seikkala derivative was done in several worthy works [8, 9, 10, 11, 12]. Also, Dubois and Prade used the extension principle for solving FDE [13]. However, the research findings have shown that these derivatives have several significant drawbacks, the most critical of which is that the diameter of the fuzzy function being studied must essentially be non-decreasing. This restriction causes the determined solution of an FDE to frequently differ from what could be intuitively deduced from the features of the system or phenomenon that the FDE is modelling. P. Diamond [14] has shown that the solutions of FDE using H-derivative is sometimes unbounded as $t \to \infty$. So, it does not reflect the rich behaviour of uncertain systems and it is quite different from the crisp solution. Hullermeier [15] overcame this problem by introducing Fuzzy Differential Inclusion (FDI). In FDI, he replaced FDE by a family of differential equations at each $\alpha-$ level for $0 \leqslant \alpha \leqslant 1$, where the solution is defined by the $\alpha-$ level sets. Recently, Min et al. [16] discussed the existence of the solutions for fuzzy implicit differential inclusions generalizing some established results of fuzzy differential equations and inclusions. The applications of fuzzy differential equations are also found in different literature. Several authors are already shown interest in modelling inventory control problems [17, 18, 19, 20], diabetes modelling [21, 22], mechanics problem [23], Arms Race problem [24, 25], Bio mathematical modelling [26, 27, 28, 29, 30] (See the work [31] for more theoretical development).

Fuzzy sets have long been used as an effective tool in many fields, including pattern recognition, medicinal applications, engineering difficulties, etc. Although using fuzzy sets makes sense in many situations, it does not appear acceptable in more uncertain settings. While occasionally, an object's membership in the universe of discourse is unknown, fuzzy sets assign each member of the primary domain a single value from the range $[0, 1]$ as their membership value. This characteristic allowed the researchers to discover type-2 fuzzy sets with fuzzy membership functions rather than single values. As type-2 fuzzy differential inclusion can handle more uncertainties than FDE, it earns the interests of many researchers [32, 33, 34, 35, 36, 37, 38, 39]. Despite a few noted developments in this direction, we observed that there are no such works in which the differential inclusion method has been manifested in a type

2 fuzzy setting. Making an end to this hiatus, we have studied the Type-2 fuzzy differential inclusion(T2FDI) by replacing T2FDE by a family of differential equations at each $(\alpha-\tilde{\alpha})$-level sets in this article. The solution of T2FDI has been obtained as $(\alpha-\tilde{\alpha})$-level sets. We have considered an initial value problem and a second order boundary value problem to solve them by using T2FDI. Numerical examples with graphical representations of the solutions has been shown. As mentioned earlier, type-2 fuzzy sets generalize the uncertainty carried by fuzzy sets, and fuzzy differential inclusions replace the fuzzy differential equation with broader perspectives; this present work contributes a significant content amalgamating two noted theories.

The remaining of this article is designed as follows. Section 2 presents some preliminaries on type-2 fuzzy sets. In section 3, non-linear type-2 fuzzy numbers are defined, and their arithmetic properties are introduced. Section 4 is about the theory of type 2 fuzzy differential inclusions. The conclusions at the end of this study are made in section 5.

2. Preliminaries

This present article focus to demonstrate the solution of the uncertain differential equation using differential inclusion method for type-2 fuzzy valued variables. Thus, a few fundamental definitions of type-2 fuzzy sets in this part are connected to the work introduced.

Definition 2.1. Fuzzy set: [2] Let U be a given collection which is taken to be the universal discourse. A *fuzzy set* $\tilde{A}$ is denoted by an ordered pair

$$\tilde{A} = \{(x, \mu_A(x)) \,|\, x \in U\}.$$

In the ordered pair, x is an element of $\tilde{A}$ and $\mu_A(x)$ is representing the measure of belonging of x in the set $\tilde{A}$. $\mu_A(x)$ is called the *membership function* having range $[0,1]$. The membership function substitutes the characteristic function of the crisp set for generalizing it introducing fuzzy set.

Further generalization can be made by introducing type-2 fuzzy set. The membership values are real numbers lying in $[0,1]$ in classical fuzzy set. Now, the notion is extended to a wider phenomena in which the membership values are themselves fuzzy sets. This extension defines type 2 fuzzy sets as following.

Definition 2.2. Type-2 fuzzy set: [40] Let $\tilde{A}$ be a classical fuzzy set defined on U with its membership function $\mu_A(x) = u$. If, furthermore, the membership value $\mu_A(x) = u$ is a fuzzy set on $[0,1]$, then $\tilde{A}$ is called a *type-2 fuzzy set* (T2FS) in U and it is defined as

$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u)) : x \in U, u \in [0,1]\}. \tag{2.1}$$

In equation 2.1, $\mu_{\tilde{A}}(x,u)$ is representing the type-2 membership function (T2MF) of the type-2 fuzzy set $\tilde{A}$ defined in U and $\mu_{\tilde{A}}(x,u)$ always belongs to $[0,1]$.

Example 2.3. Let $U = \{x_1, x_2, x_3\}$ be discrete universe of discourse with primary membership grades as follows: $\mu_A(x_1) = \{0, 0.2\} = u_1$, $\mu_A(x_2) = \{0.2, 0.4, 0.6\} = u_2$ and $\mu_A(x_3) = \{0.6, 0.8, 1\} = u_3$.

Suppose u_1, u_2 and u_3 are all fuzzy sets on $[0,1]$. Also, let the secondary membership grades of u_1 is given by

$$\mu_A(x_1, 0) = 1,\ \mu_A(x_1, 0.2) = 0.3.$$

Let the secondary membership grades of elements of u_2 is given by

$$\mu_A(x_2, 0.2) = 0.6,\ \mu_A(x_2, 0.4) = 1,\ \mu_A(x_3, 0.6) = 0.7.$$

Let the secondary membership grades of elements of u_3 is given by

$$\mu_A(x_3, 0.6) = 0.4,\ \mu_A(x_3, 0.8) = 0.7,\ \mu_A(x_3, 1) = 1.$$

Then the type-2 fuzzy set $\tilde{A}$ can be described by

$$(1/0 + 0.3/0.2)/x_1 + (0.6/0.2 + 1/0.4 + 0.7/0.6)/x_2 + (0.4/0.6 + 0.7/0.8 + 1/1)/x_3.$$

Definition 2.4. Support of a T2FS: [41] The *support* of a T2FS $\tilde{A}$ given by equation 2.1 is denoted by J_x^u and it is defined as

$$J_x^u = [u : u \in [0,1], \mu_{\tilde{A}}(x,u) > 0]. \tag{2.2}$$

Example 2.5. Suppose a T2FS is given by $\tilde{A} = (1/0 + 0.3/0.2)/x_1 + (0.6/0.2 + 1/0.4 + 0.7/0.6)/x_2 + (0.4/0.6 + 0.7/0.8 + 1/1)/x_3$. Then its support is $J_x^u = \{u_1, u_2, u_3\}$.

Definition 2.6. $\tilde{\alpha}$-plane of T2FS: [42] Let $\tilde{A}$ be a T2FS in the universal discourse U. Then its *$\tilde{\alpha}$-plane* is defined as

$$\tilde{A}_{\tilde{\alpha}} = \{((x,u), \mu_{\tilde{A}}(x,u)); \mu_{\tilde{A}}(x,u) \geq \tilde{\alpha}, \tilde{\alpha} \in [0,1]\}.$$

Example 2.7. If we take $\tilde{\alpha} = 0$, then $\tilde{A}_0 = \{((x_1,0),1), ((x_1,0.2),0.3), ((x_2,0.2),0.6), ((x_2,0.4),1), ((x_3,0.6),0.7), ((x_3,0.6),0.4), ((x_3,0.8),0.7), ((x_3,1),1)\}$.
If $\tilde{\alpha} = 0.5$, then $\tilde{A}_{0.5} = \{((x_1,0),1), ((x_2,0.2),0.6), ((x_2,0.4),1), ((x_3,0.6),0.7), ((x_3,0.8),0.7), ((x_3,1),1)\}$.
If $\tilde{\alpha} = 1$, then $\tilde{A}_1 = \{((x_1,0),1), ((x_2,0.4),1), ((x_3,1),1)\}$.

Definition 2.8. Footprint of uncertainty (FOU): [41] Let $\tilde{A}$ be a T2FS in the universal discourse U. Following is the set denoting the *footprint of uncertainty*

$$\tilde{A}_0 = \{((x,u), \mu_{\tilde{A}}(x,u)) \geq 0\}.$$

Definition 2.9. Type-2 triangular fuzzy number (T2TFN): A *Type-2 triangular fuzzy number* (T2TFN) $\tilde{A}$ is denoted by $\tilde{A} = (a_1, a_2, a_3, m, b_1, b_2, b_3)$ with seven real numbers, where $a_1 \leq a_2 \leq a_3 \leq m \leq b_1 \leq b_2 \leq b_3$; $m, a_i, b_i \in \mathbb{R}$ for $i = 1,2,3$. Here, (a_1, m, b_3), (a_3, m, b_1) and (a_2, m, b_2) are three triangular fuzzy numbers (TFNs) and their membership functions are called *upper, lower* and *primary membership function* respectively. Let, $\tilde{A}_1 = (a_1, m, b_3)$, $\tilde{A}_2 = (a_2, m, b_2)$ and $\tilde{A}_3 = (a_3, m, b_1)$, then the following membership profiles are provided:

$$\mu_{A_1}(x) = \begin{cases} \frac{x-a_1}{m-a_1}; & \text{if } a_1 \leq x \leq m \\ \frac{b_3-x}{b_3-m}; & \text{if } m \leq x \leq b_3 \\ 0; & \text{otherwise,} \end{cases} \tag{2.3}$$

$$\mu_{A_2}(x) = \begin{cases} \frac{x-a_2}{m-a_2}; & \text{if } a_2 \le x \le m \\ \frac{b_3-x}{b_3-m}; & \text{if } m \le x \le b_2 \\ 0; & \text{otherwise,} \end{cases} \tag{2.4}$$

$$\mu_{A_3}(x) = \begin{cases} \frac{x-a_3}{m-a_3}; & \text{if } a_3 \le x \le m \\ \frac{b_3-x}{b_3-m}; & \text{if } m \le x \le b_1 \\ 0; & \text{otherwise.} \end{cases} \tag{2.5}$$

α-cuts of $\tilde{A}_1$ is $\tilde{A}_{1\alpha} = [L^\alpha_{A_1}, R^\alpha_{A_3}]$, α-cuts of $\tilde{A}_2$ is $\tilde{A}_{2\alpha} = [L^\alpha_{A_2}, R^\alpha_{A_2}]$ and α-cuts of $\tilde{A}_3$ is $\tilde{A}_{3\alpha} = [L^\alpha_{A_3}, R^\alpha_{A_1}]$.

To put it another way, the type-2 triangular fuzzy number previously defined is sometimes referred as the linear type-2 triangular fuzzy number (LT2TFN).

Definition 2.10. **$\tilde{\alpha}$-plane representation:** *$\tilde{\alpha}$-plane representation* of a T2TFN $\tilde{A}$ is $[\underline{A_{\tilde{\alpha}}}, \overline{A_{\tilde{\alpha}}}]$, where $\tilde{\alpha} \in [0, 1]$.

The components of a T2TFN $\tilde{A}$ in $\tilde{\alpha}$-plane depiction is further given by the succeeding definition.

Definition 2.11. **α-cut representation of $\tilde{\alpha}$-plane:** *α-cut representation of* $\underline{A_{\tilde{\alpha}}}$ is $[\underline{L}^\alpha_{\underline{A_{\tilde{\alpha}}}}, \underline{R}^\alpha_{\underline{A_{\tilde{\alpha}}}}]$ and *α-cut representation of* $\overline{A_{\tilde{\alpha}}}$ is $[\overline{L}^\alpha_{\overline{A_{\tilde{\alpha}}}}, \overline{R}^\alpha_{\overline{A_{\tilde{\alpha}}}}]$, where

$$\underline{L}^\alpha_{\underline{A_{\tilde{\alpha}}}} = L^\alpha_{A_3} - \tilde{\alpha}(L^\alpha_{A_3} - L^\alpha_{A_2}),\ \underline{R}^\alpha_{\underline{A_{\tilde{\alpha}}}} = R^\alpha_{A_1} + \tilde{\alpha}(R^\alpha_{A_2} - R^\alpha_{A_1}),$$

$$\overline{L}^\alpha_{\overline{A_{\tilde{\alpha}}}} = L^\alpha_{A_1} + \tilde{\alpha}(L^\alpha_{A_2} - L^\alpha_{A_1})\ , \overline{R}^\alpha_{\overline{A_{\tilde{\alpha}}}} = R^\alpha_{A_3} - \tilde{\alpha}(R^\alpha_{A_3} - R^\alpha_{A_2}),$$

$$L^\alpha_{A_1} = a_1 + \alpha(m - a_1)\ , L^\alpha_{A_2} = a_2 + \alpha(m - a_2),\ L^\alpha_{A_3} = a_3 + \alpha(m - a_3)$$

and

$$R^\alpha_{A_1} = b_1 - \alpha(b_1 - m),\ R^\alpha_{A_2} = b_2 - \alpha(b_2 - m),\ R^\alpha_{A_3} = b_3 - \alpha(b_3 - m).$$

Example 2.12. Let a linear type-2 triangular fuzzy number with its seven entries is given as $\tilde{A} = \langle 50, 70, 80, 100, 120, 130.150\rangle$. Then, using the above two definitions, its $(\alpha - \tilde{\alpha})$-cuts are given by

$$\begin{cases} \underline{L}^\alpha_{\underline{A_{\tilde{\alpha}}}} = 100 - 30(1-\alpha) + 10(1-\alpha)(1-\tilde{\alpha}) = 80 + 20\alpha - 10\tilde{\alpha} + 10\alpha\tilde{\alpha}, \\ \underline{R}^\alpha_{\underline{A_{\tilde{\alpha}}}} = 100 + 30(1-\alpha) - 10(1-\alpha)(1-\tilde{\alpha}) = 120 - 20\alpha + 10\tilde{\alpha} - 10\alpha\tilde{\alpha}, \\ \overline{L}^\alpha_{\overline{A_{\tilde{\alpha}}}} = 100 - 30(1-\alpha) - 20(1-\alpha)(1-\tilde{\alpha}) = 50 + 50\alpha + 20\tilde{\alpha} - 20\alpha\tilde{\alpha}, \\ \overline{R}^\alpha_{\overline{A_{\tilde{\alpha}}}} = 100 + 30(1-\alpha) + 20(1-\alpha)(1-\tilde{\alpha}) = 150 - 50\alpha - 20\tilde{\alpha} + 20\alpha\tilde{\alpha}. \end{cases} \tag{2.6}$$

Remark 2.1. In the above example, it is perceived that a T2FN can be easily represent in its parametric form where α and $\tilde{\alpha} \in [0, 1]$ are the parameters. The pictorial representation of $(\alpha - \tilde{\alpha})$-cut of the T2FN taken in Example 2.12 is given in figure 1.

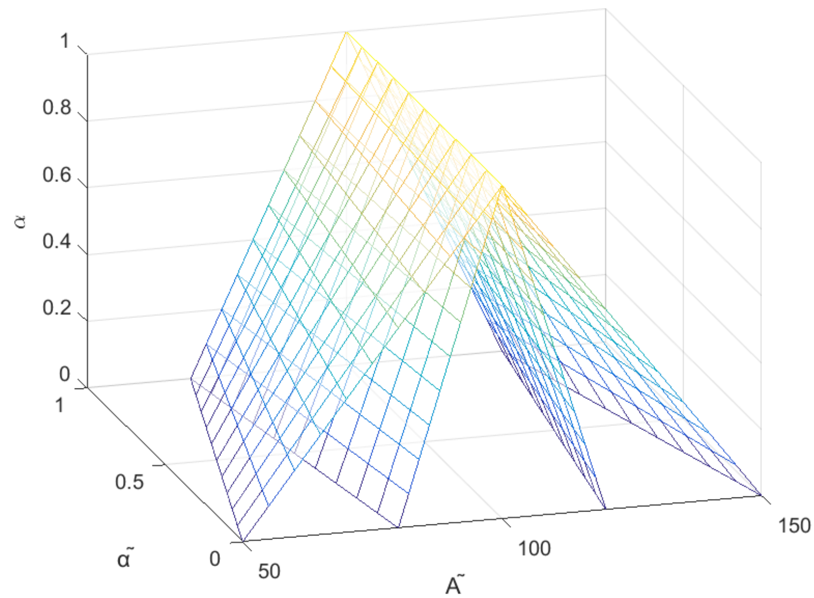


FIGURE 1. $(\alpha - \tilde{\alpha})$-cut of the T2FN taken in Example 2.12

3. Nonlinear type-2 triangular fuzzy number (NT2TFN) and its arithmetic operations

In this section, we propose some definitions and operations of nonlinear type-2 triangular fuzzy numbers (NT2TFN) with numerical examples. At first the definition of Type-2 fuzzy number (T2FN) is extended to its non-linear counterpart in the following definition.Our literature review found that most research had ignored the non-linearity in the case of type-2 fuzzy numbers and instead concentrated on linear type-2 fuzzy numbers. When the membership function contains any geometric concavity or convexity, we usually need non-linear type-2 membership functions. To identify and control the underlying uncertainties, we must consider non-linear type-2 fuzzy numbers in various circumstances. In this study, a generalized NT2TFN has been introduced to offer more flexibility on the chosen option.

Definition 3.1. Characterization of membership functions of NT2TFN: A NT2TFN is characterized by three membership functions (MFs):

(1) upper membership function (UMF),
(2) lower membership function (LMF),
(3) primary membership function (PMF).

Let $\tilde{A}$ be a NT2TFN and denoted by $\tilde{A} = (a_1, a_2, a_3, m, b_1, b_2, b_3; \, p, q)$, where $a_1 \leq a_2 \leq a_3 \leq m \leq b_1 \leq b_2 \leq b_3$; $m, a_i, b_i \in \mathbb{R}$ for $i = 1, 2, 3$ and p and q $(\in \mathbb{R})$ are the left degree of non-linearity (LDNL) and right degree of non-linearity (RDNL) respectively.

Here $(a_1, m, b_3;\, p, q)$, $(a_3, m, b_1;\, p, q)$ and $(a_2, m, b_2;\, p, q)$ are three Nonlinear Triangular Fuzzy Numbers (NTFNs) and their MFs are called UMF, LMF and PMF respectively. Let $\tilde{A}_1 = (a_1, m, b_3;\, p, q)$, $\tilde{A}_2 = (a_2, m, b_2;\, p, q)$ and $\tilde{A}_3 = (a_3, m, b_1;\, p, q)$. Then their MFs are given as follows:

$$\mu_{A_1}(x) = \begin{cases} \left(\frac{x-a_1}{m-a_1}\right)^p; & \text{if } a_1 \le x \le m \\ \left(\frac{b_3-x}{b_3-m}\right)^q; & \text{if } m \le x \le b_3 \\ 0; & \text{otherwise,} \end{cases} \tag{3.1}$$

$$\mu_{A_2}(x) = \begin{cases} \left(\frac{x-a_2}{m-a_2}\right)^p; & \text{if } a_2 \le x \le m \\ \left(\frac{b_3-x}{b_3-m}\right)^q; & \text{if } m \le x \le b_2 \\ 0; & \text{otherwise,} \end{cases} \tag{3.2}$$

$$\mu_{A_3}(x) = \begin{cases} \left(\frac{x-a_3}{m-a_3}\right)^p; & \text{if } a_3 \le x \le m \\ \left(\frac{b_3-x}{b_3-m}\right)^q; & \text{if } m \le x \le b_1 \\ 0; & \text{otherwise.} \end{cases} \tag{3.3}$$

Thus $\alpha-$ cuts of $\tilde{A}_1$ is $\tilde{A}_{1\alpha} = [L^{\alpha}_{A_1}, R^{\alpha}_{A_3}]$, $\alpha-$ cuts of $\tilde{A}_2$ is $\tilde{A}_{2\alpha} = [L^{\alpha}_{A_2}, R^{\alpha}_{A_2}]$ and $\alpha-$ cuts of $\tilde{A}_3$ is $\tilde{A}_{3\alpha} = [L^{\alpha}_{A_3}, R^{\alpha}_{A_1}]$, where $L^{\alpha}_{A_1} = a_1 + \alpha^{\frac{1}{q}}(m-a_1)$, $L^{\alpha}_{A_2} = a_2 + \alpha^{\frac{1}{q}}(m-a_2)$, $L^{\alpha}_{A_3} = a_3 + \alpha^{\frac{1}{q}}(m-a_3)$
and $R^{\alpha}_{A_1} = b_1 - \alpha^{\frac{1}{q}}(b_1 - m)$, $R^{\alpha}_{A_2} = b_2 - \alpha^{\frac{1}{q}}(b_2 - m)$, $R^{\alpha}_{A_3} = b_3 - \alpha^{\frac{1}{q}}(b_3 - m)$.

Note 3.1. α-cut and α-plane representations of the discussed type of non-linear type-2 fuzzy number are little bit different. This can be written in the following way.

Definition 3.2. α-cut representation of $\tilde{\alpha}$-planes: Let $\tilde{A} = (a_1, a_2, a_3, m, b_1, b_2, b_3;\, p, q)$, $p, q \in \mathbb{R}$ represent a NT2TFN, where p is the left degree of non-linearity (LDNL) and q is the right degree of non-linearity (RDNL). Then the

(i) *α-cut representation of* $\underline{A_{\tilde{\alpha}}}$ is $[\underline{L^{\alpha}_{A_{\tilde{\alpha}}}}, \underline{R^{\alpha}_{A_{\tilde{\alpha}}}}]$, where
$\underline{L^{\alpha}_{A_{\tilde{\alpha}}}} = L^{\alpha}_{A_3} - \tilde{\alpha}^{\frac{1}{p}}(L^{\alpha}_{A_3} - L^{\alpha}_{A_2}) = a_3 + \alpha^{\frac{1}{q}}(m - a_3) - \tilde{\alpha}^{\frac{1}{p}}(a_3 - a_2)(1 - \alpha^{\frac{1}{q}})$ and
$\underline{R^{\alpha}_{A_{\tilde{\alpha}}}} = R^{\alpha}_{A_1} + \tilde{\alpha}^{\frac{1}{p}}(R^{\alpha}_{A_2} - R^{\alpha}_{A_1}) = b_1 - \alpha^{\frac{1}{q}}(b_1 - m) + \tilde{\alpha}^{\frac{1}{p}}(b_2 - b_1)(1 - \alpha^{\frac{1}{q}})$.

(ii) *α-cut representation of* $\overline{A_{\tilde{\alpha}}}$ is $[\overline{L^{\alpha}_{A_{\tilde{\alpha}}}}, \overline{R^{\alpha}_{A_{\tilde{\alpha}}}}]$, where
$\overline{L^{\alpha}_{A_{\tilde{\alpha}}}} = L^{\alpha}_{A_1} + \tilde{\alpha}^{\frac{1}{p}}(L^{\alpha}_{A_2} - L^{\alpha}_{A_1}) = a_1 + \alpha^{\frac{1}{q}}(m - a_1) + \tilde{\alpha}^{\frac{1}{p}}(a_2 - a_1)(1 - \alpha^{\frac{1}{q}})$ and
$\overline{R^{\alpha}_{A_{\tilde{\alpha}}}} = R^{\alpha}_{A_3} - \tilde{\alpha}^{\frac{1}{p}}(R^{\alpha}_{A_3} - R^{\alpha}_{A_2}) = b_3 - \tilde{\alpha}^{\frac{1}{p}}(b_3 - m) - \tilde{\alpha}^{\frac{1}{p}}(b_3 - b_2)(1 - \alpha^{\frac{1}{q}})$.

Definition 3.3. Membership function of a generalised triangular type -2 fuzzy number: The generalised membership function of triangular type -2 fuzzy number defined as

$$\mu_A(u, x) = \begin{cases} \left(\frac{u(x) - \underline{u}(x)}{Apex - \underline{u}(x)}\right)^p; & \text{if } \underline{u}(x) \le u(x) \le Apex \\ \left(\frac{\overline{u}(x) - u(x)}{\overline{u}(x) - Apex}\right)^q; & \text{if } Apex \le u(x) \le \overline{u}(x) \\ 0; & \text{otherwise,} \end{cases} \tag{3.4}$$

where $Apex = \underline{u}(x) + \omega(x)(\overline{u}(x) - \underline{u}(x))$, $\omega(x) \in [0, 1]$, $\underline{u}(x)$ = Lower bound of the triangular type -2 fuzzy number, $\overline{u}(x)$ = Upper bound of the triangular type -2 fuzzy number (in this study, $Apex = m$ which is the middle term of the triangular type -2 fuzzy number).

Definition 3.4. Equality of two NT2TFN: Let $\tilde{A} = (a_1, a_2, a_3, m, b_1, b_2, b_3;\, p_1, p_2)$ and $\tilde{B} = (c_1, c_2, c_3, n, d_1, d_2, d_3;\, q_1, q_2)$ are two NT2TFNs. Then $\tilde{A} = \tilde{B}$ if and only if $a_i = c_i$, $b_i = d_i$ for all $i = 1, 2, 3$, $m = n$ and $p_j = q_j$ for $j = 1, 2$.

Definition 3.5. Non negative and non positive NT2TFN: A NT2TFN $\tilde{A} = (a_1, a_2, a_3, m, b_1, b_2, b_3;\, p, q)$ is said to be non-negative when $a_1 \geq 0$. On contrary, it is said to be non-positive when $b_3 \leq 0$.

Definition 3.6. Addition of NT2TFNs: Addition of two nonlinear type-2 triangular fuzzy number $\tilde{A}$ and $\tilde{B}$ is $\tilde{A} \oplus \tilde{B} = \tilde{C}$ given in term of the secondary membership function $\mu_{\tilde{C}}(z, u) = sup\{min\{\mu_{\tilde{A}}(x, u), \mu_{\tilde{B}}(y, u)\} : x + y = z\}$
and if $\tilde{A} = (a_1, a_2, a_3, m, b_1, b_2, b_3;\, p_1, p_2)$ and $\tilde{B} = (c_1, c_2, c_3, n, d_1, d_2, d_3;\, q_1, q_2)$ NT2TFNs, then $\tilde{C}$ is defined as $\tilde{C} = \tilde{A} \oplus \tilde{B} = (a_1 + c_1, a_2 + c_2, a_3 + c_3, m + n, b_1 + d_1, b_2 + d_2, b_3 + d_3;\, p, q)$, where $p = min\{p_1, q_1\}$, $q = min\{p_2, q_2\}$, $C_{\tilde{\alpha}^{\alpha}} = A_{\tilde{\alpha}^{\alpha}} + B_{\tilde{\alpha}^{\alpha}}$. Thus

$$\begin{aligned}
&\left([\underline{L}^{\alpha}_{\underline{C}_{\tilde{\alpha}}}, \underline{R}^{\alpha}_{\underline{C}_{\tilde{\alpha}}}];\, [\overline{L}^{\alpha}_{\overline{C}_{\tilde{\alpha}}}, \overline{R}^{\alpha}_{\overline{C}_{\tilde{\alpha}}}]\right) \\
&= \left([\underline{L}^{\alpha}_{\underline{A}_{\tilde{\alpha}}}, \underline{R}^{\alpha}_{\underline{A}_{\tilde{\alpha}}}];\, [\overline{L}^{\alpha}_{\overline{A}_{\tilde{\alpha}}}, \overline{R}^{\alpha}_{\overline{A}_{\tilde{\alpha}}}]\right) + \left([\underline{L}^{\alpha}_{\underline{B}_{\tilde{\alpha}}}, \underline{R}^{\alpha}_{\underline{B}_{\tilde{\alpha}}}];\, [\overline{L}^{\alpha}_{\overline{B}_{\tilde{\alpha}}}, \overline{R}^{\alpha}_{\overline{B}_{\tilde{\alpha}}}]\right) \\
&= \left([\underline{L}^{\alpha}_{\underline{A}_{\tilde{\alpha}}} + \underline{L}^{\alpha}_{\underline{B}_{\tilde{\alpha}}}, \underline{R}^{\alpha}_{\underline{A}_{\tilde{\alpha}}} + \underline{R}^{\alpha}_{\underline{B}_{\tilde{\alpha}}}];\, [\overline{L}^{\alpha}_{\overline{A}_{\tilde{\alpha}}} + \overline{L}^{\alpha}_{\overline{B}_{\tilde{\alpha}}}, \overline{R}^{\alpha}_{\overline{A}_{\tilde{\alpha}}} + \overline{R}^{\alpha}_{\overline{B}_{\tilde{\alpha}}}]\right),
\end{aligned}$$

where $\alpha, \tilde{\alpha} \in [0, 1]$.

Definition 3.7. Scalar multiplication of NT2TFN: A NT2TFN $\tilde{A} = (a_1, a_2, a_3, m, b_1, b_2, b_3;\, p, q)$ is multiplied by a scalar k as follows

$$k\tilde{A} = (ka_1, ka_2, ka_3, km, kb_1, kb_2, kb_3;\, p, q), \text{ when } k \geq 0 \text{ and}$$
$$k\tilde{A} = (kb_3, kb_2, kb_1, km, ka_3, ka_2, ka_1;\, p, q), \text{ when } k < 0.$$

Definition 3.8. Subtraction of NT2TFNs: Subtraction of two NT2TFNs $\tilde{A}$ and $\tilde{B}$ is defined as $\tilde{A} \ominus \tilde{B} = \tilde{C}$ where the secondary membership function is defined as $\mu_{\tilde{C}}(z, u) = sup\{min\{\mu_{\tilde{A}}(x, u), \mu_{\tilde{B}}(y, u)\} : x - y = z\}$
and if $\tilde{A} = (a_1, a_2, a_3, m, b_1, b_2, b_3;\, p_1, p_2)$ and $\tilde{B} = (c_1, c_2, c_3, n, d_1, d_2, d_3;\, q_1, q_2)$ is defined as $\tilde{C} = \tilde{A} \ominus \tilde{B} = (a_1 - d_3, a_2 - d_2, a_3 - d_1, m - n, b_1 - c_3, b_2 - c_2, b_3 - c_1;\, p, q)$, where $p = min\{p_1, q_1\}$, $q = min\{p_2, q_2\}$ and $C_{\tilde{\alpha}^{\alpha}} = A_{\tilde{\alpha}^{\alpha}} + B_{\tilde{\alpha}^{\alpha}}$. Then

$$\begin{aligned}
&\left([\underline{L}^{\alpha}_{\underline{C}_{\tilde{\alpha}}}, \underline{R}^{\alpha}_{\underline{C}_{\tilde{\alpha}}}];\, [\overline{L}^{\alpha}_{\overline{C}_{\tilde{\alpha}}}, \overline{R}^{\alpha}_{\overline{C}_{\tilde{\alpha}}}]\right) \\
&= \left([\underline{L}^{\alpha}_{\underline{A}_{\tilde{\alpha}}}, \underline{R}^{\alpha}_{\underline{A}_{\tilde{\alpha}}}];\, [\overline{L}^{\alpha}_{\overline{A}_{\tilde{\alpha}}}, \overline{R}^{\alpha}_{\overline{A}_{\tilde{\alpha}}}]\right) - \left([\underline{L}^{\alpha}_{\underline{B}_{\tilde{\alpha}}}, \underline{R}^{\alpha}_{\underline{B}_{\tilde{\alpha}}}];\, [\overline{L}^{\alpha}_{\overline{B}_{\tilde{\alpha}}}, \overline{R}^{\alpha}_{\overline{B}_{\tilde{\alpha}}}]\right) \\
&= \left([\underline{L}^{\alpha}_{\underline{A}_{\tilde{\alpha}}} - \underline{R}^{\alpha}_{\underline{B}_{\tilde{\alpha}}}, \underline{R}^{\alpha}_{\underline{A}_{\tilde{\alpha}}} - \underline{L}^{\alpha}_{\underline{B}_{\tilde{\alpha}}}];\, [\overline{L}^{\alpha}_{\overline{A}_{\tilde{\alpha}}} - \overline{R}^{\alpha}_{\overline{B}_{\tilde{\alpha}}}, \overline{R}^{\alpha}_{\overline{A}_{\tilde{\alpha}}} - \overline{L}^{\alpha}_{\overline{B}_{\tilde{\alpha}}}]\right),
\end{aligned}$$

where $\alpha, \tilde{\alpha} \in [0, 1]$.

Definition 3.9. Multiplication of NT2TFNs: Multiplication of two NT2TFNs $\tilde{A}$ and $\tilde{B}$ is $\tilde{A} \odot \tilde{B} = \tilde{C}$ described by the secondary membership function $\mu_{\tilde{C}}(z, u) = sup\{min\{\mu_{\tilde{A}}(x, u), \mu_{\tilde{B}}(y, u)\} : x \times y = z\}$
and if $\tilde{A} = (a_1, a_2, a_3, m, b_1, b_2, b_3;\, p_1, p_2)$ and $\tilde{B} = (c_1, c_2, c_3, n, d_1, d_2, d_3;\, q_1, q_2)$,

then $\tilde{C} = \tilde{A} \odot \tilde{B} = (a_1c_1, a_2c_2, a_3c_3, mn, b_1d_1, b_2d_2, b_3d_3;\ p, q)$, where $p = min\{p_1, q_1\}$, $q = min\{p_2, q_2\}$ and $C_{\tilde{\alpha}^\alpha} = A_{\tilde{\alpha}^\alpha} \cdot B_{\tilde{\alpha}^\alpha}$. Thus

$$\begin{aligned}
&\Big([\underline{L}^{\alpha}_{\underline{C}_{\tilde{\alpha}}}, \underline{R}^{\alpha}_{\underline{C}_{\tilde{\alpha}}}];\ [\overline{L}^{\alpha}_{\overline{C}_{\tilde{\alpha}}}, \overline{R}^{\alpha}_{\overline{C}_{\tilde{\alpha}}}]\Big)\\
&= \Big([\underline{L}^{\alpha}_{\underline{A}_{\tilde{\alpha}}}, \underline{R}^{\alpha}_{\underline{A}_{\tilde{\alpha}}}];\ [\overline{L}^{\alpha}_{\overline{A}_{\tilde{\alpha}}}, \overline{R}^{\alpha}_{\overline{A}_{\tilde{\alpha}}}]\Big) \cdot \Big([\underline{L}^{\alpha}_{\underline{B}_{\tilde{\alpha}}}, \underline{R}^{\alpha}_{\underline{B}_{\tilde{\alpha}}}];\ [\overline{L}^{\alpha}_{\overline{B}_{\tilde{\alpha}}}, \overline{R}^{\alpha}_{\overline{B}_{\tilde{\alpha}}}]\Big)\\
&= \Big([\underline{L}^{\alpha}_{\underline{A}_{\tilde{\alpha}}} \cdot \underline{L}^{\alpha}_{\underline{B}_{\tilde{\alpha}}}, \underline{R}^{\alpha}_{\underline{A}_{\tilde{\alpha}}} \cdot \underline{R}^{\alpha}_{\underline{B}_{\tilde{\alpha}}}];\ [\overline{L}^{\alpha}_{\overline{A}_{\tilde{\alpha}}} \cdot \overline{L}^{\alpha}_{\overline{B}_{\tilde{\alpha}}}, \overline{R}^{\alpha}_{\overline{A}_{\tilde{\alpha}}} \cdot \overline{R}^{\alpha}_{\overline{B}_{\tilde{\alpha}}}]\Big),
\end{aligned}$$

where $\alpha, \tilde{\alpha} \in [0, 1]$.

Definition 3.10. Division of NT2TFNs: Division of two NT2TFNs $\tilde{A}$ and $\tilde{B}$ is $\tilde{A} \oslash \tilde{B} = \tilde{C}$ manifested by the secondary membership function $\mu_{\tilde{C}}(z, u) = sup\{min\{\mu_{\tilde{A}}(x, u), \mu_{\tilde{B}}(y, u)\} : x = y \times z\}$
and if $\tilde{A} = (a_1, a_2, a_3, m, b_1, b_2, b_3;\ p_1, p_2)$ and $\tilde{B} = (c_1, c_2, c_3, n, d_1, d_2, d_3;\ q_1, q_2)$, then $\tilde{C} = \tilde{A} \oslash \tilde{B} = (\frac{a_1}{d_3}, \frac{a_2}{d_2}, \frac{a_3}{d_1}, \frac{m}{n}, \frac{b_1}{c_3}, \frac{b_2}{c_2}, \frac{b_3}{c_1};\ p, q)$ where $p = min\{p_1, q_1\}$, $q = min\{p_2, q_2\}$ and $C_{\tilde{\alpha}^\alpha} = \frac{A^{\alpha}_{\tilde{\alpha}}}{B^{\alpha}_{\tilde{\alpha}}}$. Thus

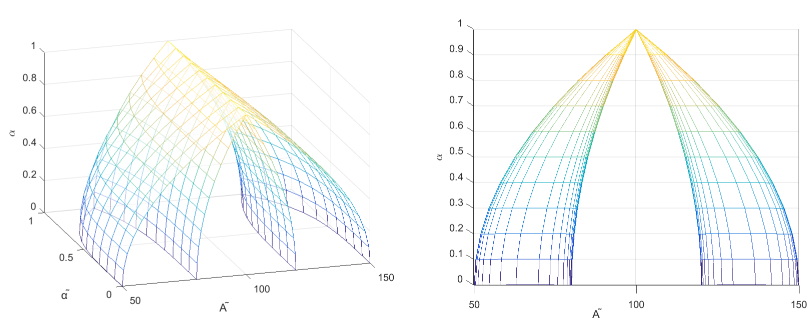
$$\begin{aligned}
\Big([\underline{L}^{\alpha}_{\underline{C}_{\tilde{\alpha}}}, \underline{R}^{\alpha}_{\underline{C}_{\tilde{\alpha}}}];\ [\overline{L}^{\alpha}_{\overline{C}_{\tilde{\alpha}}}, \overline{R}^{\alpha}_{\overline{C}_{\tilde{\alpha}}}]\Big) &= \frac{\Big([\underline{L}^{\alpha}_{\underline{A}_{\tilde{\alpha}}}, \underline{R}^{\alpha}_{\underline{A}_{\tilde{\alpha}}}];[\overline{L}^{\alpha}_{\overline{A}_{\tilde{\alpha}}}, \overline{R}^{\alpha}_{\overline{A}_{\tilde{\alpha}}}]\Big)}{\Big([\underline{L}^{\alpha}_{\underline{B}_{\tilde{\alpha}}}, \underline{R}^{\alpha}_{\underline{B}_{\tilde{\alpha}}}];[\overline{L}^{\alpha}_{\overline{B}_{\tilde{\alpha}}}, \overline{R}^{\alpha}_{\overline{B}_{\tilde{\alpha}}}]\Big)}\\
&= \left(\left[\frac{\underline{L}^{\alpha}_{\underline{A}_{\tilde{\alpha}}}}{\underline{R}^{\alpha}_{\underline{B}_{\tilde{\alpha}}}}, \frac{\underline{R}^{\alpha}_{\underline{A}_{\tilde{\alpha}}}}{\underline{L}^{\alpha}_{\underline{B}_{\tilde{\alpha}}}}\right];\ \left[\frac{\overline{L}^{\alpha}_{\overline{A}_{\tilde{\alpha}}}}{\overline{R}^{\alpha}_{\overline{B}_{\tilde{\alpha}}}}, \frac{\overline{R}^{\alpha}_{\overline{A}_{\tilde{\alpha}}}}{\overline{L}^{\alpha}_{\overline{B}_{\tilde{\alpha}}}}\right]\right),
\end{aligned}$$

where $\alpha, \tilde{\alpha} \in [0, 1]$.

Example 3.11. Let a NT2TFN is taken as $\tilde{A} = \langle 50, 70, 80, 100, 120, 130.150;\ 0.25, 0.5\rangle$. Then, using the definition 3.2, its $(\alpha - \tilde{\alpha})$-cuts are given by

$\underline{L}^{\alpha}_{\underline{A}_{\tilde{\alpha}}} = 100-30(1-\alpha^2)+10(1-\alpha^2)(1-\alpha^4)$, $\underline{R}^{\alpha}_{\underline{A}_{\tilde{\alpha}}} = 100+30(1-\alpha^2)-10(1-\alpha^2)(1-\alpha^4)$,

$\overline{L}^{\alpha}_{\overline{A}_{\tilde{\alpha}}} = 100-30(1-\alpha^2)-20(1-\alpha^2)(1-\alpha^4)$, $\overline{R}^{\alpha}_{\overline{A}_{\tilde{\alpha}}} = 100+30(1-\alpha^2)+20(1-\alpha^2)(1-\alpha^4)$.



(a) (b)

FIGURE 2. (a) and (b) represents the $(\alpha - \tilde{\alpha})$-cut of nonlinear triangular type-2 fuzzy number taken in the example 3.11

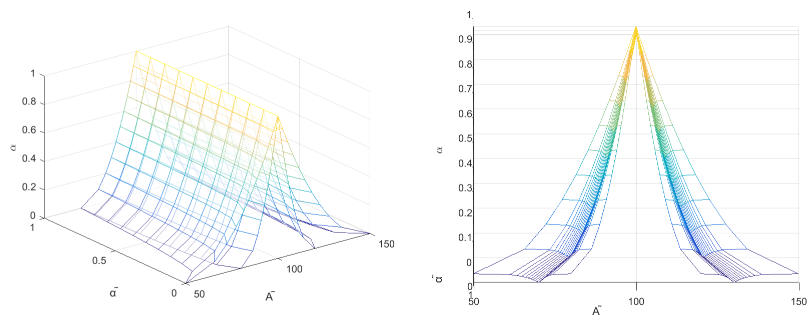
Example 3.12. Let a NT2TFN is taken as $\tilde{A} = \langle 50, 70, 80, 100, 120, 130, 150;\ 4, 2\rangle$. Then, using the definition 3.2, its $(\alpha - \tilde{\alpha})$-cuts are given by

$$\underline{L}^{\alpha}_{A_{\tilde{\alpha}}} = 100 - 30(1-\alpha^{0.5}) + 10(1-\alpha^{0.5})(1-\alpha^{0.25}),$$

$$\underline{R}^{\alpha}_{A_{\tilde{\alpha}}} = 100 + 30(1-\alpha^{0.5}) - 10(1-\alpha^{0.5})(1-\alpha^{0.25}),$$

$$\overline{L}^{\alpha}_{A_{\tilde{\alpha}}} = 100 - 30(1-\alpha^{0.5}) - 20(1-\alpha^{0.5})(1-\alpha^{0.25}),$$

$$\overline{R}^{\alpha}_{A_{\tilde{\alpha}}} = 100 + 30(1-\alpha^{0.5}) + 20(1-\alpha^{0.5})(1-\alpha^{0.25}).$$



(a) (b)

FIGURE 3. (a) and (b) represents the $(\alpha - \tilde{\alpha})$-cut of nonlinear triangular type-2 fuzzy number taken in example 3.12

Example 3.13. Let, a NT2TFN is taken as $\tilde{A} = \langle 50, 70, 80, 100, 120, 130, 150;\ 4, 0.5\rangle$. Then, using the definition 3.2, its $(\alpha - \tilde{\alpha})$-cuts are given by

$$\underline{L}^{\alpha}_{A_{\tilde{\alpha}}} = 100 - 30(1-\alpha^{2}) + 10(1-\alpha^{2})(1-\alpha^{0.25}),$$

$$\underline{R}^{\alpha}_{A_{\tilde{\alpha}}} = 100 + 30(1-\alpha^{2}) - 10(1-\alpha^{2})(1-\alpha^{0.25}),$$

$$\overline{L}^{\alpha}_{A_{\tilde{\alpha}}} = 100 - 30(1-\alpha^{2}) - 20(1-\alpha^{2})(1-\alpha^{0.25}),$$

$$\overline{R}^{\alpha}_{A_{\tilde{\alpha}}} = 100 + 30(1-\alpha^{2}) + 20(1-\alpha^{2})(1-\alpha^{0.25}).$$

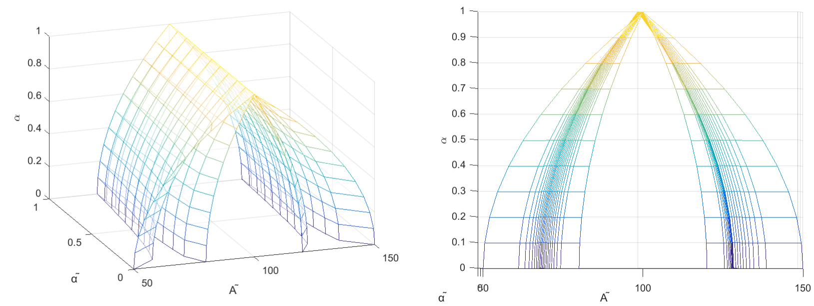
Example 3.14. Let, a NT2TFN is taken as $\tilde{A} = \langle 50, 70, 80, 100, 120, 130, 150;\ 0.5, 4\rangle$. Then, using the definition 3.2, its $(\alpha - \tilde{\alpha})$-cuts are given by

$$\underline{L}^{\alpha}_{A_{\tilde{\alpha}}} = 100 - 30(1-\alpha^{0.25}) + 10(1-\alpha^{0.25})(1-\alpha^{2}),$$

$$\underline{R}^{\alpha}_{A_{\tilde{\alpha}}} = 100 + 30(1-\alpha^{0.25}) - 10(1-\alpha^{0.25})(1-\alpha^{2}),$$

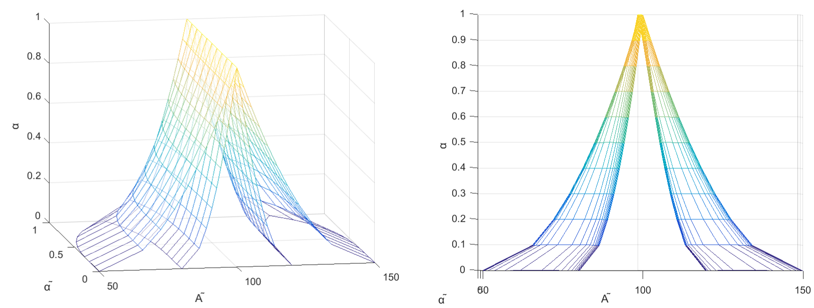
$$\overline{L}^{\alpha}_{A_{\tilde{\alpha}}} = 100 - 30(1-\alpha^{0.25}) - 20(1-\alpha^{0.25})(1-\alpha^{2}),$$

$$\overline{R}^{\alpha}_{A_{\tilde{\alpha}}} = 100 + 30(1-\alpha^{0.25}) + 20(1-\alpha^{0.25})(1-\alpha^{2}).$$



(a) (b)

FIGURE 4. (a) and (b) represents the $(\alpha - \tilde{\alpha})$-cut of nonlinear triangular type-2 fuzzy number taken in example 3.13



(a) (b)

FIGURE 5. (a) and (b) represents the $(\alpha - \tilde{\alpha})$-cut of non-linear triangular type-2 fuzzy number taken in example 3.14

Remarks: In figure 2(a) and 2(b) we draw the pictorial representation of Type-2 nonlinear fuzzy number whereas in figure 3(a) and 3(b) we draw the pictorial representation of generalized Type-2 non linear fuzzy number. Similarly In figure 4(a) and 4(b) we draw the pictorial view of Type-2 nonlinear fuzzy number whereas in figure 5(a) and 5(b) we draw the pictorial insight of generalized Type-2 non linear fuzzy number.

4. Type-2 fuzzy differential inclusion

Differential inclusion theory was created to address specific types of uncertainty not covered by traditional dynamical systems. These uncertainties result from various factors, such as incomplete information brought on by the difficulties of fully comprehending a phenomenon or ignorance of the principles governing system control. Control can be exerted by steering, acceleration, fuel, temperature, weight, or other system-affecting factors.

A set of differential equations is used in the mathematical model

$$\begin{cases} \frac{dx}{dt} = f(x(t), u(t)) \\ u(t) \in U(x(t)) \end{cases} . \tag{4.1}$$

In the above equation, $x \in R^n$,$u \in R^m$ and U are called state variable, control and subset of admissible control respectively. As an extension, a fuzzy differential inclusion can be given by

$$\begin{cases} \frac{dx}{dt} \in f(t, x(t)) \\ x(0) \in X_0 \end{cases} \tag{4.2}$$

in which X_0 is a set of fuzzy numbers and x is a fuzzy state variables. Further in level cut representation, above fuzzy differential inclusion takes the form

$$\begin{cases} \frac{dx}{dt} \in [f(t, x(t))]_\alpha \\ x(0) \in [X_0]_\alpha \end{cases} \tag{4.3}$$

in which $[f(t, x(t))]_\alpha$ and $[X_0]_\alpha$ represents the level sets for the fuzzy valued function $f(t, x(t))$ and fuzzy number X_0 respectively. The existence of the solution is provided by the following theorem.

Theorem 4.1[39] Let, X_0 be a fuzzy number defined on R^n, ω be an open set on $R \times R^n$ containing $0 \times [X_0]_\alpha$ and f is a fuzzy valued function defined on $\omega \times F(R^n)$. Furthermore, let $x \in [X_0]_\alpha$ preserves the bounded criteria. Then, all the solution of the system given by Equation 4.6 are compact subsets in $Z_T(R^n)$, for each aspiration level. Moreover, these solutions are level cuts of fuzzy subsets in $Z_T(R^n)$, which is a solution of the system given by Equation 4.2.

In this present section, we are going to extend the idea of fuzzy differential inclusion in type-2 fuzzy domain. For solving type-2 fuzzy differential equation by differential inclusion approach we follow the following steps:

(1) First replace the DE with a family of DEs,
(2) Take the $(\alpha - \tilde{\alpha})-$ cut of the fuzzy numbers and the fuzzy functions,
(3) Solve the family of differential equations by any well known method,
(4) The solutions are obtained as $(\alpha - \tilde{\alpha})-$ cut.

4.1. **Type-2 fuzzy differential inclusion for Type-2 fuzzy initial value problem:** Let us consider initial valued first order non homogeneous constant coefficient linear differential equation as

$$\begin{cases} \frac{dx}{dt} = px + q \\ x(0) = \tilde{x}_0. \end{cases} \tag{4.4}$$

In equation 4.4, $x(t)$ is the state variable which is a smooth function of t, $\tilde{x}(0) = \tilde{x_0}$ is a non-linear type-2 fuzzy number and $p,\ q \in \mathbb{R}$ are the coefficients of the differential equation.

The research has demonstrated that H-derivative and Seikkala derivative derivatives have several severe limitations, the most important of which is that the diameter of the fuzzy function under study must be non-decreasing. Due to this constraint, the determined solution of an FDE frequently deviates from what could be inferred intuitively from the characteristics of the system or phenomena that the FDE is modelling. The solutions to FDE employing the H-derivative can occasionally be unbounded as $t \to \infty$. In this way, it differs significantly from the crisp solution and does not accurately reflect the complex behavior of uncertain systems. In context, fuzzy differential inclusions provide fruitful purposes. As this current article is employing to demonstrate the properties of nonlinear type-2 fuzzy numbers, here, in this pocket, we develop type-2 fuzzy differential inclusion technique which would bring a novel sense to solve type-2 fuzzy differential equation with initial information availed.

Let $x0^{\alpha}_{\tilde{\alpha}} = \Big([\underline{L}^{\alpha}_{\underline{x0}_{\tilde{\alpha}}}, \underline{R}^{\alpha}_{\underline{x0}_{\tilde{\alpha}}}]; [\overline{L}^{\alpha}_{\overline{x0}_{\tilde{\alpha}}}, \overline{R}^{\alpha}_{\overline{x0}_{\tilde{\alpha}}}]\Big)$ is the $(\alpha - \tilde{\alpha})$-cut of $\tilde{x_0}$ on NT2TFN. Then the differential equation (4.4) of $x(t)$ on NT2TFN is the family of inclusion

$$\begin{cases} x'(t) \in x(t)p + q \\ x(0) \in x0^{\alpha}_{\tilde{\alpha}}. \end{cases} \tag{4.5}$$

Let $\zeta_\alpha(t)$ represents the column vector $\begin{pmatrix} \underline{L}^{\alpha}_{\underline{x}_{\tilde{\alpha}}}(t) \\ \underline{R}^{\alpha}_{\underline{x}_{\tilde{\alpha}}}(t) \\ \overline{L}^{\alpha}_{\overline{x}_{\tilde{\alpha}}}(t) \\ \overline{R}^{\alpha}_{\overline{x}_{\tilde{\alpha}}}(t) \end{pmatrix}$.

We consider two scenarios for signs of the coefficient p in discussion.

Case I: when $p > 0$. Then, the ordinary value problem is obtained

$$\begin{cases} \zeta'_\alpha(t) = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \zeta_\alpha(t) + \begin{pmatrix} q \\ q \\ q \\ q \end{pmatrix} \\ \zeta_\alpha(0) = \begin{pmatrix} \underline{L}^{\alpha}_{\underline{x0}_{\tilde{\alpha}}} \\ \underline{R}^{\alpha}_{\underline{x0}_{\tilde{\alpha}}} \\ \overline{L}^{\alpha}_{\overline{x0}_{\tilde{\alpha}}} \\ \overline{R}^{\alpha}_{\overline{x0}_{\tilde{\alpha}}} \end{pmatrix}. \end{cases} \tag{4.6}$$

It can be easily verified that equation (4.6) has the solution

$$\begin{cases} \underline{L}^{\alpha}_{\underline{x}_{\tilde{\alpha}}}(t) = e^{pt}\Big\{a_3 + \frac{\alpha^{\frac{1}{q}}}{\omega}(m - a_3) - \frac{\tilde{\alpha}^{\frac{1}{p}}}{\omega}(a_3 - a_2)(1 - \frac{\alpha^{\frac{1}{q}}}{\omega}) + \frac{q}{p} - \frac{qe^{-pt}}{p}\Big\} \\ \underline{R}^{\alpha}_{\underline{x}_{\tilde{\alpha}}}(t) = e^{pt}\Big\{b_1 - \frac{\alpha^{\frac{1}{q}}}{\omega}(b_1 - m) + \frac{\tilde{\alpha}^{\frac{1}{p}}}{\omega}(b_2 - b_1)(1 - \frac{\alpha^{\frac{1}{q}}}{\omega}) + \frac{q}{p} - \frac{qe^{-pt}}{p}\Big\} \\ \overline{L}^{\alpha}_{\overline{x}_{\tilde{\alpha}}}(t) = e^{pt}\Big\{a_1 + \frac{\alpha^{\frac{1}{q}}}{\omega}(m - a_1) - \frac{\tilde{\alpha}^{\frac{1}{p}}}{\omega}(a_2 - a_1)(1 - \frac{\alpha^{\frac{1}{q}}}{\omega}) + \frac{q}{p} - \frac{qe^{-pt}}{p}\Big\} \\ \overline{R}^{\alpha}_{\overline{x}_{\tilde{\alpha}}}(t) = e^{pt}\Big\{b_3 - \frac{\alpha^{\frac{1}{q}}}{\omega}(b_3 - m) + \frac{\tilde{\alpha}^{\frac{1}{p}}}{\omega}(b_3 - b_2)(1 - \frac{\alpha^{\frac{1}{q}}}{\omega}) + \frac{q}{p} - \frac{qe^{-pt}}{p}\Big\} \end{cases} \tag{4.7}$$

Hence the solution to equation (4.4) is $x^{\alpha}_{\tilde{\alpha}} =$

$$\Bigg(\Bigg[e^{pt}\Bigg\{a_3 + \frac{\alpha^{\frac{1}{q}}}{\omega}(m-a_3) - \frac{\tilde{\alpha}^{\frac{1}{p}}}{\omega}(a_3-a_2)(1-\frac{\alpha^{\frac{1}{q}}}{\omega}) + \frac{q}{p} - \frac{qe^{-pt}}{p}\Bigg\},$$
$$e^{pt}\Bigg\{b_1 - \frac{\alpha^{\frac{1}{q}}}{\omega}(b_1-m) + \frac{\tilde{\alpha}^{\frac{1}{p}}}{\omega}(b_2-b_1)(1-\frac{\alpha^{\frac{1}{q}}}{\omega}) + \frac{q}{p} - \frac{qe^{-pt}}{p}\Bigg\}\Bigg];$$
$$\Bigg[e^{pt}\Bigg\{a_1 + \frac{\alpha^{\frac{1}{q}}}{\omega}(m-a_1) - \frac{\tilde{\alpha}^{\frac{1}{p}}}{\omega}(a_2-a_1)(1-\frac{\alpha^{\frac{1}{q}}}{\omega}) + \frac{q}{p} - \frac{qe^{-pt}}{p}\Bigg\},$$
$$e^{pt}\Bigg\{b_3 - \frac{\alpha^{\frac{1}{q}}}{\omega}(b_3-m) + \frac{\tilde{\alpha}^{\frac{1}{p}}}{\omega}(b_3-b_2)(1-\frac{\alpha^{\frac{1}{q}}}{\omega}) + \frac{q}{p} - \frac{qe^{-pt}}{p}\Bigg\}\Bigg]\Bigg).$$

Case II: when $p < 0$. Then, we obtain the ordinary value problems

$$\begin{cases} \zeta'_{\alpha}(t) = \begin{pmatrix} 0 & p & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & p \\ 0 & 0 & p & 0 \end{pmatrix} \zeta_{\alpha}(t) + \begin{pmatrix} q \\ q \\ q \\ q \end{pmatrix} \\ \zeta_{\alpha}(0) = \begin{pmatrix} \underline{L}^{\alpha}_{\underline{x_0}_{\tilde{\alpha}}} \\ \underline{R}^{\alpha}_{\underline{x_0}_{\tilde{\alpha}}} \\ \overline{L}^{\alpha}_{\overline{x_0}_{\tilde{\alpha}}} \\ \overline{R}^{\alpha}_{\overline{x_0}_{\tilde{\alpha}}} \end{pmatrix}. \end{cases} \tag{4.8}$$

If we put $A = e^{pt}\Bigg(a_3 + \frac{\alpha^{\frac{1}{q}}}{\omega}(m-a_3) - \frac{\tilde{\alpha}^{\frac{1}{p}}}{\omega}(a_3-a_2)(1-\frac{\alpha^{\frac{1}{q}}}{\omega})\Bigg)$, then

$B = e^{pt}\Bigg(b_1 - \frac{\alpha^{\frac{1}{q}}}{\omega}(b_1-m) + \frac{\tilde{\alpha}^{\frac{1}{p}}}{\omega}(b_2-b_1)(1-\frac{\alpha^{\frac{1}{q}}}{\omega})\Bigg)$,

$C = e^{pt}\Bigg(a_1 + \frac{\alpha^{\frac{1}{q}}}{\omega}(m-a_1) - \frac{\tilde{\alpha}^{\frac{1}{p}}}{\omega}(a_2-a_1)(1-\frac{\alpha^{\frac{1}{q}}}{\omega})\Bigg)$ and

$D = e^{pt}\Bigg(b_3 - \frac{\alpha^{\frac{1}{q}}}{\omega}(b_3-m) + \frac{\tilde{\alpha}^{\frac{1}{p}}}{\omega}(b_3-b_2)(1-\frac{\alpha^{\frac{1}{q}}}{\omega})\Bigg)$.

It can be easily verified that equation (4.8) has the solution

$$\begin{cases} \underline{L}^{\alpha}_{\underline{x}_{\tilde{\alpha}}}(t) = e^{-pt}\Bigg\{\frac{A}{2} - \frac{B}{2} + e^{2pt}\Bigg(\frac{A}{2} + \frac{B}{2} + \frac{q}{p} - \frac{qe^{-pt}}{p}\Bigg)\Bigg\} \\ \underline{R}^{\alpha}_{\underline{x}_{\tilde{\alpha}}}(t) = e^{-pt}\Bigg\{\frac{B}{2} - \frac{A}{2} + e^{2pt}\Bigg(\frac{A}{2} + \frac{B}{2} + \frac{q}{p} - \frac{qe^{-pt}}{p}\Bigg)\Bigg\} \\ \overline{L}^{\alpha}_{\overline{x}_{\tilde{\alpha}}}(t) = e^{-pt}\Bigg\{\frac{C}{2} - \frac{D}{2} + e^{2pt}\Bigg(\frac{C}{2} + \frac{D}{2} + \frac{q}{p} - \frac{qe^{-pt}}{p}\Bigg)\Bigg\} \\ \overline{R}^{\alpha}_{\overline{x}_{\tilde{\alpha}}}(t) = e^{-pt}\Bigg\{\frac{D}{2} - \frac{C}{2} + e^{2pt}\Bigg(\frac{C}{2} + \frac{D}{2} + \frac{q}{p} - \frac{qe^{-pt}}{p}\Bigg)\Bigg\} \end{cases} \tag{4.9}$$

Hence the solution to equation (4.4) is $x^{\alpha}_{\tilde{\alpha}}(t) =$

$$\Bigg(\Bigg[e^{-pt}\Bigg\{\frac{A}{2} - \frac{B}{2} + e^{2pt}\Bigg(\frac{A}{2} + \frac{B}{2} + \frac{q}{p} - \frac{qe^{-pt}}{p}\Bigg)\Bigg\},$$
$$e^{-pt}\Bigg\{\frac{B}{2} - \frac{A}{2} + e^{2pt}\Bigg(\frac{A}{2} + \frac{B}{2} + \frac{q}{p} - \frac{qe^{-pt}}{p}\Bigg)\Bigg\}\Bigg];$$
$$\Bigg[e^{-pt}\Bigg\{\frac{C}{2} - \frac{D}{2} + e^{2pt}\Bigg(\frac{C}{2} + \frac{D}{2} + \frac{q}{p} - \frac{qe^{-pt}}{p}\Bigg)\Bigg\},$$
$$e^{-pt}\Bigg\{\frac{D}{2} - \frac{C}{2} + e^{2pt}\Bigg(\frac{C}{2} + \frac{D}{2} + \frac{q}{p} - \frac{qe^{-pt}}{p}\Bigg)\Bigg\}\Bigg]\Bigg).$$

Example 4.1. For better understanding of the method discussed in the section 4.1, we here give an example of initial value problem with the initial condition as non linear type-2 fuzzy number.

$$\begin{cases} y'(t) = -5y(t) \\ y(0) = (1, 1.5, 2, 3, 4, 4.5, 5; 0.2, 0.5). \end{cases} \tag{4.10}$$

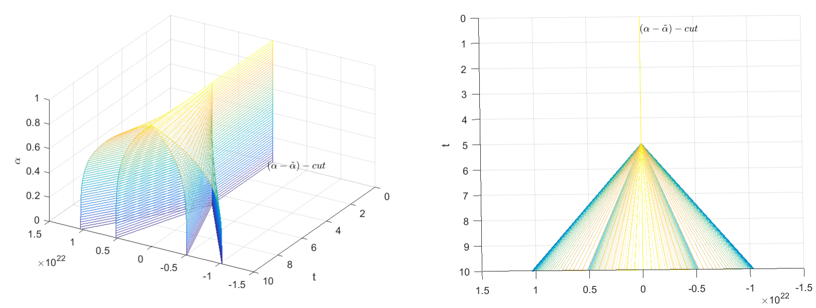
In equation 4.10, the coefficient is negative. Actually, the problem on example (4.1) is the family of inclusion of the following differential equations

$$\begin{cases} \zeta'_\alpha(t) = \begin{pmatrix} 0 & -5 & 0 & 0 \\ -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & -5 & 0 \end{pmatrix} \zeta_\alpha(t) \\ \zeta_\alpha(0) = \begin{pmatrix} \underline{L}^{\alpha}_{x_{0}{}_{\tilde{\alpha}}} \\ \underline{R}^{\alpha}_{x_{0}{}_{\tilde{\alpha}}} \\ \overline{L}^{\alpha}_{\overline{x_0}_{\tilde{\alpha}}} \\ \overline{R}^{\alpha}_{\overline{x_0}_{\tilde{\alpha}}} \end{pmatrix}. \end{cases} \tag{4.11}$$

Now, solving the differential inclusion on equation (4.11) we obtained the solution of the initial value problem on example (4.1) in terms of $(\alpha, \tilde{\alpha})$-cut as follows

$$y^{\alpha}_{\tilde{\alpha}}(t) = \left(\left[\tfrac{A+B}{2} e^{-5t} + \tfrac{A-B}{2} e^{5t}, \tfrac{A+B}{2} e^{-5t} - \tfrac{A-B}{2} e^{5t} \right]; \left[\tfrac{C+D}{2} e^{-5t} + \tfrac{C-D}{2} e^{5t}, \tfrac{C+D}{2} e^{-5t} - \tfrac{C-D}{2} e^{5t} \right] \right),$$

where $A = 2 + \alpha^4 - 0.5\tilde{\alpha}^2(1 - \alpha^4)$, $B = 4 - \alpha^4 + 0.5\tilde{\alpha}^2(1 - \alpha^4)$, $C = 1 + 2\alpha^4 - 0.5\tilde{\alpha}^2(1 - \alpha^4)$, and $D = 5 - 2\alpha^4 + 0.5\tilde{\alpha}^2(1 - \alpha^4)$.



(a) (b)

FIGURE 6. (a) and (b) represents the $(\alpha - \tilde{\alpha})-$ cut of the solution of the example (4.1) via type-2 fuzzy differential inclusions.

4.2. **Type-2 fuzzy differential inclusion for solving type-2 fuzzy boundary Value Problem:** We take a boundary value problem having coefficients to be crisp constants as follows

$$\begin{cases} y''(t) + Py'(t) + Qy(t) = R(t) \\ y'(a) = \tilde{A} = (a_1, a_2, a_3, m, b_1, b_2, b_3;\, p, q), y(b) = \tilde{B} = (c_1, c_2, c_3, n, d_1, d_2, d_3;\, p, q). \end{cases} \tag{4.12}$$

Here, $y(t)$ is NT2TFN and the state variable which is differential function of t up to second order at least, $P, Q \in \mathbb{R}$ are the coefficients of the differential equation and $R(t)$ is a continuous function of t. Let $A^{\alpha}_{\tilde{\alpha}} = \Big(\big[\underline{L}^{\alpha}_{\underline{A}_{\tilde{\alpha}}}, \underline{R}^{\alpha}_{\underline{A}_{\tilde{\alpha}}}\big]; \big[\overline{L}^{\alpha}_{\overline{A}_{\tilde{\alpha}}}, \overline{R}^{\alpha}_{\overline{A}_{\tilde{\alpha}}}\big] \Big)$ and $B^{\alpha}_{\tilde{\alpha}} = \Big(\big[\underline{L}^{\alpha}_{\underline{B}_{\tilde{\alpha}}}, \underline{R}^{\alpha}_{\underline{B}_{\tilde{\alpha}}}\big]; \big[\overline{L}^{\alpha}_{\overline{B}_{\tilde{\alpha}}}, \overline{R}^{\alpha}_{\overline{B}_{\tilde{\alpha}}}\big] \Big)$ are the $(\alpha - \tilde{\alpha})$-cuts of $\tilde{A}$ and $\tilde{B}$.

Case I: When $P < 0, Q < 0$, the differential equation (4.9) is the family of inclusion

$$(4.13)\qquad \begin{cases} \zeta_{\alpha}(t) \in \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ Q & 0 & 0 & 0 & P & 0 & 0 & 0 \\ 0 & Q & 0 & 0 & 0 & P & 0 & 0 \\ 0 & 0 & Q & 0 & 0 & 0 & P & 0 \\ 0 & 0 & 0 & Q & 0 & 0 & 0 & P \end{pmatrix} \zeta_{\alpha}(t) + \begin{pmatrix} R(t) \\ R(t) \\ R(t) \\ R(t) \\ R(t) \\ R(t) \\ R(t) \\ R(t) \end{pmatrix} \\ x(a) \in x^{\alpha}_{\tilde{\alpha}}(a), y(b) \in y^{\alpha}_{\tilde{\alpha}}(b). \end{cases}$$

Case II: If $P > 0,\ Q > 0$, then the differential equation (4.9) is the family of inclusion

$$(4.14)\qquad \begin{cases} \zeta_{\alpha}(t) \in \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & Q & 0 & 0 & 0 & P & 0 & 0 \\ Q & 0 & 0 & 0 & P & 0 & 0 & 0 \\ 0 & 0 & 0 & Q & 0 & 0 & 0 & P \\ 0 & 0 & Q & 0 & 0 & 0 & P & 0 \end{pmatrix} \zeta_{\alpha}(t) + \begin{pmatrix} R(t) \\ R(t) \\ R(t) \\ R(t) \\ R(t) \\ R(t) \\ R(t) \\ R(t) \end{pmatrix} \\ x(a) \in x^{\alpha}_{\tilde{\alpha}}(a), y(b) \in y^{\alpha}_{\tilde{\alpha}}(b), \end{cases}$$

where $\zeta_{\alpha}(t)$ represents the vector $\begin{pmatrix} \underline{L}^{\alpha}_{\underline{x}_{\tilde{\alpha}}}(t) \\ \underline{R}^{\alpha}_{\underline{x}_{\tilde{\alpha}}}(t) \\ \overline{L}^{\alpha}_{\overline{x}_{\tilde{\alpha}}}(t) \\ \overline{R}^{\alpha}_{\overline{x}_{\tilde{\alpha}}}(t) \\ \underline{L}^{\alpha}_{\underline{y}_{\tilde{\alpha}}}(t) \\ \underline{R}^{\alpha}_{\underline{y}_{\tilde{\alpha}}}(t) \\ \overline{L}^{\alpha}_{\overline{y}_{\tilde{\alpha}}}(t) \\ \overline{R}^{\alpha}_{\overline{y}_{\tilde{\alpha}}}(t) \end{pmatrix}$.

Thus the solution of the boundary value problem on equation (4.12) is obtained as $x^{\alpha}_{\tilde{\alpha}}(t) = \Big(\big[\underline{L}^{\alpha}_{\underline{x}_{\tilde{\alpha}}}(t), \underline{R}^{\alpha}_{\underline{x}_{\tilde{\alpha}}}(t)\big]; \big[\overline{L}^{\alpha}_{\overline{x}_{\tilde{\alpha}}}(t), \overline{R}^{\alpha}_{\overline{x}_{\tilde{\alpha}}}(t)\big] \Big)$, $y^{\alpha}_{\tilde{\alpha}}(t) = \Big(\big[\underline{L}^{\alpha}_{\underline{y}_{\tilde{\alpha}}}(t), \underline{R}^{\alpha}_{\underline{y}_{\tilde{\alpha}}}(t)\big]; \big[\overline{L}^{\alpha}_{\overline{y}_{\tilde{\alpha}}}(t), \overline{R}^{\alpha}_{\overline{y}_{\tilde{\alpha}}}(t)\big] \Big)$.

Example 4.2. We take a second ordered differential equation with boundary value as

$$(4.15)\qquad \begin{cases} y''(t) + 9y(t) = cos(t) \\ y(\frac{\pi}{2}) = (0.5, 1, 1.5, 2, 3, 4, 5;\ 4, 2), y'(0) = (1, 3, 4, 5, 6, 7, 9;\ 0.25, 0.5). \end{cases}$$

The non linear type-2 fuzzy boundary value problem on example (4.2) is the family of inclusion of the following differential equations

$$
\left\{\begin{array}{l}
\zeta_{\tilde{\alpha}}^{\alpha}(t) \in \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -9 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -9 & 0 & 0 & 0 & 0 \\ 0 & 0 & -9 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \zeta_{\alpha}(t) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ cos(t) \\ cos(t) \\ cos(t) \\ cos(t) \end{pmatrix} \\
x(a) \in x_{\tilde{\alpha}}^{\alpha}(a), y(b) \in y_{\tilde{\alpha}}^{\alpha}(b).
\end{array}\right. \tag{4.16}
$$

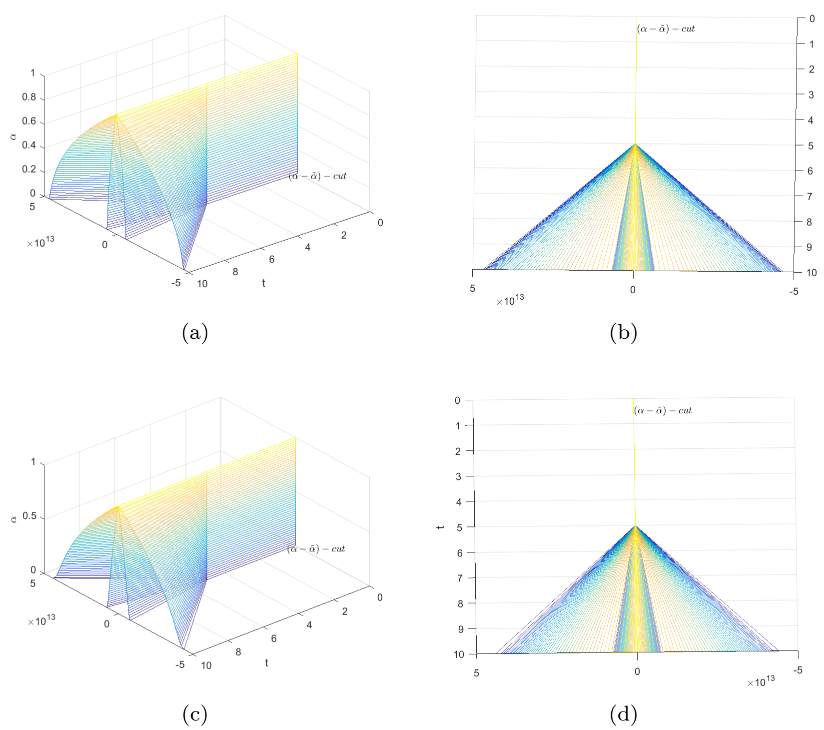


FIGURE 7. (a), (b), (c) and (d) represents the $(\alpha-\tilde{\alpha})$-cut of the solution of example (4.2) via type-2 fuzzy differential inclusions.

Now, solving the differential inclusion (4.16) we obtained the solution problem given in example (4.2) in terms of $(\alpha, \tilde{\alpha})-$ cut as follows $y_{\tilde{\alpha}}^{\alpha}(t) =$

$$
\Bigg(\bigg[\frac{cost}{8} - \frac{cos3t}{8} + (\frac{A}{2}+\frac{B}{2})cos3t + (\frac{A}{4}-\frac{B}{4}+\frac{E}{12}-\frac{F}{12})e^{3t} + (\frac{A}{4}-\frac{B}{4}-\frac{E}{12}+\frac{F}{12})e^{-3t} + (\frac{E}{6}+\frac{F}{6})sin3t,
$$

$$
\frac{cost}{8} - \frac{cos3t}{8} + (\frac{A}{2}+\frac{B}{2})cos3t - (\frac{A}{4}-\frac{B}{4}+\frac{E}{12}-\frac{F}{12})e^{3t} - (\frac{A}{4}-\frac{B}{4}-\frac{E}{12}+\frac{F}{12})e^{-3t} + (\frac{E}{6}+\frac{F}{6})sin3t\bigg];
$$

$$
\bigg[\frac{cost}{8} - \frac{cos3t}{8} + (\frac{C}{2}+\frac{D}{2})cos3t + (\frac{C}{4}-\frac{D}{4}+\frac{G}{12}-\frac{H}{12})e^{3t} + (\frac{C}{4}-\frac{D}{4}-\frac{G}{12}+\frac{H}{12})e^{-3t} + (\frac{G}{6}+\frac{H}{6})sin3t,
$$

$\frac{cost}{8}-\frac{cos3t}{8}+(\frac{C}{2}+\frac{D}{2})cos3t-(\frac{C}{4}-\frac{D}{4}+\frac{G}{12}-\frac{H}{12})e^{3t}-(\frac{C}{4}-\frac{D}{4}-\frac{G}{12}+\frac{H}{12})e^{-3t}+(\frac{G}{6}+\frac{H}{6})sin3t\Big]\Big)$,

$\Big(\Big[\frac{3sin3t}{8}-\frac{sint}{8}+(\frac{E}{2}+\frac{F}{2})cos3+(\frac{3A}{4}-\frac{3B}{4}+\frac{E}{4}-\frac{F}{4})e^{3t}+(\frac{3B}{4}-\frac{3A}{4}+\frac{E}{4}-\frac{F}{4})e^{-3t}-(\frac{3A}{2}+\frac{3B}{2})sin3t,$

$\frac{3sin3t}{8}-\frac{sint}{8}+(\frac{E}{2}+\frac{F}{2})cos3t-(\frac{3A}{4}-\frac{3B}{4}+\frac{E}{4}-\frac{F}{4})e^{3t}-(\frac{3B}{4}-\frac{3A}{4}+\frac{E}{4}-\frac{F}{4})e^{-3t}-(\frac{3A}{2}+\frac{3B}{2})sin3t\Big];$

$\Big[\frac{3sin3t}{8}-\frac{sint}{8}+(\frac{G}{2}+\frac{H}{2})cos3t+(\frac{3C}{4}-\frac{3D}{4}+\frac{G}{4}-\frac{H}{4})e^{3t}+(\frac{3D}{4}-\frac{3C}{4}+\frac{G}{4}-\frac{H}{4})e^{-3t}-(\frac{3C}{2}+\frac{3D}{2})sin3t,$

$\frac{3sin3t}{8}-\frac{sint}{8}+(\frac{G}{2}+\frac{H}{2})cos3t-(\frac{3C}{4}-\frac{3D}{4}+\frac{G}{4}-\frac{H}{4})e^{3t}+(\frac{3D}{4}-\frac{3C}{4}+\frac{G}{4}-\frac{H}{4})e^{-3t}-(\frac{3C}{2}+\frac{3D}{2})sin3t\Big]\Big)$,

where $A=4+\alpha^2-\tilde{\alpha}^4(1-\alpha^2)$, $B=6-\alpha^2+\tilde{\alpha}^4(1-\alpha^2)$, $C=1+4\alpha^2-2\tilde{\alpha}^4(1-\alpha^2)$, $D=9-4\alpha^2+2\tilde{\alpha}^4(1-\alpha^2)$, $E=1.5+0.5\alpha^{0.5}+0.5\tilde{\alpha}^{0.25}(1-\alpha^{0.5})$, $F=3-\alpha^{0.5}+\tilde{\alpha}^{0.25}(1-\alpha^{0.5})$, $G=0.5+1.5\alpha^{0.5}-0.5\tilde{\alpha}^{0.25}(1-\alpha^{0.5})$ and $H=5-3\alpha^{0.5}+\tilde{\alpha}^{0.25}(1-\alpha^{0.5})$.

5. Conclusion

We have solved the type 2 fuzzy differential equation. The distinct arithmetic features of the nonlinear type-2 triangular fuzzy numbers (NT2TFNs) are demonstrated. The type-2 fuzzy initial and boundary value problem have been chosen and are solved using type-2 fuzzy differential inclusion techniques. Numerical examples has been shown along with the solution graphs. The uncertainty that fuzzy sets carry is generalized by type-2 fuzzy sets. Again, the fuzzy differential inclusions take a more comprehensive approach to the fuzzy differential equation. The current work makes a substantial contribution by combining two well-known notions.

In future, the type-2 fuzzy differential inclusion methods may be used for solving intuitionistic type 2 fuzzy differential equation. Case-study based works on suitable applications of the type-2 fuzzy differential inclusion techniques may contribute significant investigations in this direction.

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