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Some remarks on results of Pant and Pant along with generalization for hybrid mappings

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Abstract. In the present note, we pose some remarks on results of Pant and Pant [23]. After resolving some errors; we provide their results using common limit in the range property. Further, we generalize their results for hybrid pairs of mappings in fuzzy metric space. we also provide some examples for the validity of our results.

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1. Introduction

A novel approach to represent '*vagueness*' or '*uncertainty*' in a mathematical structure in everyday life, Zadeh [31] introduced the idea of fuzzy set in 1965. In fact the idea of a fuzzy set is born as natural expansion of the concept of set, since then, this idea has been used in mathematics and its applications like analysis, algebra, topology, logic etc. In particular many authors have expansively developed the theory of fuzzy metric spaces in special directions. In 1975, Kramosil and Michalek [16] introduced the notion of fuzzy metric space (in short FMS), which opened the way for further development of analysis. Further, George and Veeramani [10] modified the concept of FMS introduced by Kramosil and Michalek [16] with a view to obtain a Hausdorff topology on it. Consequently, different mappings have been used by various authors to obtain fixed point theorems in FMS.

Some remarkable conditions on the pairs of mappings like weakly compatible mappings [14], E. A. property [1], faintly compatible mappings [5] and common limit range in the range property [27], have been introduced and used to prove common fixed point theorems in various spaces by researchers.

Some significant and interesting results using these conditions in fuzzy spaces are [6, 7, 8, 25, 29]. In 2002, Aamri and Moutawakil [1] defined the idea of property (E.A) for a pair of self mappings which contains the class of non-compatible mappings. Pant and Pant [23] defined the notion of R-weakly commuting maps of type (A_g) and the property (E.A) in the FMS and then obtained common fixed point theorems in FMS for a pair of selfmaps by using the concept of pointwise R-weak commutativity but without assuming the completeness of the space or continuity of the mappings involved.

In an attractive paper of Ali and Imdad [4], it was indicated that property (E.A) allows replacing the completeness requirement of the space with a more natural condition of closedness of the range. Afterwards, there are a number of consequences proved for contraction mappings satisfying property (E.A) in fuzzy metric spaces. Some recent results using property (E.A) are [12, 17, 19]. The notion of common limit range property is given by Sintunavarat and Kumam [27] in 2011, which relaxes the condition of closedness of the underlying subspace. Some important results using common limit in the range property are [3, 6, 7, 9, 30].

The study of fixed points for multivalued contraction mappings with the Hausdorff metric was initiated by Nadler [22] and Markin [20]. Further, Singh et al. [26] and Khan et al. [15] studied the contraction types involving single-valued and multivalued mappings. Some recent results related to multivalued mappings can be found in [3, 11, 13, 28].

Here, we pose some remarks on results of Pant and Pant [23]. We also resolve some errors and then prove their results using common limit in the range property. Further, we also generalize their results for hybrid pairs of mappings in FMS along with some examples.

2. Preliminaries and notations

In this section, we have recalled some definitions and useful results which have already been in the literature.

Definition 2.1 ([23, 24])**.** A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm, if ([0, 1], $*$) is an abelian topological monoid with the unit 1 such that $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$, $\forall a, b, c, d \in [0,1]$.

Definition 2.2 ([10])**.** A 3-tuple $(X, M, *)$ is called a fuzzy metric space (FMS), if X is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set in $X^2 \times (0,\infty)$ satisfying the following conditions: $\forall x, y \in X$ and $t > 0$,

(i) $M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$, $t > 0$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t)^* M(y, z, s) \leq M(x, z, t + s)$,
(v) $M(x, y, \cdot) : (0,\infty) \to [0,1]$ is continuous.

Example 2.3 ([10])**.** Let (X, d) be a metric space and define $a*b = min\{a, b\}, \forall a, b \in [0,1]$,

$$M(x, y, t) = \frac{t}{t+d(x,y)}, \forall t > 0.$$

Then $(X, M, *)$ is a FMS. We call this fuzzy metric M induced by the metric d the standard fuzzy metric.

Definition 2.4 ([23]). Two mappings A and S of a FMS $(X, M, *)$ into itself are R-weakly commuting provided there exists some real number R such that

$$M(ASx, SAx, t) \geq M(Ax, Sx, t/R),$$

for each $x \in X$ and $t > 0$.

Definition 2.5 ([18]). The self mappings A and S of a FMS $(X, M, *)$ are called pointwise R-weakly commuting, if there exists $R > 0$, such that

$$M(ASx, SAx, t) \geq M(Ax, Sx, t/R),$$

for all x in X and $t > 0$.

Definition 2.6 ([14]). f and g are said to be weakly compatible, if they commute at their coincidence points, i.e, $fx = gx$, for some $x \in X$ implies that $fgx = gfx$.

Definition 2.7 ([23]). Let f and g be two selfmappings of a FMS $(X, M, *)$. We say that A and S satisfy the property (E.A), if there exists a sequence x_n such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t, \text{ for some } t \in X.$$

Definition 2.8 ([3, 27]). A pair (f, g) of self-mappings of a FMS $(X, M, *)$ is said to satisfy the common limit in the range property with respect to mapping g (briefly, (CLR_g) property), if there exists a sequence x_n in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gz, \text{ for some } z \in X.$$

Definition 2.9 ([3]). Let $CB(X)$ be the set of all nonempty closed bounded subsets of FMS $(X, M, *)$. Then for every $A, B, C \in CB(X)$ and $t > 0$,

$$M(A, B, t) = min\{min_{a\in A} M(a, B, t), min_{b\in B} M(A, b, t)\},$$

where $M(C, y, t) = max\{M(z, y, t) : z \in C\}$. Obviously, $M(A, B, t) \leq M(a, B, t)$, whenever $a \in A$ and $M(A, B, t) = 1$ if and only if $A = B$.

Definition 2.10 ([2, 28]). A point in X is a coincidence point (fixed point) of f and T, if $fx = Tx$ ($Tx = fx = x$).

Definition 2.11 ([2, 28]). A point x in X is a coincidence point of $f : X \to X$ and $F : X \to CB(X)$, if $fx \in Fx$. We denote the set of all coincidence points of f and F by C_{fF}.

Definition 2.12. A point $x \in X$ is a coincidence point (fixed point) of a hybrid pair (f, T) of single valued mapping $f : X \to X$ and multivalued mapping $T : X \to CB(X)$, if $fx \in Tx$ ($x = fx \in Tx$).

Definition 2.13 ([28]). Let $F : X \to CB(X)$. The map $f : X \to X$ is said to be F-weakly commuting at $x \in X$, if $ffx \in Ffx$.

Definition 2.14 ([3]). Let $(X, M, *)$ be a FMS. Two mappings $f : X \to X$ and $F : X \to CL(X)$, where $CL(X)$ is the set of all nonempty closed subsets, are said to be satisfy the (CLR_g) property, if there exists a sequence x_n in X such that

$$\lim_{n\to\infty} fx_n = u \in A = \lim_{n\to\infty} Fx_n$$

with $u = fv$, for some $u, v \in X$.

Definition 2.15 ([3]). Let $(X, M, *)$ be a FMS, $f, g : Y \subseteq X \to X$ and $F, G : Y \subseteq X \to CL(X)$. Then the hybrid pairs (f, F) and (g, G) are said to have the property $(CLR_{(f,g)})$, if there exist two sequences x_n and y_n in X such that $\lim_{n\to\infty} Gy_n \in CL(X)$, and

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = u \in A = \lim_{n\to\infty} Fx_n$$

with $u = fv = gw$, for some $u, v, w \in X$.

Lemma 2.16 ([21]). *Let $(X, M, *)$ be a FMS such that for all $x, y \in X$, $M(x, y, t) \to 1$ as $t \to \infty$. If there exists a constant $k \in (0, 1)$, such that $\forall t > 0$,*

$$M(x, y, kt) \geq M(x, y, t),\ \forall x, y \in X,\ \text{then } x = y.$$

Pant and Pant [23], proved the following results:

Theorem 2.17. *Let f and g be pointwise R-weakly commuting selfmappings of a FMS $(X, M, *)$ satisfying the property (E.A) and*
(i) $fX \subset gX$,
(ii) $M(fx, fy, kt) \geq M(gx, gy, t)$, $k \geq 0$, and
(iii) $M(fx, f^2x, t) > max \left\{ \begin{array}{c} M(gx, gfx, t), M(fx, gx, t), M(f^2x, gfx, t), \\ M(fx, gfx, t), M(gx, f^2x, t) \end{array} \right\},$
whenever $fx \neq f^2x$.
If the range of f or g is a complete subspace of X, then f and g have a common fixed point.

Theorem 2.18. *Let f and g be pointwise R-weakly commuting selfmappings of a FMS $(X, M, *)$ satisfying the property (E.A) and*
(i) $fX \subset gX$,
(ii) $M(fx, fy, t) \geq M(gx, gy, t)$, and
(iii) $M(fx, f^2x, t) > max \left\{ \begin{array}{c} M(gx, gfx, t), M(fx, gx, t), M(f^2x, gfx, t), \\ M(fx, gfx, t), M(gx, f^2x, t) \end{array} \right\},$
whenever $fx \neq f^2x$.
If the range of f or g is a complete subspace of X, then f and g have a common fixed point.

Theorem 2.19. *Let f and g be noncopmatible pointwise R-weakly commuting selfmappings of type A_g of a FMS $(X, M, *)$ satisfying*
(i) $fX \subset gX$,
(ii) $M(fx, fy, kt) \geq M(gx, gy, t)$, $k \geq 0$, and
(iii) $M(fx, f^2x, t) > max \left\{ \begin{array}{c} M(gx, gfx, t), M(fx, gx, t), M(f^2x, gfx, t), \\ M(fx, gfx, t), M(gx, f^2x, t) \end{array} \right\},$
whenever $fx \neq f^2x$.
If the range of f or g is a complete subspace of X, then f and g have a common fixed point and the fixed point is the point of discontinuity.

Now we give our main results.

3. Main section

The arrangement of the main section is as follows: Section 4, consists some remarks on results of Pant and Pant [23]. In section 5, we prove their results using common limit in the range property in FMS. In section 6, we generalize their results for hybrid pairs of mappings in FMS. Throughout this paper, straightforward proofs are dropped.

4. Some remarks on results of Pant and Pant [23]

Remark 4.1. In the proof of Theorem 2.17 in [23] (page 650-651) contractive condition (iii) implies,

$$\begin{aligned} M(fu, ffu, t) &> max\left\{ \begin{array}{c} M(gu, gfu, t), M(fu, gu, t), M(ffu, gfu, t), \\ M(fu, gfu, t), M(gu, ffu, t) \end{array} \right\} \\ &= M(fu, ffu, t) \end{aligned}$$

which is not correct. Because using $fu = gu$, we have,

$$M(fu, ffu, t) > max\{M(fu, ffu, t), 1, 1, M(fu, ffu, t), M(fu, ffu, t)\} = 1.$$

In the proof of Theorem 2.17 in [23],

$$M(fu, ffu, t) > M(fu, ffu, t)$$

can be obtained by taking minimum contractive condition instead of maximal.

Remark 4.2. There is neither any need nor any use of $k \geq 0$ in the proof of Theorem 2.17 in [23].

Remark 4.3. Theorem 2.17 in [23] can also be found without using contractive condition (iii), with some correction in condition (ii) as follows:

Theorem 4.4. *Let f and g be pointwise R-weakly commuting selfmappings of a FMS $(X, M, *)$ satisfying the property (E.A) and*

(4.4.1) $fX \subset gX$,

(4.4.2) *there exists a constant $k \in (0, 1)$ such that*

$$M(fx, fy, kt) \geq M(gx, gy, t),\ \forall x, y \in X.$$

If the range of f or g is a complete subspace of X, then f and g have a common fixed point.

Proof. Property E.A. of the pair (f, g) implies that there exists a sequence x_n in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = p, \text{ for some } p \in X.,$$

Since $p \in f(X)$, $f(X) \subset g(X)$, there exists some point $u \in X$ such that $p = gu$.
To show $fu = gu$. Suppose that $fu \neq gu$. Then the condition (4.4.2) with $x = x_n$ and $y = u$. implies that

$$M(gu, fu, kt) \geq 1, \text{ as } n \to +\infty.$$

Then, $fu = gu$.
Since the mappings f and g are pointwise R-weak commutating, there exists $R > 0$ such that

$$M(fgx, gfx, t) \geq M(fx, gx, t/R) = 1,$$
$$\text{i.e., } fgu = gfu \text{ and thus } ffu = fgu = gfu = ggu.$$

To show that, $ffu = fu$. Suppose that $ffu \neq fu$. Then the condition (4.4.2) implies that

$$M(ffu, fu, kt) \geq M(ffu, fu, t).$$

By Lemma 2.16, we have $ffu = fu$. Thus, $fu = ffu = gfu$, i.e., fu is a common fixed point of f and g in X. The case when $f(X)$ is a complete subspace of X is similar to the above case since $fX \subset gX$. □

5. Improved result of Pant and Pant [23] using CLR_g property

Here, we improve the result of [23] using CLR_g property in the same breath of Remarks 4.1 and 4.2 as follows:

Theorem 5.1. *Let f and g be selfmappings of a FMS $(X, M, *)$ satisfying the following conditions:*
(5.1.1) *the pair (f, g) enjoys the CLR_g property,*
(5.1.2) $M(fx, fy, t) \geq M(gx, gy, t)$ *and*
(5.1.3) $M(fx, f^2x, t) > min \left\{ \begin{array}{c} M(gx, gfx, t), M(fx, gx, t), M(f^2x, gfx, t), \\ M(fx, gfx, t), M(gx, f^2x, t) \end{array} \right\}$,
whenever $fx \neq f^2x$.
Then f and g have a point of coincidence. If the mappings f and g are weakly compatible, then f and g have a common fixed point in X.

Proof. Since the pair (f, g) enjoys the CLR_g property, there exists a sequence x_n in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z, \text{ where } z \in g(X).$$

Since $z \in g(X)$, there exist points $u \in X$ such that $gu = z$.
We show that $fu = gu$. Suppose that $fu \neq gu$. Using (5.1.2), we get,

$$M(fx_n, fu, t) \geq M(gx_n, gu, t).$$

Take the limit as $n \to +\infty$, we get,

$$M(gu, fu, t) \geq M(gu, gu, t) = 1, \text{ i.e.,}$$

$M(gu, fu, t) \geq 1$. Then, $fu = gu$, i.e., u is a coincidence point of f and g.

Suppose that the mappings f and g are weakly compatible. Then the weak compatibility of f and g implies that $fgu = gfu$. So $ffu = fgu = gfu = ggu$.

Next we assert that $ffu = fu$. Suppose that $ffu \neq fu$. Using (5.1.3), we get,

$$M(fu, ffu, t) > min \left\{ \begin{array}{c} M(gu, gfu, t), M(fu, gu, t), M(ffu, gfu, t), \\ M(fu, gfu, t), M(gu, ffu, t) \end{array} \right\};$$

Take the limit as $n \to +\infty$ we get,

$$\begin{aligned} M(fu, ffu, t) &> min\{M(fu, ffu, t), 1, 1, M(fu, ffu, t), M(fu, ffu, t)\}, \\ M(fu, ffu, t) &> M(fu, ffu, t), \end{aligned}$$

a contradiction. Hence, $fu = ffu$. Therefore, $ffu = gfu = fu$, i.e., fu is a common fixed point of f and g in X. □

Example 5.2. Let $X = [1, 20]$. Define $a * b = min\{a, b\}$, for all $a, b \in [0, 1]$, and
$$M(x, y, t) = \frac{t}{t+|x-y|},$$
for all $x, y \in X$, $t > 0$. Then $(X, M, *)$ is FMS. Define $f, g\colon X \to X$ as follows:
$$fx = \begin{cases} 1, & \text{if } x = 1 \text{ or } x > 5; \\ 7, & \text{if } 1 < x \leq 5, \end{cases} \quad \text{and} \quad gx = \begin{cases} 1, & \text{if } x = 1; \\ 5, & \text{if } 1 < x \leq 5; \\ \frac{x+1}{6}, & \text{if } x > 5. \end{cases}$$
Clearly, the pair (f, g) enjoys the CLR_g property for the sequence $x_n = 5 + \frac{1}{n} \in X$, since
$$\lim_{n\to\infty} fx_n = 1 (= g(1)) = \lim_{n\to\infty} gx_n.$$
Also, the pair (f, g) is weakly compatible, since, for $x = 1 \in X$, $f(1) = g(1)$ implies that $fg(1) = gf(1)$. One can easily verify that f and g satisfy condition (5.1.2) and (5.1.3). Thus f and g satisfy all the conditions of Theorem 5.1 and have a common fixed point $1 \in X$.

Example 5.3. Let $X = [0, 2]$. Define $a * b = min\{a, b\}$, for all $a, b \in [0, 1]$, and
$$M(x, y, t) = \frac{t}{t+|x-y|},$$
for all $x, y \in X$, $t > 0$. Then $(X, M, *)$ is FMS. Define $f, g\colon X \to X$ as follows:
$$fx = \begin{cases} \frac{1}{2}, & \text{if } 0 < x < 1; \\ x, & \text{if } x \geq 1, \end{cases} \quad \text{and} \quad gx = \begin{cases} \frac{1}{3}, & \text{if } 0 < x < 1; \\ 2 - x, & \text{if } x \geq 1. \end{cases}$$
Clearly, the pair (f, g) enjoys the CLR_g property for the sequence $x_n = 1 + \frac{1}{n} \in X$, since
$$\lim_{n\to\infty} f(1 + \tfrac{1}{n}) = 1, \text{ and } \lim_{n\to\infty} g(1 + \tfrac{1}{n}) = 1 = g(1).$$
Also, the pair (f, g) is weakly compatible, since, for $x = 1 \in X$, $f(1) = g(1)$ implies that $fg(1) = gf(1)$. One can easily verify that f and g satisfy condition (5.1.2) and (5.1.3). Thus f and g satisfy all the conditions of Theorem 5.1 and have a common fixed point $1 \in X$.

Note 5.4. In Example 5.2 and 5.3, f and g are discontinuous at their common fixed point, i.e., fixed point of f and g is the point of discontinuity.

Remark 5.5. Theorem 5.1 is improved result in the following sense:
(1) containment of ranges and completeness of the subspace has been completely removed,
(2) pointwise R-weakly commuting mapping is replaced by weakly compatible mapping,
(3) property E.A. is replaced by CLR_g property.

6. Generalization for hybrid pair of mappings

Here, we generalize the result of [23] for hybrid pairs of single and multivalued maps under contrative condition (in the light of Remark 4.3) as follows:

Theorem 6.1. *Let f be a self mapping from a FMS $(X, M, *)$ and $F : X \to CB(X)$ satisfy the following conditions:*
(6.1.1) *hybrid pair (f, F) enjoys the CLR_f property,*
(6.1.2) *there exists a constant $k \in (0, 1)$ such that*

$$M(Fx, Fy, kt) \geq M(fx, fy, t), \textit{ for all } x, y \in X,\ t > 0.$$

Then ($\alpha - 1$) *hybrid pair* (f, F) *have a coincidence point* $v \in X$,

($\alpha - 2$) *hybrid pair* (f, F) *have a common fixed point in* X*, provided that* f *is F-weakly commuting at* $v \in X$.

Proof. Since the hybrid pair (f, F) enjoys the property CLR_f, there exists a sequence x_n in X and $A \in CB(X)$ such that

$$\lim_{n\to\infty} fx_n = u \in A = \lim_{n\to\infty} Fx_n$$

with $u = fv$, for some $u, v \in X$. We show that $A = Fv$. Let $A \neq Fv$. Using (6.1.2), we get,

$$M(Fx_n, Fv, kt) \geq M(fx_n, fv, t).$$

Take the limit as $n \to +\infty$, we get,

$$M(A, Fv, kt) \geq M(u, u, t) = 1.$$

Then, $A = Fv$. Since $fv \in A = Fv$, $fv \in Fv$. Thus, v is a coincidence point of the hybrid pair (f, F). This proves ($\alpha - 1$).

Using F-weakly commutativity of f (from condition ($\alpha - 2$)), we get $ffv \in Ffv$. To show, $fu = u$ (i.e. $ffv = fv$). Let $fu \neq u$. Using (6.1.2), we get,

$$M(Ffv, Fv, kt) \geq M(ffv, fv, t).$$

Since, $fv \in Fv$ and $ffv \in Ffv$, we have,

$$M(ffv, fv, kt) \geq M(ffv, fv, t),$$
$$M(fu, u, kt) \geq M(fu, u, t).$$

By Lemma 2.16, we have, $fu = u$. So, $u = fu \in Fu$, i.e., u is a common fixed point of the hybrid pair (f, F) in X. This proves ($\alpha - 2$). □

Example 6.2. Let $X = [0, 1]$. Define $a * b = min\{a, b\}$, for all $a, b \in [0, 1]$, and

$$M(x, y, t) = \frac{t}{t+|x-y|},$$

for all $x, y \in X$, $t > 0$. Then $(X, M, *)$ is FMS. Define $f, g \colon X \to X$ as follows:

$$fx = \begin{cases} x, & \text{if } x \in [0, \frac{1}{2}]; \\ 0, & otherwise, \end{cases} \quad \text{and} \quad Fx = \begin{cases} [\frac{x}{2}, \frac{1}{2}], & \text{if } x \in [0, \frac{1}{2}]; \\ [0, \frac{1}{2}], & otherwise. \end{cases}$$

Clearly, the hybrid pair (f, F) enjoys the CLR_f property for the sequence $x_n = \frac{1}{2n} \in X$, since

$$\lim_{n\to\infty} fx_n = 0 \in A = \lim_{n\to\infty} Fx_n = [0, \tfrac{1}{2}],$$

Also, f is F-weakly commuting, since $ff(0) \in Ff(0)$ at $0 \in X$. One can easily verify that f and g satisfy condition (6.1.2). Then the hybrid pair (f, F) satisfy all the conditions of the Theorem 6.1 and have a common fixed point $0 \in X$.

Theorem 6.3. *Let* f *and* g *be a two self mapping from a FMS* $(X, M, *)$ *and* $F, G : X \to CB(X)$ *satisfy the following conditions:*

(6.3.1) *hybrid pairs* (f, F) *and* (g, G) *enjoy the* $CLR_{(f,g)}$ *property,*

(6.3.2) *there exists a constant* $k \in (0, 1)$ *such that*

$$M(Fx, Gy, kt) \geq M(fx, gy, t), \textit{ for all } x, y \in X,\ t > 0.$$

Then $(\beta-1)$ *hybrid pair* (f,F) *have a coincidence point* $v\in X$,

$(\beta-2)$ *hybrid pair* (g,G) *have a coincidence point* $w\in X$,

$(\beta-3)$ *hybrid pair* (f,F) *have a common fixed point in* X, *provided that* f *is F-weakly commuting at* $v\in X$,

$(\beta-4)$ *hybrid pair* (g,G) *have a common fixed point in* X, *provided that* g *is G-weakly commuting at* $w\in X$,

$(\beta-5)$ f,g,F *and* G *have a common fixed point in* X, *provided that both* $(\beta-3)$ *and* $(\beta-4)$ *are true.*

Proof. Since the hybrid pair (f,F) and (g,G) enjoys the property $CLR_{(f,g)}$, there exist sequences x_n and y_n in X and $A,B\in CB(X)$ such that $\lim_{n\to\infty} Gy_n=B$ and

$$\lim_{n\to\infty} fx_n=\lim_{n\to\infty} gy_n=u\in A=\lim_{n\to\infty} Fx_n$$

with $u=fv=gw$, for some $u,v,w\in X$.

We show that $A=B$. Let $A\neq B$. Using (6.3.2), we get,

$$M(Fx_n,Gy_n,kt)\geq M(fx_n,gy_n,t).$$

Take the limit as $n\to+\infty$, we get, $M(A,B,kt)\geq M(u,u,t)=1$. Then, $A=B$.

To show that $A=Fv$. Let $A\neq Fv$. Using (6.3.2), we get,

$$M(Fv,Gy_n,kt)\geq M(fv,gy_n,t).$$

Take the limit as $n\to+\infty$, we get, $M(Fv,B,kt)\geq M(u,u,t)=1$.
Thus, $Fv=B=A$. Since $fv\in A=Fv$, $fv\in Fv$, i.e., v is a coincidence point of the hybrid pair (f,F). This proves $(\beta-1)$.

To show that $B=Gw$. Let $B\neq Gw$. Using (6.3.2), we get,

$$M(Fx_n,Gw,kt)\geq M(fx_n,gw,t).$$

Take the limit as $n\to+\infty$, we get, $M(A,Gw,kt)\geq M(u,u,t)=1$.
So, $A=B=Gw$. Since $gw\in A=Gw$, $gw\in Gw$, i.e., w is a coincidence point of the hybrid pair (g,G). This proves $(\beta-2)$.

Using F-weakly commutativity of f (from condition $(\beta-3)$), we get $ffv\in Ffv$. Using G-weakly commutativity of g (from condition $(\beta-4)$), we get $ggw\in Ggw$.

To show, $fu=u$ (i.e. $ffv=fv$). Let $fu\neq u$. Using (6.3.2), we get,

$$M(Ffv,Gw,kt)\geq M(ffv,gw,t),$$

Since, $u=fv=gw$, $gw\in Gw$ and $ffv\in Ffv$. Then we have,

$$M(ffv,gw,kt)\geq M(ffv,fv,t),$$
$$M(fu,u,kt)\geq M(fu,u,t).$$

By Lemma 2.16, we have, $fu=u$. Thus, $u=fu\in Fu$, i.e., u is a common fixed point of the hybrid pair (f,F) in X. This proves $(\beta-3)$. Similarly $(\beta-4)$ can be proved. Then $(\beta-5)$ follows immediately. □

Example 6.4. Let $X=[0,1]$. Define $a*b=min\{a,b\}$, for all $a,b\in[0,1]$, and

$$M(x,y,t)=\frac{t}{t+|x-y|},$$

for all $x,y\in X$, $t>0$. Then $(X,M,*)$ is FMS. Define $f,g\colon X\to X$ as follows:

$$fx = x, \qquad gx = 0, \qquad Fx = [\tfrac{x}{3}, 1] \qquad \text{and} \qquad Gx = [\tfrac{x^2}{2}, 1].$$

Clearly, the hybrid pairs (f, F) and (g, G) enjoy the $CLR_{(f,g)}$ property for the sequences $x_n = \frac{1}{n}$ and $y_n = \frac{1}{4n} \in X$, since

$$\lim_{n\to\infty} Gy_n = \lim_{n\to\infty} G(\tfrac{1}{4n}) = [0, 1]$$

and

$$\lim_{n\to\infty} f(\tfrac{1}{n}) = 0 = f(0) \in A = \lim_{n\to\infty} F(\tfrac{1}{n}) = [0, 1],$$
$$\lim_{n\to\infty} g(\tfrac{1}{4n}) = 0 = g(0) \in A = \lim_{n\to\infty} F(\tfrac{1}{n}) = [0, 1].$$

Also, f is F-weakly commuting, since $ff(0) \in Ff(0)$ at $0 \in X$, and g is G-weakly commuting, since $gg(0) \in Gg(0)$ at $0 \in X$. One can easily verify that the hybrid pairs (f, F) and (g, G) satisfy the condition (6.3.2). Thus the hybrid pairs (f, F) and (g, G) satisfy all the conditions of Theorem 6.3 and have a common fixed point $0 \in X$.

7. Conclusion

In this paper, we pose some remarks on results of Pant and Pnat [23] in order to refine and improve their results. Further, we prove common fixed point theorem using common limit in the range property. We also generalized their result by proving fixed point theorem for hybrid pair of single and multivaued mappings under hybrid contractive condition.

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