Hesitant fuzzy soft sets over UP-algebras

Phakawat Mosrijai, Aiyared Iampan

Received 24 June 2018; Revised 28 July 2018; Accepted 6 August 2018

Abstract. This paper aims to extend the notion of hesitant fuzzy sets on UP-algebras to hesitant fuzzy soft sets over UP-algebras by merging the notions of hesitant fuzzy sets and soft sets. Further, we discuss the notions of hesitant fuzzy soft strongly UP-ideals, hesitant fuzzy soft UP-ideals, hesitant fuzzy soft UP-filters, and hesitant fuzzy soft UP-subalgebras of UP-algebras, and provide some properties.

2010 AMS Classification: 03G25

Keywords: UP-algebra, Hesitant fuzzy soft UP-subalgebra, Hesitant fuzzy soft UP-filter, Hesitant fuzzy soft UP-ideal, Hesitant fuzzy soft strongly UP-ideal.

Corresponding Author: Aiyared Iampan (aiyared.ia@up.ac.th)

1. Introduction

The branch of the logical algebra, UP-algebras was introduced by Iampan [3] in 2017, and it is known that the class of KU-algebras [14] is a proper subclass of the class of UP-algebras. It have been examined by several researchers, for example, Somjanta et al. [19] introduced the notion of fuzzy sets in UP-algebras, the notion of intuitionistic fuzzy sets in UP-algebras was introduced by Kesorn et al. [7], the notion of Q-fuzzy sets in UP-algebras was introduced by Tanamoong et al. [20], Senapati et al. [17, 18] applied cubic set and interval-valued intuitionistic fuzzy structure in UP-algebras, etc.

A soft set over a universe set is a parametrized family of subsets of the universe set. Molodtsov [8] introduced the notion of soft sets over a universe set in 1999.

A hesitant fuzzy set on a set is a function from a reference set to a power set of the unit interval. The notion of a hesitant fuzzy set on a set was first considered by Torra [21] in 2010. Recently hesitant fuzzy sets theory has been applied to the different algebraic structures (see [6, 11, 12, 13]). In UP-algebras, Mosrijai et al. [9] extended the notion of fuzzy sets in UP-algebras to hesitant fuzzy sets on UP-algebras, and Satirad et al. [16] considered level subsets of a hesitant fuzzy set on UP-algebras.
in 2017. The notion of partial constant hesitant fuzzy sets on UP-algebras was introduced by Mosrijai et al. [10] afterwards.

The notion of hesitant fuzzy soft sets that is a link between classical soft sets and hesitant fuzzy sets is introduced by Babitha and John [1] in 2013. There exists some researchers, such as Jun et al. [5], applied hesitant fuzzy soft set theory to some algebraic structures, which are BCK and BCI algebras.

In this paper, we extend the notion of hesitant fuzzy sets on UP-algebras to hesitant fuzzy soft sets over UP-algebras by merging the notion of hesitant fuzzy sets and soft sets. Further, we discuss the notion of hesitant fuzzy soft strongly UP-ideals, hesitant fuzzy soft UP-ideals, hesitant fuzzy soft UP-filters and hesitant fuzzy soft UP-subalgebras of UP-algebras, and provide some properties.

2. Basic results on UP-algebras

Before we begin our study, we will introduce the definition of a UP-algebra.

**Definition 2.1 ([3]).** An algebra \( A = (A, \cdot, 0) \) of type \((2, 0)\) is called a UP-algebra, where \( A \) is a nonempty set, \( \cdot \) is a binary operation on \( A \), and \( 0 \) is a fixed element of \( A \) (i.e., a nullary operation), if it satisfies the following axioms: for any \( x, y, z \in A \),

\[
\begin{align*}
\text{(UP-1)} \quad & (y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0, \\
\text{(UP-2)} \quad & 0 \cdot x = x, \\
\text{(UP-3)} \quad & x \cdot 0 = 0, \\
\text{(UP-4)} \quad & x \cdot y = 0 \text{ and } y \cdot x = 0 \text{ imply } x = y.
\end{align*}
\]

From [3], we know that the notion of UP-algebras is a generalization of KU-algebras.

**Example 2.2 ([15]).** Let \( X \) be a universal set and let \( \Omega \in \mathcal{P}(X) \). Let \( \mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\} \). Define a binary operation \( \cdot \) on \( \mathcal{P}_\Omega(X) \) by : for all \( A, B \in \mathcal{P}_\Omega(X) \),

\[
A \cdot B = B \cap (A' \cup \Omega).
\]

Then \( (\mathcal{P}_\Omega(X), \cdot, \Omega) \) is a UP-algebra and we shall call it the generalized power UP-algebra of type 1 with respect to \( \Omega \).

In particular, \((\mathcal{P}(X), \cdot, \emptyset)\) is the power UP-algebra of type 1.

**Example 2.3 ([15]).** Let \( X \) be a universal set and let \( \Omega \in \mathcal{P}(X) \). Let \( \mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\} \). Define a binary operation \( * \) on \( \mathcal{P}^\Omega(X) \) by : for all \( A, B \in \mathcal{P}^\Omega(X) \),

\[
A * B = B \cup (A' \cap \Omega).
\]

Then \( (\mathcal{P}^\Omega(X), *, \Omega) \) is a UP-algebra and we shall call it the generalized power UP-algebra of type 2 with respect to \( \Omega \).

In particular, \((\mathcal{P}(X), *, X)\) is the power UP-algebra of type 2.

In a UP-algebra \( A = (A, \cdot, 0) \), the following assertions are valid (see [3, 4]):

\[
\begin{align*}
(2.1) & \quad (\forall x \in A)(x \cdot x = 0), \\
(2.2) & \quad (\forall x, y, z \in A)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0), \\
(2.3) & \quad (\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0),
\end{align*}
\]
In what follows, let $A$ denote UP-algebras unless otherwise specified.

On a UP-algebra $A = (A, \cdot, 0)$, we define a binary relation $\leq$ on $A$ [3] as follows: for any $x, y \in A$,

$x \leq y$ if and only if $x \cdot y = 0$.

**Definition 2.4** ([3]). A subset $S$ of $A$ is called a UP-subalgebra of $A$, if the constant 0 of $A$ is in $S$ and $(S, \cdot, 0)$ itself forms a UP-algebra.

Iampan [3] proved the useful criteria that a nonempty subset $S$ of $A$ is a UP-subalgebra of $A$ if and only if $S$ is closed under the $\cdot$ multiplication on $A$.

**Definition 2.5** ([2, 3, 19]). A subset $S$ of $A$ is called:

1. a UP-filter of $A$, if
   (i) the constant 0 of $A$ is in $S$, and
   (ii) for any $x, y \in A$, $x \cdot y \in S$ and $x \in S$ imply $y \in S$,

2. a UP-ideal of $A$, if
   (i) the constant 0 of $A$ is in $S$, and
   (ii) for any $x, y, z \in A$, $x \cdot (y \cdot z) \in S$ and $y \in S$ imply $x \cdot z \in S$,

3. a strongly UP-ideal of $A$, if
   (i) the constant 0 of $A$ is in $S$, and
   (ii) for any $x, y, z \in A$, $(z \cdot y) \cdot (z \cdot x) \in S$ and $y \in S$ imply $x \in S$.

Guntasow et al. [2] proved the generalization that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra $X$ is the only one strongly UP-ideal of itself.
3. Basic results on hesitant fuzzy sets

**Definition 3.1** ([21]). Let $X$ be a reference set. A hesitant fuzzy set on $X$ is defined in term of a function $h_H$ that when applied to $X$ return a subset of $[0, 1]$, that is, $h_H: X \rightarrow \mathcal{P}([0, 1])$. A hesitant fuzzy set $h_H$ can also be viewed as the following mathematical representation:

$$H := \{(x, h_H(x)) \mid x \in X\}$$

where $h_H(x)$ is a set of some values in $[0, 1]$, denoting the possible membership degrees of the elements $x \in X$ to the set $H$.

**Definition 3.2** ([9]). A hesitant fuzzy set $H$ on $A$ is called:

1. a hesitant fuzzy UP-subalgebra of $A$, if it satisfies the following property: for any $x, y \in A$, $h_H(x \cdot y) \supseteq h_H(x) \cap h_H(y)$,
2. a hesitant fuzzy UP-filter of $A$, if it satisfies the following properties: for any $x, y \in A$,
   - (i) $h_H(0) \supseteq h_H(x)$, and
   - (ii) $h_H(y) \supseteq h_H(x \cdot y) \cap h_H(x)$,
3. a hesitant fuzzy UP-ideal of $A$, if it satisfies the following properties: for any $x, y, z \in A$,
   - (i) $h_H(0) \supseteq h_H(x)$, and
   - (ii) $h_H(x \cdot z) \supseteq h_H(x \cdot (y \cdot z)) \cap h_H(y)$,
4. a hesitant fuzzy strongly UP-ideal of $A$, if it satisfies the following properties: for any $x, y, z \in A$,
   - (i) $h_H(0) \supseteq h_H(x)$, and
   - (ii) $h_H(x) \supseteq h_H((z \cdot y) \cdot (z \cdot x)) \cap h_H(y)$.

Mosrijai et al. [9] proved that the notion of hesitant fuzzy UP-subalgebras of UP-algebras is a generalization of hesitant fuzzy UP-filters, the notion of hesitant fuzzy UP-filters of UP-algebras is a generalization of hesitant fuzzy UP-ideals, and the notion of hesitant fuzzy UP-ideals of UP-algebras is a generalization of hesitant fuzzy strongly UP-ideals.

**Theorem 3.3** ([9]). A hesitant fuzzy set $H$ on $A$ is a hesitant fuzzy strongly UP-ideal of $A$ if and only if it is a constant hesitant fuzzy set on $A$.

**Proposition 3.4.** Let $H$ be a hesitant fuzzy UP-filter (and also hesitant fuzzy UP-ideal, hesitant fuzzy strongly UP-ideal) of $A$. Then for any $x, y \in A$,

$$x \leq y \text{ implies } h_H(x) \subseteq h_H(y) \subseteq h_H(x \cdot y).$$

**Proof.** Let $x, y \in A$ be such that $x \leq y$. Then $x \cdot y = 0$. Thus

$$h_H(y) \supseteq h_H(x \cdot y) \cap h_H(x) = h_H(0) \cap h_H(x) = h_H(x).$$

By (2.5), we have $y \leq x \cdot y$. So $h_H(y) \subseteq h_H(x \cdot y)$. \hfill $\square$

4. Hesitant fuzzy soft UP-subalgebras

**Definition 4.1** ([1]). Let $X$ be a reference set (or an initial universe set) and $P$ be a set of parameters. Let $HFS(X)$ be the set of all hesitant fuzzy sets on $X$ and $Y$
be a nonempty subset of \( P \). A pair \((\bar{H}, Y)\) is called a hesitant fuzzy soft set over \( X \), where \( \bar{H} \) is a mapping given by:
\[
\bar{H}: Y \rightarrow \text{HFS}(X), p \mapsto \bar{H}[p].
\]

**Definition 4.2.** Let \( Y \) be a nonempty subset of \( P \). A hesitant fuzzy soft set \((\bar{H}, Y)\) over \( A \) is called a hesitant fuzzy soft UP-subalgebra based on \( p \in Y \) (we shortly call a \( p \)-hesitant fuzzy soft UP-subalgebra) of \( A \), if the hesitant fuzzy set
\[
\bar{H}[p] := \{(a, h_{\bar{H}[p]}(a)) \mid a \in A\}
\]
on \( A \) is a hesitant fuzzy UP-subalgebra of \( A \). If \((\bar{H}, Y)\) is a \( p \)-hesitant fuzzy soft UP-subalgebra of \( A \) for all \( p \in Y \), we state that \((\bar{H}, Y)\) is a hesitant fuzzy soft UP-subalgebra of \( A \).

**Theorem 4.3.** If \((\bar{H}, Y)\) is a hesitant fuzzy soft UP-subalgebra of \( A \), then it satisfies the property: for any \( p \in Y \) and \( x \in A \),
\[
(4.1) \quad h_{\bar{H}[p]}(0) \supseteq h_{\bar{H}[p]}(x).
\]

**Proof.** Assume that \((\bar{H}, Y)\) is a hesitant fuzzy soft UP-subalgebra of \( A \) and let \( p \in Y \) and \( x \in A \). Then \( \bar{H}[p] \) is a hesitant fuzzy UP-subalgebra of \( A \). Thus,
\[
h_{\bar{H}[p]}(0) = h_{\bar{H}[p]}(x \cdot x) \supseteq h_{\bar{H}[p]}(x) \cap h_{\bar{H}[p]}(x) = h_{\bar{H}[p]}(x).
\]

\( \square \)

**Example 4.4.** Let \( \mathcal{P}_\varnothing(\{a, b\}) \) is the power UP-algebra of type 1 which a binary operation \( \cdot \) defined by the following Cayley table:
\[
\begin{array}{c|cccc}
\cdot & \varnothing & \{a\} & \{b\} & X \\
\hline
\varnothing & \varnothing & \{a\} & \{b\} & X \\
\{a\} & \varnothing & \varnothing & \{b\} & \{b\} \\
\{b\} & \varnothing & \{a\} & \varnothing & \{a\} \\
X & \varnothing & \varnothing & \varnothing & \varnothing
\end{array}
\]

Let \( Y = \{p_1, p_2, p_3, p_4\} \) be a parameter set. We define a hesitant fuzzy soft set \((\bar{H}, Y)\) over \( \mathcal{P}_\varnothing(\{a, b\}) \) by the following table:
\[
\begin{array}{cc|cc|c}
\bar{H} & \varnothing & \{a\} & \{b\} & X \\
\hline
p_1 & \{0.3, 0.4\} & \{0.3\} & \{0.4\} & \varnothing \\
p_2 & [0.6, 0.9] & \{0.9\} & [0.6, 0.9] & [0.6, 0.9] \\
p_3 & (0.3, 0.8) & \{0.3, 0.5\} & \{0.4, 0.5\} & \{0.4, 0.5\} \\
p_4 & [0.1] & [0.1] & [0.1] & [0.1]
\end{array}
\]

Then \((\bar{H}, Y)\) satisfies the property \((4.1)\), but not a hesitant fuzzy soft UP-subalgebra of \( A \) based on parameter \( p_2 \). Indeed,
\[
h_{\bar{H}[p_2]}(\{b\} \cdot X) = h_{\bar{H}[p_2]}(\{a\}) = \{0.9\} \not\supseteq [0.6, 0.9]
\]
\[
= [0.6, 0.9] \cap [0.6, 0.9] = h_{\bar{H}[p_2]}(\{b\}) \cap h_{\bar{H}[p_2]}(X).
\]
Theorem 4.5. Let \((\bar{H}, Y)\) be a hesitant fuzzy soft set over \(A\) which satisfies the condition: for any \(p \in Y\) and \(x, y, z \in A\),
\[
(4.2) \quad z \leq x \cdot y \implies h_{\bar{H}[p]}(y) \supseteq h_{\bar{H}[p]}(z) \cap h_{\bar{H}[p]}(x).
\]
Then \((\bar{H}, Y)\) is a hesitant fuzzy soft UP-subalgebra of \(A\).

Proof. Let \(p \in Y\) and \(x, y \in A\). Then by (2.5) and (UP-3), we have \(x \cdot (y \cdot (x \cdot y)) = x \cdot 0 = 0\). Thus \(x \leq y \cdot (x \cdot y)\). It follows form (4.2) that
\[
h_{\bar{H}[p]}(x \cdot y) \supseteq h_{\bar{H}[p]}(x) \cap h_{\bar{H}[p]}(y).
\]
So, \(\bar{H}[p]\) is a hesitant fuzzy UP-subalgebra of \(A\). Hence, \((\bar{H}, Y)\) is a \(p\)-hesitant fuzzy soft UP-subalgebra of \(A\). Since \(p\) is arbitrary, we know that \((\bar{H}, Y)\) is a hesitant fuzzy soft UP-subalgebra of \(A\). \(\square\)

Corollary 4.6. If \((\bar{H}, Y)\) is a hesitant fuzzy soft set over \(A\) which satisfies the condition (4.2), then it satisfies the property (4.1).

Proof. It is straightforward form Theorems 4.5 and 4.3. \(\square\)

Theorem 4.7. If \((\bar{H}, Y)\) is a hesitant fuzzy soft UP-subalgebra of \(A\) and \(N\) is a nonempty subset of \(Y\), then \((\bar{H}|_N, N)\) is a hesitant fuzzy soft UP-subalgebra of \(A\).

Proof. Assume that \((\bar{H}, Y)\) is a hesitant fuzzy soft UP-subalgebra of \(A\) and \(\emptyset \neq N \subseteq Y\). Then \((\bar{H}, Y)\) is a \(p\)-hesitant fuzzy soft UP-subalgebra of \(A\), for all \(p \in Y\). Since \(N \subseteq Y\), we have \((\bar{H}|_N, N)\) is a \(p\)-hesitant fuzzy soft UP-subalgebra of \(A\), for all \(p \in N\). Then, \((\bar{H}|_N, N)\) is a hesitant fuzzy soft UP-subalgebra of \(A\). \(\square\)

The following example shows that there exists a nonempty subset \(N\) of \(Y\) such that \((\bar{H}|_N, N)\) is a hesitant fuzzy soft UP-subalgebra of \(A\), but \((\bar{H}, Y)\) is not a hesitant fuzzy soft UP-subalgebra of \(A\).

Example 4.8. Let \(A = \{0, 1, 2, 3\}\) be a set with a binary operation \(\cdot\) defined by the following Cayley table:

<table>
<thead>
<tr>
<th>(\cdot)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \((A, \cdot, 0)\) is a UP-algebra. Let \(Y = \{p_1, p_2, p_3, p_4, p_5\}\) be a parameter set. We define a hesitant fuzzy soft set \((\bar{H}, Y)\) over \(A\) as the following table:

<table>
<thead>
<tr>
<th>(\bar{H})</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p_1)</td>
<td>{0.5,0.6,0.7}</td>
<td>{0.5}</td>
<td>{0.6}</td>
<td>{0.6}</td>
<td>\emptyset</td>
</tr>
<tr>
<td>(p_2)</td>
<td>{0.4,0.6}</td>
<td>{0.4,0.6}</td>
<td>{0.5,0.6}</td>
<td>{0.5,0.55}</td>
<td>{0.5}</td>
</tr>
<tr>
<td>(p_3)</td>
<td>{0.1,0.2,0.3}</td>
<td>{0.1,0.2}</td>
<td>{0.1,0.2}</td>
<td>{0.2}</td>
<td>{0.2}</td>
</tr>
<tr>
<td>(p_4)</td>
<td>{0.7,1}</td>
<td>{0.7,1}</td>
<td>{0.7}</td>
<td>{0.5,0.7}</td>
<td>{0.5,0.7}</td>
</tr>
<tr>
<td>(p_5)</td>
<td>{0.9}</td>
<td>{0.9}</td>
<td>{0.9}</td>
<td>{0.9}</td>
<td>{0.9}</td>
</tr>
</tbody>
</table>
Then $\widetilde{H}[p_4]$ is not a hesitant fuzzy UP-subalgebra of $A$. Indeed,
\[
h_{\widetilde{H}[p_4]}(1 \cdot 1) = h_{\widetilde{H}[p_4]}(0) = [0.7, 1] \not\supset [0.7, 1] = [0.7, 1] \cap [0.7, 1] = h_{\widetilde{H}[p_4]}(1) \cap h_{\widetilde{H}[p_4]}(1).
\]
Thus, $(\widetilde{H}, Y)$ is not a hesitant fuzzy soft UP-subalgebra of $A$. But if we choose $N = \{p_1, p_2, p_3, p_5\}$, then $(\widetilde{H}|_N, N)$ is a hesitant fuzzy soft UP-subalgebra of $A$.

### 5. Hesitant fuzzy soft UP-filters

**Definition 5.1.** Let $Y$ be a nonempty subset of $P$. A hesitant fuzzy soft set $(\widetilde{H}, Y)$ over $A$ is called a hesitant fuzzy soft UP-filter based on $p \in Y$ (we shortly call a $p$-hesitant fuzzy soft UP-filter) of $A$, if the hesitant fuzzy set
\[
\widetilde{H}[p] := \{(a, h_{\widetilde{H}[p]}(a)) \mid a \in A\}
\]
on $A$ is a hesitant fuzzy UP-filter of $A$. If $(\widetilde{H}, Y)$ is a $p$-hesitant fuzzy soft UP-filter of $A$ for all $p \in Y$, we state that $(\widetilde{H}, Y)$ is a hesitant fuzzy soft UP-filter of $A$.

From [9], we known that every hesitant fuzzy UP-filter of $A$ is a hesitant fuzzy UP-subalgebra. Then we have the following theorem:

**Theorem 5.2.** Every $p$-hesitant fuzzy soft UP-filter of $A$ is a $p$-hesitant fuzzy soft UP-subalgebra.

*Proof.* Assume that $(\widetilde{H}, Y)$ is a $p$-hesitant fuzzy soft UP-filter of $A$. Then $\widetilde{H}[p]$ is a hesitant fuzzy UP-filter of $A$. Thus $\widetilde{H}[p]$ is a hesitant fuzzy UP-subalgebra of $A$. So, $(\widetilde{H}, Y)$ is a $p$-hesitant fuzzy soft UP-subalgebra of $A$. \(\square\)

The following example shows that the converse of Theorem 5.2 is not true, in general.

**Example 5.3.** Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Then $(A, \cdot, 0)$ is a UP-algebra. Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{H}, Y)$ over $A$ by the following table:

<table>
<thead>
<tr>
<th>$\widetilde{H}$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>${0.5, 0.6}$</td>
<td>${0.5}$</td>
<td>${0.6}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$p_2$</td>
<td>$[0.4, 0.6]$</td>
<td>$[0.4, 0.6]$</td>
<td>$[0.4, 0.6]$</td>
<td>$[0.4, 0.6]$</td>
</tr>
<tr>
<td>$p_3$</td>
<td>$(0.2, 0.7)$</td>
<td>$(0.3, 0.5)$</td>
<td>$[0.4, 0.5]$</td>
<td>${0.5}$</td>
</tr>
<tr>
<td>$p_4$</td>
<td>${0.8}$</td>
<td>${0.8}$</td>
<td>${0.8}$</td>
<td>${0.8}$</td>
</tr>
</tbody>
</table>
Then \((\tilde{H}, Y)\) is a hesitant fuzzy soft UP-subalgebra of \(A\), but not a hesitant fuzzy soft UP-filter of \(A\) based on parameter \(p_1\). Indeed,
\[
\begin{align*}
\tilde{h}_{\tilde{H}[p_1]}(1) &= \{0.5\} \nsubseteq \{0.6\} \\
&= \{0.5, 0.6\} \cap \{0.6\} \\
&= \tilde{h}_{\tilde{H}[p_1]}(0) \cap \tilde{h}_{\tilde{H}[p_1]}(2) \\
&= \tilde{h}_{\tilde{H}[p_1]}(2 \cdot 1) \cap \tilde{h}_{\tilde{H}[p_1]}(2).
\end{align*}
\]

**Theorem 5.4.** A hesitant fuzzy soft set \((\tilde{H}, Y)\) over \(A\) is a hesitant fuzzy soft UP-filter of \(A\) if and only if it satisfies the condition \((4.2)\).

**Proof.** Assume that \((\tilde{H}, Y)\) is a hesitant fuzzy soft UP-filter of \(A\). Let \(p \in Y\) and let \(x, y, z \in A\) be such that \(z \leq x \cdot y\). Then \(\tilde{H}[p]\) is a hesitant fuzzy UP-filter of \(A\). By Proposition 3.4, we have \(\tilde{h}_{\tilde{H}[p]}(z) \subseteq \tilde{h}_{\tilde{H}[p]}(x \cdot y)\). Thus,
\[
\tilde{h}_{\tilde{H}[p]}(y) \supseteq \tilde{h}_{\tilde{H}[p]}(x \cdot y) \cap \tilde{h}_{\tilde{H}[p]}(x) \supseteq \tilde{h}_{\tilde{H}[p]}(z) \cap \tilde{h}_{\tilde{H}[p]}(x).
\]

Conversely, assume that \((\tilde{H}, Y)\) satisfies the condition \((4.2)\). Let \(p \in Y\) and let \(x \in A\). By Corollary 4.6, we have \(\tilde{h}_{\tilde{H}[p]}(0) \supseteq \tilde{h}_{\tilde{H}[p]}(x)\). Let \(x, y \in A\). Then by \((2.1)\), we have \((x \cdot y) \cdot (x \cdot y) = 0\). Thus \(x \cdot y \leq x \cdot y\). It follows from \((2)\) that
\[
\tilde{h}_{\tilde{H}[p]}(y) \supseteq \tilde{h}_{\tilde{H}[p]}(x \cdot y) \cap \tilde{h}_{\tilde{H}[p]}(x).
\]

So, \(\tilde{H}[p]\) is a hesitant fuzzy UP-filter of \(A\). Hence, \((\tilde{H}, Y)\) is a \(p\)-hesitant fuzzy soft UP-filter of \(A\). Since \(p\) is arbitrary, we know that \((\tilde{H}, Y)\) is a hesitant fuzzy soft UP-filter of \(A\). \(\square\)

**Theorem 5.5.** Let \((\tilde{H}, Y)\) be a hesitant fuzzy soft set over \(A\) which satisfies the condition: for any \(p \in Y\) and \(w, x, y, z \in A\),
\[(5.1) \quad x \leq w \cdot (y \cdot z) \text{ implies } \tilde{h}_{\tilde{H}[p]}(x \cdot z) \supseteq \tilde{h}_{\tilde{H}[p]}(w) \cap \tilde{h}_{\tilde{H}[p]}(y).\]
Then it is a hesitant fuzzy soft UP-filter of \(A\).

**Proof.** Assume that \((\tilde{H}, Y)\) is a hesitant fuzzy soft set over \(A\) which satisfies the condition \((5.1)\). Let \(p \in Y\) and let \(x, y \in A\). Then by \((2.1)\), we have \(0 - ((x \cdot y) \cdot (x \cdot y)) = 0 \cdot 0 = 0\). Thus \(0 \leq (x \cdot y) \cdot (x \cdot y)\). It follows form \((5.1)\) that
\[
\tilde{h}_{\tilde{H}[p]}(y) = \tilde{h}_{\tilde{H}[p]}(0) \supseteq \tilde{h}_{\tilde{H}[p]}(x \cdot y) \cap \tilde{h}_{\tilde{H}[p]}(x).
\]

So, \((\tilde{H}, Y)\) is a hesitant fuzzy soft UP-filter of \(A\). \(\square\)

**Corollary 5.6.** If \((\tilde{H}, Y)\) is a hesitant fuzzy soft set over \(A\) which satisfies the condition \((5.1)\), then it satisfies the condition \((4.2)\).

**Proof.** It is straightforward form Theorems 5.5 and 5.4. \(\square\)

**Theorem 5.7.** Let \((\tilde{H}, Y)\) be a hesitant fuzzy soft set over \(A\) which satisfies the condition: for any \(p \in Y\) and \(w, x, y, z \in A\),
\[(5.2) \quad w \leq x \cdot (y \cdot z) \text{ implies } \tilde{h}_{\tilde{H}[p]}(x \cdot z) \supseteq \tilde{h}_{\tilde{H}[p]}(w) \cap \tilde{h}_{\tilde{H}[p]}(y).\]
Then it is a hesitant fuzzy soft UP-filter of \(A\).
Proof. Assume that \((\bar{H}, Y)\) is a hesitant fuzzy soft set over \(A\) which satisfies the condition (5.2). Let \(p \in Y\) and let \(x, y \in A\). Then by (2.1) and (UP-2), we have \((x \cdot y) \cdot (0 \cdot (x \cdot y)) = (x \cdot y) \cdot (x \cdot y) = 0\). Thus \(x \cdot y \leq 0 \cdot (x \cdot y)\). It follows form (5.2) that

\[
    h_{\bar{H}[p]}(y) = h_{\bar{H}[p]}(0 \cdot y) \supseteq h_{\bar{H}[p]}(x \cdot y) \cap h_{\bar{H}[p]}(x).
\]

So, \((\bar{H}, Y)\) is a hesitant fuzzy soft UP-filter of \(A\). \(\square\)

**Corollary 5.8.** If \((\bar{H}, Y)\) is a hesitant fuzzy soft set over \(A\) which satisfies the condition (5.2), then it satisfies the condition (4.2).

**Proof.** It is straightforward form Theorems 5.7 and 5.4. \(\square\)

6. **HESITANT FUZZY SOFT UP-IDEALS**

**Definition 6.1.** Let \(Y\) be a nonempty subset of \(P\). A hesitant fuzzy soft set \((\bar{H}, Y)\) over \(A\) is called a hesitant fuzzy soft UP-ideal based on \(p \in Y\) (we shortly call a \(p\)-hesitant fuzzy soft UP-ideal) of \(A\) if the hesitant fuzzy set

\[
    \bar{H}[p] := \{(a, h_{\bar{H}[p]}(a)) \mid a \in A\}
\]

on \(A\) is a hesitant fuzzy UP-ideal of \(A\). If \((\bar{H}, Y)\) is a \(p\)-hesitant fuzzy soft UP-ideal of \(A\) for all \(p \in Y\), we state that \((\bar{H}, Y)\) is a hesitant fuzzy soft UP-ideal of \(A\).

From [9], we known that every hesitant fuzzy UP-ideal of \(A\) is a hesitant fuzzy UP-filter. Then we have the following theorem:

**Theorem 6.2.** Every \(p\)-hesitant fuzzy soft UP-ideal of \(A\) is a \(p\)-hesitant fuzzy soft UP-filter.

**Proof.** Assume that \((\bar{H}, Y)\) is a \(p\)-hesitant fuzzy soft UP-ideal of \(A\). Then \(\bar{H}[p]\) is a hesitant fuzzy UP-ideal of \(A\). Thus \(\bar{H}[p]\) is a hesitant fuzzy UP-filter of \(A\). So, \((\bar{H}, Y)\) is a \(p\)-hesitant fuzzy soft UP-filter of \(A\). \(\square\)

The following example shows that the converse of Theorem 6.2 is not true in general.

**Example 6.3.** Let \(A = \{0, 1, 2, 3\}\) be a set with a binary operation \(\cdot\) defined by the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \((A, \cdot, 0)\) is a UP-algebra. Let \(Y = \{p_1, p_2, p_3, p_4\}\) be a parameter set. We define a hesitant fuzzy soft set \((\bar{H}, Y)\) over \(A\) by the following table:

<table>
<thead>
<tr>
<th>(\bar{H})</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p_1)</td>
<td>([0.2, 0.5])</td>
<td>({0.5})</td>
<td>(\emptyset)</td>
<td>({0.3, 0.4})</td>
</tr>
<tr>
<td>(p_2)</td>
<td>([0.4, 0.6])</td>
<td>([0.4, 0.6])</td>
<td>({0.5})</td>
<td>({0.5})</td>
</tr>
<tr>
<td>(p_3)</td>
<td>([0.5, 0.7])</td>
<td>([0.5, 0.6])</td>
<td>([0.5, 0.6])</td>
<td>([0.5, 0.6])</td>
</tr>
<tr>
<td>(p_4)</td>
<td>([0.1, 0.6])</td>
<td>([0.1, 0.6])</td>
<td>([0.1, 0.6])</td>
<td>([0.1, 0.6])</td>
</tr>
</tbody>
</table>
Then \((\tilde{H}, Y)\) is a hesitant fuzzy soft UP-filter of \(A\), but not a hesitant fuzzy soft UP-ideal of \(A\) based on parameter \(p_1\). Indeed,
\[
\begin{align*}
  h_{\tilde{H}[p_1]}(3 \cdot 2) &= h_{\tilde{H}[p_1]}(2) = \emptyset \supsetneq \{0.5\} \\
  &= [0.2, 0.5] \cap \{0.5\} \\
  &= h_{\tilde{H}[p_1]}(0) \cap h_{\tilde{H}[p_1]}(1) \\
  &= h_{\tilde{H}[p_1]}(3 \cdot (1 \cdot 2)) \cap h_{\tilde{H}[p_1]}(1) .
\end{align*}
\]

**Theorem 6.4.** If \((\tilde{H}, Y)\) is a hesitant fuzzy soft UP-ideal of \(A\), then it satisfies the condition (5.1).

**Proof.** Assume that \((\tilde{H}, Y)\) is a hesitant fuzzy soft UP-ideal of \(A\). Let \(p \in Y\) and let \(w, x, y, z \in A\) be such that \(x \leq w \cdot (y \cdot z)\). Then \(\tilde{H}[p]\) is a hesitant fuzzy UP-ideal of \(A\) and \(x \cdot (w \cdot (y \cdot z)) = 0\). Thus
\[
\begin{align*}
  h_{\tilde{H}[p]}(x \cdot (y \cdot z)) &\supseteq h_{\tilde{H}[p]}(x \cdot (w \cdot (y \cdot z))) \cap h_{\tilde{H}[p]}(w) \\
  &= h_{\tilde{H}[p]}(0) \cap h_{\tilde{H}[p]}(w) = h_{\tilde{H}[p]}(w).
\end{align*}
\]

So, \(h_{\tilde{H}[p]}(x \cdot (y \cdot z)) \supseteq h_{\tilde{H}[p]}(x \cdot (w \cdot (y \cdot z))) \cap h_{\tilde{H}[p]}(w) \supseteq h_{\tilde{H}[p]}(w) \cap h_{\tilde{H}[p]}(y)\). \(\square\)

**Example 6.5.** Let \(A = \{0, 1, 2, 3\}\) be a set with a binary operation · defined by the same Cayley table in Example 6.3. Then \((A, \cdot, 0)\) is a UP-algebra. Let \(Y = \{p_1, p_2, p_3, p_4\}\) be a parameter set. We define a hesitant fuzzy soft set \((\tilde{H}, Y)\) over \(A\) by the following table:

<table>
<thead>
<tr>
<th>(H)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p_1)</td>
<td>([0.2, 0.5])</td>
<td>{0.5}</td>
<td>(\emptyset)</td>
<td>([0.3, 0.4])</td>
</tr>
<tr>
<td>(p_2)</td>
<td>([0.4, 0.6])</td>
<td>([0.4, 0.6])</td>
<td>{0.5}</td>
<td>{0.5}</td>
</tr>
<tr>
<td>(p_3)</td>
<td>([0.5, 0.7])</td>
<td>([0.5, 0.7])</td>
<td>(\emptyset)</td>
<td>{0.7}</td>
</tr>
<tr>
<td>(p_4)</td>
<td>([0.7, 0.8, 0.9])</td>
<td>([0.7, 0.8])</td>
<td>{0.8}</td>
<td>{0.8}</td>
</tr>
</tbody>
</table>

Then \((\tilde{H}, Y)\) is a hesitant fuzzy soft UP-filter of \(A\), but does not satisfy the condition (5.1).

**Theorem 6.6.** A hesitant fuzzy soft set \((\tilde{H}, Y)\) over \(A\) is a hesitant fuzzy soft UP-ideal of \(A\) if and only if it satisfies the condition (5.2).

**Proof.** Assume that \((\tilde{H}, Y)\) is a hesitant fuzzy soft UP-ideal of \(A\). Let \(p \in Y\) and \(w, x, y, z \in A\) be such that \(w \leq x \cdot (y \cdot z)\). Then \(\tilde{H}[p]\) is a hesitant fuzzy UP-ideal of \(A\). Thus by Proposition 3.4, we have \(h_{\tilde{H}[p]}(w) \subseteq h_{\tilde{H}[p]}(x \cdot (y \cdot z))\). So,
\[
\begin{align*}
  h_{\tilde{H}[p]}(x \cdot z) &\supseteq h_{\tilde{H}[p]}(x \cdot (y \cdot z)) \cap h_{\tilde{H}[p]}(y) \supseteq h_{\tilde{H}[p]}(w) \cap h_{\tilde{H}[p]}(y) .
\end{align*}
\]

Conversely, assume that \((\tilde{H}, Y)\) satisfies the condition (5.2). Let \(p \in Y\) and let \(x \in A\). By Corollary 5.8 and 4.6, we have \(h_{\tilde{H}[p]}(0) \supseteq h_{\tilde{H}[p]}(x)\). Let \(x, y, z \in A\). Then by (2.1), we have \((x \cdot (y \cdot z)) \cdot (x \cdot (y \cdot z)) = 0\). Thus \(x \cdot (y \cdot z) \leq x \cdot (y \cdot z)\). It follows form (5.2) that
\[
\begin{align*}
  h_{\tilde{H}[p]}(x \cdot z) &\supseteq h_{\tilde{H}[p]}(x \cdot (y \cdot z)) \cap h_{\tilde{H}[p]}(y) .
\end{align*}
\]
So, $\tilde{H}[p]$ is a hesitant fuzzy UP-ideal of $A$. Hence, $(H,Y)$ is a $p$-hesitant fuzzy soft UP-ideal of $A$. Since $p$ is arbitrary, we know that $(H,Y)$ is a hesitant fuzzy soft UP-ideal of $A$. □

**Theorem 6.7.** Let $(\tilde{H},Y)$ be a hesitant fuzzy soft set over $A$ which satisfies the condition: for any $p \in Y$ and $w, x, y, z \in A$,

\begin{equation}
(6.1) \quad w \leq (z \cdot y) \cdot (z \cdot x) \implies h_{\tilde{H}[p]}(x) \supseteq h_{\tilde{H}[p]}(w) \cap h_{\tilde{H}[p]}(y).
\end{equation}

Then it is a hesitant fuzzy soft UP-ideal of $A$.

**Proof.** Assume that $(\tilde{H},Y)$ is a hesitant fuzzy soft set over $A$ which satisfies the condition (6.1). Let $p \in Y$ and let $x, y, z \in A$. Then by (2.1) and (UP-3), we have $(x \cdot (y \cdot z)) \cdot ((x \cdot z) \cdot (x \cdot z)) = (x \cdot (y \cdot z)) \cdot (x \cdot (z \cdot y) \cdot 0) = (x \cdot (y \cdot z)) \cdot 0 = 0$. Thus $x \cdot (y \cdot z) \leq ((x \cdot z) \cdot (x \cdot z))$. It follows form (6.1) that

\[ h_{\tilde{H}[p]}(x \cdot z) \supseteq h_{\tilde{H}[p]}(x \cdot (y \cdot z)) \cap h_{\tilde{H}[p]}(y). \]

So, $(\tilde{H},Y)$ is a hesitant fuzzy soft UP-ideal of $A$. □

**Corollary 6.8.** If $(\tilde{H},Y)$ is a hesitant fuzzy soft set over $A$ which satisfies the condition (6.1), then it satisfies the conditions (5.1) and (5.2).

**Proof.** It is straightforward from Theorems 6.7, 6.4, and 6.6. □

### 7. Hesitant Fuzzy Soft Strongly UP-ideals

**Definition 7.1.** Let $Y$ be a nonempty subset of $P$. A hesitant fuzzy soft set $(\tilde{H},Y)$ over $A$ is called a hesitant fuzzy soft strongly UP-ideal based on $p \in Y$ (we shortly call a $p$-hesitant fuzzy soft strongly UP-ideal) of $A$, if the hesitant fuzzy set

\[ \tilde{H}[p] := \{(a, h_{\tilde{H}[p]}(a)) \mid a \in A\} \]

on $A$ is a hesitant fuzzy strongly UP-ideal of $A$. If $(\tilde{H},Y)$ is a $p$-hesitant fuzzy soft strongly UP-ideal of $A$ for all $p \in Y$, we state that $(\tilde{H},Y)$ is a hesitant fuzzy soft strongly UP-ideal of $A$.

From [9], we known that every hesitant fuzzy strongly UP-ideal of $A$ is a hesitant fuzzy UP-ideal. Then we have the following theorem:

**Theorem 7.2.** Every $p$-hesitant fuzzy soft strongly UP-ideal of $A$ is a $p$-hesitant fuzzy soft UP-ideal.

**Proof.** Assume that $(\tilde{H},Y)$ is a $p$-hesitant fuzzy soft strongly UP-ideal of $A$. Then $\tilde{H}[p]$ is a hesitant fuzzy strongly UP-ideal of $A$. Thus $\tilde{H}[p]$ is a hesitant fuzzy UP-ideal of $A$. So, $(\tilde{H},Y)$ is a $p$-hesitant fuzzy soft UP-ideal of $A$. □

The following example shows that the converse of Theorem 7.2 is not true in general.
Example 7.3. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

\[
\begin{array}{c|cccc}
\cdot & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & 3 \\
1 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 \\
3 & 0 & 1 & 2 & 0 \\
\end{array}
\]

Then $(A, \cdot, 0)$ is a UP-algebra. Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\tilde{H}, Y)$ over $A$ by the following table:

<table>
<thead>
<tr>
<th>$\tilde{H}$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>[0.2,0.5]</td>
<td>{0.4}</td>
<td>{0.2,0.4}</td>
<td>{0.2,0.4}</td>
</tr>
<tr>
<td>$p_2$</td>
<td>[0.9,1]</td>
<td>{1}</td>
<td>{1}</td>
<td>{1}</td>
</tr>
<tr>
<td>$p_3$</td>
<td>[0.0,2]</td>
<td>{0.0,2}</td>
<td>{0.0,2}</td>
<td>{0.0,2}</td>
</tr>
<tr>
<td>$p_4$</td>
<td>(0,1)</td>
<td>(0,1)</td>
<td>(0,1)</td>
<td>(0,1)</td>
</tr>
</tbody>
</table>

Then $(\tilde{H}, Y)$ is a hesitant fuzzy soft UP-ideal of $A$, but not a hesitant fuzzy soft strongly UP-ideal of $A$ based on parameter $p_2$. Indeed,

\[
h_{\tilde{H}[p_2]}(3) = \{1\} \not\subseteq [0.9,1]
\]

\[
= [0.9,1] \cap [0.9,1]
\]

\[
= h_{\tilde{H}[p_2]}(0) \cap h_{\tilde{H}[p_2]}(0)
\]

\[
= h_{\tilde{H}[p_2]}((1 \cdot 0) \cdot (1 \cdot 3)) \cap h_{\tilde{H}[p_2]}(0).
\]

By Theorems 5.2, 6.2, and 7.2 and Examples 5.3, 6.3, and 7.3, we have that the notion of $p$-hesitant fuzzy soft UP-subalgebras is a generalization of $p$-hesitant fuzzy soft UP-filters, the notion of $p$-hesitant fuzzy soft UP-filters is a generalization of $p$-hesitant fuzzy soft UP-ideals, and the notion of $p$-hesitant fuzzy soft UP-ideals is a generalization of $p$-hesitant fuzzy soft strongly UP-ideals.

**Theorem 7.4.** A hesitant fuzzy soft set $(\tilde{H}, Y)$ over $A$ is a hesitant fuzzy soft strongly UP-ideal of $A$ if and only if it satisfies the condition (6.1).

**Proof.** Assume that $(\tilde{H}, Y)$ is a hesitant fuzzy soft strongly UP-ideal of $A$. Let $p \in Y$ and let $w, x, y, z \in A$ be such that $w \leq (z \cdot y) \cdot (z \cdot x)$. Then $\tilde{H}[p]$ is a hesitant fuzzy strongly UP-ideal of $A$. Thus by Proposition 3.4, we have $h_{\tilde{H}[p]}(w) \subseteq h_{\tilde{H}[p]}((z \cdot y) \cdot (z \cdot x))$. So,

\[
h_{\tilde{H}[p]}(x) \supseteq h_{\tilde{H}[p]}((z \cdot y) \cdot (z \cdot x)) \cap h_{\tilde{H}[p]}(y) \supseteq h_{\tilde{H}[p]}(w) \cap h_{\tilde{H}[p]}(y).
\]

Conversely, assume that $(\tilde{H}, Y)$ satisfies the condition (6.1). Let $p \in Y$ and let $x \in A$. By Corollary 6.8, 5.8 and 4.6, respectively, we have $h_{\tilde{H}[p]}(0) \supseteq h_{\tilde{H}[p]}(x)$. Let $x, y, z \in A$. Since $((z \cdot y) \cdot (z \cdot x)) \cdot ((z \cdot y) \cdot (z \cdot x)) = 0$, we have $(z \cdot y) \cdot (z \cdot x) \leq (z \cdot y) \cdot (z \cdot x)$. Then it follows from (6.1) that

\[
h_{\tilde{H}[p]}(x) \supseteq h_{\tilde{H}[p]}((z \cdot y) \cdot (z \cdot x)) \cap h_{\tilde{H}[p]}(y).
\]
Thus, $\tilde{H}[p]$ is a hesitant fuzzy strongly UP-ideal of $A$. So, $(\tilde{H}, Y)$ is a $p$-hesitant fuzzy soft strongly UP-ideal of $A$. Since $p$ is arbitrary, we know that $(\tilde{H}, Y)$ is a hesitant fuzzy soft strongly UP-ideal of $A$. □

**Theorem 7.5.** Let $(\tilde{H}, Y)$ be a hesitant fuzzy soft set over $A$ such that $\emptyset \neq N \subseteq Y$. Then the following statements are hold:

1. If $(\tilde{H}, Y)$ is a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of $A$, then $(\tilde{H}|N, N)$ is a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of $A$.

2. There exists $(\tilde{H}|N, N)$ is a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of $A$, but $(\tilde{H}, Y)$ is not a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of $A$. □

**Definition 7.6.** Let $Y$ be a non-empty subset of $P$. A hesitant fuzzy soft set $(\tilde{H}, Y)$ over $A$ is called a constant hesitant fuzzy soft set based on $p \in Y$ (we shortly call a $p$-constant hesitant fuzzy soft set) over $A$, if the hesitant fuzzy set

$$\tilde{H}[p] := \{(a, h_{\tilde{H}[p]}(a)) \mid a \in A\}$$

on $A$ is a constant hesitant fuzzy set on $A$. If $(\tilde{H}, Y)$ is a $p$-constant hesitant fuzzy soft set over $A$ for all $p \in Y$, we state that $(\tilde{H}, Y)$ is a constant hesitant fuzzy soft set over $A$.

**Theorem 7.7.** A hesitant fuzzy soft set $(\tilde{H}, Y)$ over $A$ is a hesitant fuzzy soft strongly UP-ideal of $A$ if and only if is a constant hesitant fuzzy soft set over $A$.

**Proof.** Assume that $(\tilde{H}, Y)$ is a hesitant fuzzy soft strongly UP-ideal of $A$ and let $p \in Y$. Then $\tilde{H}[p]$ is a hesitant fuzzy strongly UP-ideal of $A$. By Theorem 3.3, we obtain $\tilde{H}[p]$ is a constant hesitant fuzzy set on $A$. Thus $(\tilde{H}, Y)$ is a $p$-constant hesitant fuzzy soft set over $A$. Since $p$ is arbitrary, we know that $(\tilde{H}, Y)$ is a constant hesitant fuzzy soft set over $A$.

Conversely, let $p \in Y$. Assume that $(\tilde{H}, Y)$ is a constant hesitant fuzzy soft set over $A$. Then $\tilde{H}[p]$ is a constant hesitant fuzzy set on $A$. By Theorem 3.3, we have $\tilde{H}[p]$ is a hesitant fuzzy strongly UP-ideal of $A$. Since $p$ is arbitrary, we state that $(\tilde{H}, Y)$ is a hesitant fuzzy soft strongly UP-ideal of $A$. □
8. Conclusions and Future Work

In this paper, we have introduced the notion of hesitant fuzzy soft sets which is a new extension of hesitant fuzzy sets over UP-algebras and the notions of hesitant fuzzy soft UP-subalgebras, hesitant fuzzy soft UP-filters, hesitant fuzzy soft UP-ideals and hesitant fuzzy soft strongly UP-ideals of UP-algebras and investigated some of its important properties. Then we have the diagram of hesitant fuzzy soft sets over UP-algebras below.

\[
\begin{align*}
(4.1) \\
\text{Hesitant Fuzzy Soft UP-Subalgebra} \\
(4.2) \\
\text{Hesitant Fuzzy Soft UP-Filter} \\
(5.1) \\
\text{Hesitant Fuzzy Soft UP-Ideal} \\
(5.2) \\
\text{Hesitant Fuzzy Soft Strongly UP-Ideal} \\
(6.1) \\
\text{Constant Hesitant Fuzzy Soft Set}
\end{align*}
\]

In our future study of UP-algebras, may be the following topics should be considered:

- To get more results in hesitant fuzzy soft sets over UP-algebras.
- To define hesitant anti-fuzzy soft sets over UP-algebras.
- To define operations of hesitant (anti-)fuzzy soft sets over UP-algebras.

Acknowledgements. This work was financially supported by the University of Phayao.

References


Phakawat Mosrijai (phakawat.mo@gmail.com)
Department of Mathematics, School of Science, University of Phayao, Phayao, postal code 56000, Thailand

Aiyared Iampan (aiyared.ia@up.ac.th)
Department of Mathematics, School of Science, University of Phayao, Phayao, postal code 56000, Thailand