Annals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (Month 201y), pp. 1–xx
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
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The characterized fuzzy spaces represented by characterized fuzzy $R_{2\frac{1}{2}}$ and $T_{3\frac{1}{2}}$ spaces

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Received 23 August 2016; Accepted 30 November 2016

Abstract. In this research work, four new notions are proposed and investigated. The notions are named characterized global fuzzy neighborhood space, characterized global fuzzy neighborhood pre space, characterized fuzzy uniform space and characterized perfect fuzzy topoeneous structure. The properties of such characterized fuzzy spaces are deeply studied. Some sorts of relationship were introduced among such characterized fuzzy spaces and other published characterized fuzzy spaces presented by the authors. Each global fuzzy neighborhood structure is identified with characterized global fuzzy neighborhood space, however, each global fuzzy neighborhood pre structure is identified with characterized global fuzzy neighborhood pre space. The mappings between characterized fuzzy pre spaces are $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous if the related mappings between the associated global fuzzy neighborhood pre spaces are (h, k)-continuous. The vise versa is true when h and k are coincide up to identifications with $\varphi_{1,2}.\mathrm{int}_{\tau_h}$ and $\psi_{1,2}.\mathrm{int}_{\tau_k}$. For each fuzzy uniform structure on a set X, there is induced stratified fuzzy proximity on L^X. Both the fuzzy uniform structure and this induced stratified fuzzy proximity are associated with the same stratified characterized fuzzy uniform space. The associated characterized fuzzy uniform space is characterized fuzzy $R_{2\frac{1}{2}}$-space and it is characterized fuzzy $T_{3\frac{1}{2}}$-space when the related fuzzy uniform space is separated. Moreover, the relation between characterized fuzzy compact spaces which introduced in [7] and some of our characterized fuzzy T_s-spaces for $s \in \{2, 3\frac{1}{2}, 4\}$ are introduced. Finally, the characterized fuzzy compact spaces and the characterized fuzzy $T_{3\frac{1}{2}}$-spaces are equivalent.

2010 AMS Classification: Primary 54E05, 54E15 ; Secondary 54A40, 54A15

Keywords: Fuzzy filter, Fuzzy topological space, Operations, Characterized fuzzy proximity and fuzzy compact spaces, Characterized global fuzzy neighborhood space, Characterized fuzzy uniform space, Characterized fuzzy perfect topoeneous structure, Characterized FT_s-space and $FR_{2\frac{1}{2}}$-space for $s \in \{1, 2, 3, 3\frac{1}{2}, 4\}$.

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1. Introduction

The notion of fuzzy filter has been introduced by Eklund et al.[12]. By means of this notion the point-based approach to fuzzy topology related to usual points has been developed. The more general concept for fuzzy filter introduced by Gähler in [16] and fuzzy filters are classified by types. Because of the specific type of fuzzy filter, however, the approach of Eklund is related only to fuzzy topologies which are stratified, that is, all constant fuzzy sets are open. The more specific fuzzy filters considered in the former papers are now called homogeneous. On the ordinary topological space (X, T), the operation has been defined by Kasahara [24] as a mapping φ from T into 2^X such that $A \subseteq A^\varphi$, for all $A \in T$. Kandil et al.[23] extended Kasahars's operations by introducing a operation on the class of all fuzzy sets endowed with a fuzzy topology τ as a mapping $\varphi : L^X \to L^X$ such that $\text{int}\mu \leq \mu^\varphi$ for all $\mu \in L^X$, where μ^φ denotes the value of φ at μ. The notions of the fuzzy filters and the operations on the class of all fuzzy sets on X endowed with a fuzzy topology τ are applied in [1, 2, 3, 4, 5, 6, 7, 8, 9] to introduce a more general theory including all the weaker and stronger forms of the fuzzy topology. By means of these notions, the notion of $\varphi_{1,2}$-interior of the fuzzy set, $\varphi_{1,2}$-fuzzy convergence and $\varphi_{1,2}$-fuzzy neighborhood filters are defined and applied to introduced many general classes of separation axioms [2, 3, 4, 8, 9]. The notion of $\varphi_{1,2}$-interior operator for fuzzy sets is defined as a mapping $\varphi_{1,2}.\text{int} : L^X \to L^X$ which fulfill (I1) to (I5) in [1]. There is a one-to-one correspondence between the class of all $\varphi_{1,2}$-open fuzzy subsets of X and these operators, that is, the class $\varphi_{1,2}OF(X)$ of all $\varphi_{1,2}$-open fuzzy subsets of X is characterized by these operators. Then the triple $(X, \varphi_{1,2}.\text{int})$ as will as the triple $(X, \varphi_{1,2}OF(X))$ will be called *characterized fuzzy space* of the $\varphi_{1,2}$-open fuzzy subsets. The characterized fuzzy spaces are characterized by many of characterizing notions in [1, 6], for example by the $\varphi_{1,2}$-fuzzy neighborhood filters, the $\varphi_{1,2}$-fuzzy interior of the fuzzy filters and by the set of $\varphi_{1,2}$-inner points of the fuzzy filters. Moreover, the notions of closeness and compactness in characterized fuzzy spaces are introduced and studied in [7]. The notions of characterized fuzzy T_s-spaces, fuzzy $\varphi_{1,2}$-T_s spaces, characterized fuzzy R_k-spaces and fuzzy $\varphi_{1,2}$-R_k spaces are introduced and studied in [2, 3, 4, 8] for all $s \in \{0, 1, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4\}$ and $k \in \{0, 1, 2, 2\frac{1}{2}, 3\}$.

This paper is devoted to introduce and study four new notions of the characterized fuzzy spaces named *characterized global fuzzy neighborhood space, characterized global fuzzy neighborhood pre space, characterized fuzzy uniform space* and *characterized fuzzy perfect topoeneous structure*. Many relations between these characterized fuzzy spaces and our characterized fuzzy T_s-spaces and characterized fuzzy $R_{2\frac{1}{2}}$-spaces are investigated for $s \in \{1, 2, 3, 3\frac{1}{2}, 4\}$. In section 2, some definitions and notions related to fuzzy sets, fuzzy uniform structures and fuzzy uniform continuity, fuzzy topologies, fuzzy filters, fuzzy filter bases, fuzzy filter functor and fuzzy filter monads, fuzzy proximity space, fuzzy topogeneous orders and fuzzy topogeneous structure, operations on fuzzy sets, $\varphi_{1,2}$-fuzzy neighborhood filters, characterized fuzzy space, characterized fuzzy proximity space, $\varphi_{1,2}$-fuzzy convergence and $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous, the fuzzy function family, characterized fuzzy T_s-spaces and characterized fuzzy $R_{2\frac{1}{2}}$-spaces are given for $s \in \{0, 1, 2, 4\}$. Section 3, is devoted to introduce and study the notions of characterized global fuzzy neighborhood space and the characterized global fuzzy neighborhood pre space by means of the global fuzzy neighborhood structure and the homogenous global fuzzy neighborhood structure, respectively. Each global fuzzy neighborhood structure is identified with characterized global fuzzy neighborhood space, however each global fuzzy neighborhood pre-structure is identified with characterized global fuzzy neighborhood pre space. In case of the homogenous global fuzzy neighborhood structure and of the homogenous global fuzzy neighborhood pre structure the stratified characterized global fuzzy neighborhood space and the stratified characterized global fuzzy neighborhood pre space are introduced. We prove that the mappings between the characterized fuzzy pre spaces are $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous if the related mappings between the global fuzzy neighborhood pre spaces are (h, k)-continuous. The vise versa is true when h and k are coincide up to identifications with $\varphi_{1,2}.\mathrm{int}_{\tau_h}$ and $\psi_{1,2}.\mathrm{int}_{\tau_k}$, respectively. In section 4, the notions of characterized fuzzy uniform spaces and characterized fuzzy perfect topoeneous structures are investigated and studies. The fuzzy uniform space is separated if and only if the associated characterized fuzzy uniform space is characterized FT_1-space in sense of Abd-Allah [2]. We show that the mappings between the associated characterized fuzzy uniform spaces are $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous if the related mappings between the fuzzy uniform spaces are fuzzy uniform continuous. For each fuzzy uniform structure on a set X, there is induced stratified fuzzy proximity on L^X and both the fuzzy uniform structure and this induced stratified fuzzy proximity are associated with the same stratified characterized fuzzy uniform space. The associated stratified characterized fuzzy uniform space with the fuzzy uniform structure is characterized by fuzzy $R_{2\frac{1}{2}}$-space and therefore it is characterized by fuzzy $T_{3\frac{1}{2}}$-space when the fuzzy uniform space is separated. The mean important result in this section is that the associated stratified characterized fuzzy uniform spaces with the fuzzy uniform structures are compatible with the stratified characterized fuzzy $R_{2\frac{1}{2}}$-spaces. Finally in section 5, the relation between characterized fuzzy compact spaces which is introduced in [7] and some of our characterized fuzzy T_s-spaces for $s \in \{1, 2, 3, 3\frac{1}{2}, 4\}$ are introduced by help of the characterized fuzzy unit interval spaces and the characterized fuzzy T_2-spaces and the fuzzy T_4-cubes which are investigated in the present section. Especially, we show that the characterized fuzzy compact spaces and the characterized fuzzy $T_{3\frac{1}{2}}$-spaces are equivalent.

2. Preliminaries

We begin by recalling some facts on the fuzzy filters. Let L be a completely distributive complete lattice with different least and last elements 0 and 1, respectively. Let $L_0 = L \setminus \{0\}$. Sometimes we will assume more especially that L is complete chain, that is, L is a complete lattice whose partial ordering is a linear one. For a set X, let L^X be the set of all fuzzy subsets of X, that is, of all mappings $f : X \to L$. Assume that an order-reversing involution $\alpha \mapsto \alpha'$ of L is fixed. For each fuzzy set $\mu \in L^X$, let μ' denote the complement of μ and it is defined by: $\mu'(x) = \mu(x)'$, for all $x \in X$. Denote by $\bar{\alpha}$ to the constant fuzzy subset of X with value $\alpha \in L$. For all $x \in X$ and for all $\alpha \in L_0$, the fuzzy subset x_α of X whose value α at x and 0 otherwise is called a t fuzzy point in X. The set of all fuzzy point in X will be denoted by $S(X)$.

The fuzzy filter on the set X ([16]) is the mapping $\mathcal{M} : L^X \to L$ such that the following conditions are fulfilled:

(F1) $\mathcal{M}(\bar{\alpha}) \leq \alpha$, for all $\alpha \in L$ and $\mathcal{M}(\bar{1}) = 1$,
(F2) $\mathcal{M}(\mu \wedge \rho) = \mathcal{M}(\mu) \wedge \mathcal{M}(\rho)$, for all $\mu, \rho \in L^X$.

The fuzzy filter $\mathcal{M}$ is called homogeneous ([12]), if $\mathcal{M}(\overline{\alpha}) = \alpha$, for all $\alpha \in L$. For each $x \in X$, the mapping $\dot{x} : L^X \to L$ defined by $\dot{x}(\mu) = \mu(x)$ for all $\mu \in L^X$ is an example of a homogeneous fuzzy filter on X. For each $\mu \in L^X$, the mapping $\dot{\mu} : L^X \to L$ defined by $\dot{\mu}(\eta) = \bigwedge\limits_{0<\eta(x)} \eta(x)$, for all $\eta \in L^X$ is also homogeneous fuzzy filter on X, called homogenous fuzzy filter at the fuzzy set $\mu \in L^X$. Let $F_L X$ and $\mathsf{F}_L X$ denotes to the sets of all fuzzy filters and all of homogeneous fuzzy filters on X, respectively. If $\mathcal{M}$ and $\mathcal{N}$ are fuzzy filters on a set X, then $\mathcal{M}$ is said to be finer than $\mathcal{N}$ and it denoted by $\mathcal{M} \leq \mathcal{N}$, provided $\mathcal{M}(\mu) \geq \mathcal{N}(\mu)$ holds, for all $\mu \in L^X$. Noting that if L is a complete chain then $\mathcal{M}$ is not finer than $\mathcal{N}$ and it denoted by $\mathcal{M} \not\leq \mathcal{N}$, provided there exists $\mu \in L^X$ such that $\mathcal{M}(\mu) < \mathcal{N}(\mu)$ holds. As shown in [10], $\mu \leq \rho$ if and only if $\dot{\mu} \leq \dot{\rho}$ for all $\mu, \rho \in L^X$. For each non-empty set $\mathcal{A}$ of the fuzzy filters on X, the supremum $\bigvee\limits_{\mathcal{M} \in \mathcal{A}} \mathcal{M}$ exists [16] and it given by $(\bigvee\limits_{\mathcal{M} \in \mathcal{A}} \mathcal{M})\mu = \bigwedge\limits_{\mathcal{M} \in \mathcal{A}} \mathcal{M}(\mu)$, for all $\mu \in L^X$. Where as, the infimaum $\bigwedge\limits_{\mathcal{M} \in \mathcal{A}} \mathcal{M}$ of $\mathcal{A}$ does not exists in general as a fuzzy filter. If the infimum $\bigwedge\limits_{\mathcal{M} \in \mathcal{A}} \mathcal{M}$ exists, then we have $(\bigwedge\limits_{\mathcal{M} \in \mathcal{A}} \mathcal{M})(\mu) = \bigvee\limits_{\substack{\mu_1 \wedge \cdots \wedge \mu_n \leq \mu, \\ \mathcal{M}_1, \cdots, \mathcal{M}_n \in \mathcal{A}}} \big(\mathcal{M}_1(\mu_1) \wedge \cdots \wedge \mathcal{M}_n(\mu_n)\big)$, for all $\mu \in L^X$, where n is a positive integer, $\mu_1, \ldots, \mu_n$ is a collection of fuzzy subsets such that $\mu_1 \wedge \ldots \wedge \mu_n \leq \mu$ and $\mathcal{M}_1, \ldots, \mathcal{M}_n$ are fuzzy filters from $\mathcal{A}$. Let X be a set and $\mu \in L^X$, then the homogeneous fuzzy filter $\dot{\mu}$ at μ is the fuzzy filter on X given by:

$$\dot{\mu} = \bigvee_{0<\mu(x)} \dot{x}. \tag{2.1}$$

Fuzzy filter bases. A family $(\mathcal{B}_\alpha)_{\alpha \in L_0}$ of a non-empty subsets of L^X is called a valued fuzzy filter base [16], if the following conditions are fulfilled:

(V1) $\mu \in \mathcal{B}_\alpha$ implies $\alpha \leq \sup \mu$,

(V2) For all $\alpha, \beta \in L_0$ with $\alpha \wedge \beta \in L_0$ and all $\mu \in \mathcal{B}_\alpha$ and $\rho \in \mathcal{B}_\beta$ there are $\gamma \geq \alpha \wedge \beta$ and $\eta \leq \mu \wedge \rho$ such that $\eta \in \mathcal{B}_\gamma$.

As shown in [16], each valued fuzzy filter base $(\mathcal{B}_\alpha)_{\alpha \in L_0}$ defines the fuzzy filter $\mathcal{M}$ on X by $\mathcal{M}(\mu) = \bigvee\limits_{\rho \in \mathcal{B}_\alpha,\, \rho \leq \mu} \alpha$, for all $\mu \in L^X$ and each fuzzy filter $\mathcal{M}$ can be generated by a valued fuzzy filter base, e.g., by $(\alpha\text{-pr}\,\mathcal{M})_{\alpha \in L_0}$ with $\alpha\text{-pr}\,\mathcal{M} = \{\mu \in L^X \mid \alpha \leq \mathcal{M}(\mu)\}$. $(\alpha\text{-pr}\,\mathcal{M})_{\alpha \in L_0}$ is a family of fuzzy pre filters on X and is called the large valued fuzzy filter base of $\mathcal{M}$. Recall that a fuzzy pre filter on X [28] is a non-empty proper subset F of L^X such that (1) $\mu, \rho \in F$ implies $\mu \wedge \rho \in F$ and (2) from $\mu \in F$ and $\mu \leq \rho$ it follows $\rho \in F$.

Valued and superior principal fuzzy filters. Let a non-empty set X be fixed, $\mu \in L^X$ and $\alpha \in L$ such that $\alpha \leq \sup\mu$. Then the valued principal fuzzy filter ([16]) generated by μ and α, will be denoted by $[\,\mu, \alpha\,]$, is the fuzzy filter on X which has $(\mathcal{B}_\beta)_{\beta \in L_0}$ with $\mathcal{B}_\beta = \{\mu\}$, if $0 < \beta \leq \alpha$ and $\mathcal{B}_\beta = \{\,\overline{1}\,\}$, otherwise as a valued fuzzy filter base. For all $\eta \in L^X$, we have $[\,\mu, \alpha\,](\eta) = 0$, if $\mu \not\leq \eta$, $[\,\mu, \alpha\,](\eta) = \alpha$, if $\mu \leq \eta \neq \overline{1}$ and $[\,\mu, \alpha\,](\eta) = 1$, if $\eta = \overline{1}$. Moreover, for each $\beta \in L_0$ we have β-pr$\,[\,\mu, \alpha\,] = \{\,\eta \mid \mu \leq \eta\,\}$, if $\beta \leq \alpha$ and β-pr$\,[\,\mu, \alpha\,] = \{\,\overline{1}\,\}$ otherwise. The superior principal fuzzy filter ([18]) generated by μ, written $[\,\mu\,]$, is the homogeneous fuzzy filter on X which has $\mathcal{B} = \{\,\mu \wedge \bar{\alpha} \mid \alpha \in L\} \cup \{\bar{\alpha} \mid \alpha \in L\,\}$ as a superior fuzzy filter base. As shown in [18], the superior principal fuzzy filter $[\,\mu\,]$ is representable by a fuzzy pre filter if and only if $\sup \mu = 1$.

Fuzzy filter functors and fuzzy filter monads. The fuzzy filter functor $\mathcal{F}_L : \mathsf{SET} \to \mathsf{SET}$ is the covariant functor from the category SET of all sets to this category which assigns to each set X the set $\mathcal{F}_L X$ and to each mapping $f : X \to Y$ the mapping $\mathcal{F}_L f : \mathcal{F}_L X \to \mathcal{F}_L Y$. The homogeneous fuzzy filter functor $\mathsf{F}_L : \mathsf{SET} \to \mathsf{SET}$ is the sub fuzzy filter functor of $\mathcal{F}_L$ which assigns to each set X the set $\mathsf{F}_L X$ and to each mapping $f : X \to Y$ the domain-range restriction $\mathsf{F}_L f : \mathsf{F}_L X \to \mathsf{F}_L Y$ of the mapping $\mathcal{F}_L f : \mathcal{F}_L X \to \mathcal{F}_L Y$. For each set X, let $\eta_X : X \to \mathcal{F}_L X$ be the mapping defined by $\eta_X(x) = \dot{x}$, for all $x \in X$, and let $e_X : L^X \to L^{\mathcal{F}_L X}$ be the mapping for which $e_X(f)(\mathcal{M}) = \mathcal{M}(f)$ for all $f \in L^X$ and $\mathcal{M} \in \mathcal{F}_L X$. Moreover, let $\mu_X : \mathcal{F}_L(\mathcal{F}_L X) \to \mathcal{F}_L X$ be the mapping which assigns to each fuzzy filter $\mathcal{L}$ on $\mathcal{F}_L X$ the fuzzy filter $\mu_X(\mathcal{L}) = \mathcal{L} \circ e_X$ on X. $\eta = (\eta_X)_{X \in \mathrm{Ob}(\mathsf{SET})} : \mathrm{id} \to \mathcal{F}_L$ with id the identity set functor and $\mu = (\mu_X)_{X \in \mathrm{Ob}(\mathsf{SET})} : \mathcal{F}_L \circ \mathcal{F}_L \to \mathcal{F}_L$ are natural transformations. $(\mathcal{F}_L, \eta, \mu)$ is a monad in the categorical sense, called the fuzzy filter monad [16], that is, $\mu_X \circ \mathcal{F}_L(\eta_X) = \mu_X \circ \eta_{\mathcal{F}_L X} = 1_{\mathcal{F}_L X}$ and $\mu_X \circ \mathcal{F}_L(\mu_X) = \mu_X \circ \mu_{\mathcal{F}_L X}$, for each set X. Related to the sub functor F_L of $\mathcal{F}_L$, there are analogous natural transformations as η and μ, denoted η' and μ', respectively. η' consists of the range-restrictions $\eta'_X : X \to \mathsf{F}_L X$ of the mappings η_X. μ' is the family of all mappings $\mu'_X : \mathsf{F}_L \mathsf{F}_L X \to \mathsf{F}_L X$ defined by $\mu'_X(\mathcal{L}) = \mathcal{L} \circ e'_X$ for all homogeneous fuzzy filters $\mathcal{L}$ on $\mathsf{F}_L X$, where $e'_X : L^X \to L^{\mathsf{F}_L X}$ is the mapping given by $e'_X(f)(\mathcal{M}) = \mathcal{M}(f)$ for all $f \in L^X$ and $\mathcal{M} \in \mathsf{F}_L X$. As has been shown in [16], $(\mathsf{F}_L, \eta', \mu')$ is a sub monad of $(\mathcal{F}_L, \eta, \mu)$, that is, for the inclusion mappings $i_X : \mathsf{F}_L X \to \mathcal{F}_L X$ we have $\eta_X = i_X \circ \eta'_X$ and $\mu_X \circ \mathcal{F}_L i_X \circ i_{\mathsf{F}_L X} = i_X \circ \mu'_X$, for all sets X.

Relational fuzzy filters. For each non-empty set X, the fuzzy subset of $X \times X$ will be called fuzzy relation on X. The constant fuzzy relation on X with value α will be denoted by $\tilde{\alpha}$. The fuzzy filter $\mathcal{U}$ on $X \times X$ will also be called a relational fuzzy filter on X. According to Proposition 1.3 in [18], the family $(\mathcal{U}_\alpha)_{\alpha \in L_0}$ of fuzzy pre filters on $X \times X$ is the large valued fuzzy filter base of the relational fuzzy filter $\mathcal{U}$ on X, that is, it coincides with $(\alpha\text{-pr}\,\mathcal{U})_{\alpha \in L_0}$ if and only if the following conditions are fulfilled:

(u1) $u \in \mathcal{U}_\alpha$ implies $\alpha \leq \sup u$,
(u2) $0 < \beta \leq \alpha$ implies $\mathcal{U}_\alpha \subseteq \mathcal{U}_\beta$,

(u3) For each $\alpha \in L_0$ with $\bigvee\limits_{0<\beta<\alpha} \beta = \alpha$ we have $\mathcal{U}_\alpha = \bigcap\limits_{0<\beta<\alpha} \mathcal{U}_\beta$.

Examples of relational fuzzy filters on X are the $0,1$-fuzzy filters on $X \times X$, where the fuzzy filter is called a $0,1$-fuzzy filter, if it only has 0 and 1 as values. The $0,1$-fuzzy filters can be characterized as those fuzzy filters for which all α-fuzzy pre filters coincide. A broader class of relational fuzzy filters, being of special interest in the following, is that of relational sup fuzzy-filters, where the fuzzy filter $\mathcal{U}$ is said to be a sup fuzzy-filter if $(\mathcal{B}_\alpha)_{\alpha \in L_0}$ with $\mathcal{B}_\alpha = \{ u \in \alpha\text{-pr}\mathcal{U} \mid \mathcal{U}(u) = \sup u \}$ is a valued fuzzy filter base of $\mathcal{U}$. As shown in [18], all the $0,1$-fuzzy filters and all of the homogeneous fuzzy filter $\mathcal{U}$ are sup fuzzy-filters. As can easily be shown by examples that, there exist sup fuzzy-filters which are not $0,1$-fuzzy filters but not homogeneous. Special homogeneous relational fuzzy filters on a set X, which will appear in the sequel, are given by the pairs (x, y) of elements x, y of X. We mean the fuzzy filters $(x,y)^{\bullet} = \eta_{X\times X}(x,y)$, for which $(x,y)^{\bullet}(u) = u(x,y)$, for all $u \in L^{X\times X}$. As shown in [20], if $\mathcal{U}$ is a relational fuzzy filter on X such that $(x,x)^{\bullet} \leq \mathcal{U}$ holds for all $x \in X$ and $f : X \to Y$ is a mapping, then $\mathcal{F}_L(f\times f)(\mathcal{U})\,[\mathcal{F}_L f(\mathcal{M})]$ is fuzzy filter on Y, for all $\mathcal{M} \in \mathcal{F}_L X$. Moreover, $\mathcal{F}_L f\big(\mathcal{U}[\mathcal{M}]\big) \leq \mathcal{F}_L(f\times f)(\mathcal{U})\,[\mathcal{F}_L f(\mathcal{M})$ holds.

Fuzzy uniform structures and fuzzy uniform continuity. Let X be a set. By a fuzzy uniform structure $\mathcal{U}$ on X ([20]), we mean a relational fuzzy filter on X such that the following conditions are fulfilled:

(U1) $(x,x)^{\bullet} \leq \mathcal{U}$ for all $x \in X$,
(U2) $\mathcal{U} = \mathcal{U}^{-1}$,
(U3) $\mathcal{U} \circ \mathcal{U} \leq \mathcal{U}$.

The set X equipped with a fuzzy uniform structure $\mathcal{U}$ will be called a fuzzy uniform space. According $\mathcal{U}$ being a sup-fuzzy uniform structure or a $0,1$-fuzzy uniform structure or a homogeneous fuzzy uniform structure. The triple $(X, \mathcal{U})$ will be called a sup-fuzzy uniform space or a $0,1$-fuzzy uniform space or a homogeneous fuzzy uniform space, respectively.

If $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are fuzzy uniform spaces, then the mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is said to be uniformly fuzzy continuous, provided

$$\mathcal{F}_L(f\times f)\;(\mathcal{U}) \;\leq\; \mathcal{V} \text{ holds.} \tag{2.2}$$

Fuzzy topology. By a fuzzy topology on a set X ([11, 22]), we mean a subset of L^X which is closed with respect to all suprema and all finite infima and contains the constant fuzzy sets $\overline{0}$ and $\overline{1}$. The set X equipped with a fuzzy topology τ on X is called fuzzy topological space. For each fuzzy topological space (X, τ), the elements of τ are called open fuzzy subsets of this space. If τ_1 and τ_2 are fuzzy topologies on a set X, then τ_2 is said to be finer than τ_1 and τ_1 is said to be coarser than τ_2, provided $\tau_1 \subseteq \tau_2$ holds. The fuzzy topological space (X, τ) and also τ are said to be stratified provided $\overline{\alpha} \in \tau$ holds, for all $\alpha \in L$, that is, all constant fuzzy sets are open [27].

Fuzzy proximity space. A binary relation δ on L^X is called fuzzy proximity on X[25], provided it fulfill the following conditions:

(P1) $\mu\bar{\delta}\rho$ implies $\rho\bar{\delta}\mu$, for all $\mu, \rho \in L^X$, where $\bar{\delta}$ is the negation of δ,
(P2) $(\mu \vee \rho)\bar{\delta}\eta$ if and only if $\mu\bar{\delta}\eta$ and $\rho\bar{\delta}\eta$, for all $\mu, \rho, \eta \in L^X$,

(P3) $\mu = \bar{0}$ or $\rho = \bar{0}$ implies $\mu\bar{\delta}\rho$, for all $\mu, \rho \in L^X$,
(P4) $\mu\bar{\delta}\rho$ implies $\mu \leq \rho^{'}$, for all $\mu, \rho \in L^X$,
(P5)if $\mu\bar{\delta}\rho$, then there is an $\eta \in L^X$ such that $\mu\bar{\delta}\eta$ and $\eta^{'}\bar{\delta}\rho$.

The set X equipped with a fuzzy proximity δ on X is called a fuzzy proximity space and will be denoted by (X, δ). Every fuzzy proximity δ on a set X is associated a fuzzy topology on X denoted by τ_δ. The fuzzy proximity δ on the set X is said to be separated if and only if for all $x, y \in X$ such that $x \neq y$, we have $x_\alpha \bar{\delta} y_\beta$, for all $\alpha, \beta \in L_0$.

Fuzzy topogeneous orders and fuzzy structures. The binary relation $\ll$ on L^X is said to be fuzzy topogeneous order on X [26], if the following conditions are fulfilled:

(i) $\bar{\alpha} \ll \bar{\alpha}$, for all $\alpha \in \{0, 1\}$,
(ii) if $\mu \ll \eta$, then $\mu \leq \eta$, for all $\mu, \eta \in L^X$,
(iii) if $\mu_1 \leq \mu \ll \eta \leq \eta_1$, then $\mu_1 \ll \eta_1$,
(iv) if $\mu_1 \ll \eta_1$ and $\mu_2 \ll \eta_2$, then $\mu_1 \wedge \mu_2 \ll \eta_1 \wedge \eta_2$ and $\mu_1 \vee \mu_2 \ll \eta_1 \vee \eta_2$, for all $\mu_i, \eta_j \in L^X$, where $i, j \in \{1, 2\}$.

The fuzzy topogeneous order $\ll$ is said to be fuzzy topogeneous structure, if it fulfilled the condition:

(v) if $\mu \ll \eta$, then there is $\sigma \in L^X$ such that $\mu \ll \sigma$ and $\sigma \ll \eta$, for all $\mu, \eta \in L^X$.

The fuzzy topogeneous structure $\ll$ is said to be fuzzy topogenous complementarily symmetric, if it fulfilled the condition:

(vi) if $\mu \ll \eta$, then $\eta^{'} \ll \mu^{'}$, for all $\mu, \eta \in L^X$.

As shown in [19], every fuzzy topogeneous structure $\ll$ is identify with the mapping $\mathcal{N} : L^X \to P(L^X)$ such that $\eta \in \mathcal{N}(\mu)$ if and only if $\mu \ll \eta$ holds for all $\mu, \eta \in L^X$. The fuzzy topogeneous structures are classified by these mappings. As is easily seen, each fuzzy topogeneous order $\mathcal{N}$ can be associated a fuzzy pre topology $\operatorname{int}_\mathcal{N}$ on a set X by defining $\operatorname{int}_\mathcal{N} \mu = \bigvee\limits_{\mu \in \mathcal{N}(\eta)} \eta$, for all $\mu \in L^X$. In case of $\mathcal{N}$ is fuzzy topogeneous structure, $\operatorname{int}_\mathcal{N}$ is interior operator for fuzzy topology $\tau_\mathcal{N}$ on X associated to $\mathcal{N}$.

Operation on fuzzy sets. In the sequel, let a fuzzy topological space (X, τ) be fixed. By the operation ([23]) on the set X, we mean the mapping $\varphi : L^X \to L^X$ such that $\operatorname{int} \mu \leq \mu^\varphi$ holds, for all $\mu \in L^X$, where μ^φ denotes the value of φ at μ. The class of all operations on X will be denoted by $O_{(L^X,\tau)}$. By the identity operation on $O_{(L^X,\tau)}$, we mean the operation $1_{L^X} : L^X \to L^X$ such that $1_{L^X}(\mu) = \mu$ for all $\mu \in L^X$. Also by the constant operation on $O_{(L^X,\tau)}$, we mean the operation $\mathsf{c}_{L^X} : L^X \to L^X$ such that $\mathsf{c}_{L^X}(\mu) = \bar{1}$, for all $\mu \in L^X$. If $\leq$ is a partially ordered relation on $O_{(L^X,\tau)}$ defined as follows: $\varphi_1 \leq \varphi_2 \Longleftrightarrow \mu^{\varphi_1} \leq \mu^{\varphi_2}$ for all $\mu \in L^X$, then obviously, $(O_{(L^X,\tau)}, \leq)$ is a completely distributive lattice. As an application on this partially ordered relation, the operation $\varphi : L^X \to L^X$ will be called:

(i) isotone, if $\mu \leq \rho$ implies $\mu^\varphi \leq \rho^\varphi$, for all $\mu, \rho \in L^X$,
(ii) weakly finite intersection preserving (wfip, for short) with respect to $\mathcal{A} \subseteq L^X$, if $\rho \wedge \mu^\varphi \leq (\rho \wedge \mu)^\varphi$ holds, for all $\rho \in \mathcal{A}$ and $\mu \in L^X$,
(iii) idempotent, if $\mu^\varphi = (\mu^\varphi)^\varphi$, for all $\mu \in L^X$.

The operations $\varphi, \psi \in O_{(L^X,\tau)}$ are said to be dual, if $\psi\mu = co(\varphi\,(co\mu))$ or equivalently $\varphi\mu = co(\psi\,(co\mu))$ for all $\mu \in L^X$, where $co\mu$ denotes the complementarily of μ. The dual operation of φ is denoted by $\tilde{\varphi}$. In the classical case of $L = \{0,1\}$, by the operation on a set X we mean the mapping $\varphi : P(X) \to P(X)$ such that $\mathrm{int}\,\mathrm{A} \subseteq \mathrm{A}^{\varphi}$, for all $\mathrm{A} \in P(X)$ and the identity operation on the class of all ordinary operations $O_{(P(X),T)}$ on X will be denoted by $i_{P(X)}$ and defined by $i_{P(X)}(\mathrm{A}) = \mathrm{A}$, for all $\mathrm{A} \in P(X)$.

φ-open fuzzy sets. Let a fuzzy topological space (X,τ) be fixed and $\varphi \in O_{(L^X,\tau)}$. The fuzzy set $\mu : X \to L$ is called φ-open fuzzy set, if $\mu \le \mu^{\varphi}$ holds. We will denote to the class of all φ-open fuzzy sets on X by $\varphi OF(X)$. The fuzzy set μ is called φ-closed, if its complement $co\mu$ is φ-open. The operations $\varphi, \psi \in O_{(L^X,\tau)}$ are equivalent and written $\varphi \sim \psi$, if $\varphi OF(X) = \psi OF(X)$.

$\varphi_{1,2}$-interior of fuzzy sets. Let a fuzzy topological space (X,τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then the $\varphi_{1,2}$-interior of the fuzzy set $\mu : X \to L$ is the mapping $\varphi_{1,2}.\mathrm{int}\mu : X \to L$ defined by:

$$\varphi_{1,2}.\mathrm{int}\,\mu = \bigvee_{\rho \in \varphi_1 OF(X),\, \rho^{\varphi_2} \le \mu} \rho. \tag{2.3}$$

That is, $\varphi_{1,2}.\mathrm{int}\mu$ is the greatest φ_1-open fuzzy set ρ such that ρ^{φ_2} less than or equal to μ ([1]). The fuzzy set μ is said to be $\varphi_{1,2}$-open if $\mu \le \varphi_{1,2}.\mathrm{int}\,\mu$. The class of all $\varphi_{1,2}$-open fuzzy sets on X will be denoted by $\varphi_{1,2}OF(X)$. The complement $co\,\mu$ of a the $\varphi_{1,2}$-open fuzzy subset μ will be called $\varphi_{1,2}$-closed and the class of all $\varphi_{1,2}$-closed fuzzy subsets of X will be denoted by $\varphi_{1,2}CF(X)$. In the classical case of $L = \{\,0,1\,\}$, the fuzzy topological space (X,τ) is up to an identification by the ordinary topological space (X,T) and $\varphi_{1,2}.\mathrm{int}\,\mu$ is the classical one. Then, in this case the ordinary subset A of X is $\varphi_{1,2}$-open if $\mathrm{A} \subseteq \varphi_{1,2}.\mathrm{int}\,\mathrm{A}$. The complement of a $\varphi_{1,2}$-open subset A of X will be called $\varphi_{1,2}$-closed. The class of all $\varphi_{1,2}$-open and the class of all $\varphi_{1,2}$-closed subsets of X will be denoted by $\varphi_{1,2}O(X)$ and $\varphi_{1,2}C(X)$, respectively. Clearly, F is $\varphi_{1,2}$-closed if and only if $\varphi_{1,2}.\mathrm{cl}_T\,F = F$. As shown in [1], if the fuzzy topological space (X,τ) is fixed and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$, then the set of all $\varphi_{1,2}$-open fuzzy set of X is characterized by the $\varphi_{1,2}.\mathrm{int}$ of the fuzzy set as follows:

$$\varphi_{1,2}OF(X) = \{\,\mu \in L^X \mid \mu \le \varphi_{1,2}.\mathrm{int}\mu\,\} \tag{2.4}$$

and the following conditions are fulfilled:

(I1) if $\varphi_2 \ge 1_{L^X}$, then $\varphi_{1,2}.\mathrm{int}\mu \le \mu$ holds, for all $\mu \in L^X$,

(I2) if $\mu \le \rho$, then $\varphi_{1,2}.\mathrm{int}\mu \le \varphi_{1,2}.\mathrm{int}\rho$, for all $\mu, \rho \in L^X$,

(I3) $\varphi_{1,2}.\mathrm{int}\,\overline{1} = \overline{1}$,

(I4) if $\varphi_2 \ge 1_{L^X}$ is isotone and φ_1 is wfip with respect to $\varphi_1 OF(X)$, then $\varphi_{1,2}.\mathrm{int}\mu \wedge \varphi_{1,2}.\mathrm{int}\rho = \varphi_{1,2}.\mathrm{int}\,(\mu \wedge \rho)$, for all $\mu, \rho \in L^X$,

(I5) if φ_2 is isotone and idempotent, then $\varphi_{1,2}.\mathrm{int}\,(\varphi_{1,2}.\mathrm{int}\mu) = \varphi_{1,2}.\mathrm{int}\mu$, for all $\mu \in L^X$.

Independently on the fuzzy topologies, the notion of $\varphi_{1,2}$-interior operator for the fuzzy sets can be defined as a mapping $\varphi_{1,2}.\mathrm{int} : L^X \to L^X$ which fulfill (I1) to (I5). It is well-known that (2.3) and (2.4) give a one-to-one correspondence between the class of all $\varphi_{1,2}$-open fuzzy sets and these operators, that is, $\varphi_{1,2}OF(X)$ can be characterized by the $\varphi_{1,2}$-interior operators. In this case $(X, \varphi_{1,2}.\text{int})$ as will as $(X, \varphi_{1,2}OF(X))$ will be called characterized fuzzy space ([1]) of the all $\varphi_{1,2}$-open fuzzy subsets of X. If $(X, \varphi_{1,2}.\text{int})$ and $(X, \psi_{1,2}.\text{int})$ are two characterized fuzzy spaces, then $(X, \varphi_{1,2}.\text{int})$ is said to be finer than $(X, \psi_{1,2}.\text{int})$ and denoted by $\varphi_{1,2}.\text{int} \leq \psi_{1,2}.\text{int}$ provided $\varphi_{1,2}.\text{int}\mu \geq \psi_{1,2}.\text{int}\mu$ holds, for all $\mu \in L^X$. The characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$ is said to be stratified if and only if $\varphi_{1,2}.\text{int}\bar{\alpha} = \bar{\alpha}$, for all $\alpha \in L$. As shown in [1], the characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$ is stratified, if the related fuzzy topology is stratified. Moreover, the characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$ is said to have the weak infimum property ([21]), provided $\varphi_{1,2}.\text{int}(\mu \wedge \bar{\alpha}) = \varphi_{1,2}.\text{int}\mu \wedge \varphi_{1,2}.\text{int}\bar{\alpha}$, for all $\mu \in L^X$ and $\alpha \in L$. The characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$ is said to be strongly stratified [21], provided $\varphi_{1,2}.\text{int}$ is stratified and have the weak infimum property.

Fuzzy unit interval. The fuzzy unit interval will be denoted by I_L and it is defined in [24] as the fuzzy subset $I_L = \{x \in \mathbf{R}^\star_\mathbf{L} \mid x \leq 1^\sim\}$, where $I = [0,1]$ is the real unit interval and $\mathbf{R}^\star_\mathbf{L} = \{x \in \mathbf{R}_L \mid x(0) = 1 \,\text{and}\, 0^\sim \leq x\}$ is the set of all positive fuzzy real numbers. Note that, the binary relation $\leq$ is defined on $\mathbf{R}_L$ as follows:

$$x \leq y \Longleftrightarrow x_{\alpha_1} \leq y_{\alpha_1} \text{ and } x_{\alpha_2} \leq y_{\alpha_2},$$

for all $x, y \in \mathbf{R}_L$, where for all $\alpha \in L_0$,

$$x_{\alpha_1} = \inf\{z \in \mathbf{R} \mid x(z) \geq \alpha\} \text{ and } x_{\alpha_2} = \sup\{z \in \mathbf{R} \mid x(z) \geq \alpha\}.$$

Note that the family Ω that defined by:

$$\Omega = \{R_\delta | I_L \mid \delta \in I\} \cup \{R^\delta | I_L \mid \delta \in I\} \cup \{0^\sim | I_L\}$$

is a base for a fuzzy topology $\Im$ on I_L and the order pair $(I_L, \Im)$ is said to be fuzzy unit interval topological space, where R_δ and R^δ are the fuzzy subsets of $\mathbf{R}_L$ defined by: for all $x \in \mathbf{R}_L$ and $\delta \in \mathbf{R}$,

$$R_\delta(x) = \bigvee_{\alpha > \delta} x(\alpha) \text{ and } R^\delta = \Big(\bigvee_{\alpha \geq \delta} x(\alpha)\Big)'.$$

The restrictions of R_δ and R^δ on I_L are the fuzzy subsets $R_\delta | I_L$ and $R^\delta | I_L$, respectively. Recall that $R^\delta(x) \wedge R^\gamma(y) \leq R^{\delta+\gamma}(x+y)$, where $x+y$ is the fuzzy real number defined by:

$$(x+y)(\xi) = \bigvee_{\gamma, \zeta \in \mathbf{R}, \gamma+\zeta=\xi} (x(\gamma) \wedge y(\zeta)), \text{ for all } \xi \in \mathbf{R}.$$

Consider a fuzzy unit interval topological space $(I_L, \Im)$ is given and $\psi_1, \psi_2 \in O_{(I_L, \Im)}$. Then in this work, the characterized fuzzy space $(I_L, \psi_{1,2}.\text{int}_\Im)$ will be called characterized fuzzy unit interval space and we define the Cartesian product of a number of copies of the fuzzy unit interval I_L equipped with the product of the characterized fuzzy unit interval spaces generated by $\psi_{1,2}.\text{int}_\Im$ on it as a characterized fuzzy cube.

$\varphi_{1,2}$-fuzzy neighborhood filters. An important notion in the characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$ is that of a $\varphi_{1,2}$-fuzzy neighborhood filter at the point and at the ordinary subset in this space. Let (X, τ) be a fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. As follows by (I1) to (I5), for each $x \in X$, the mapping

$\mathcal{N}_{\varphi_{1,2}}(x) : L^X \to L$ which is defined by:

$$\mathcal{N}_{\varphi_{1,2}}(x)(\mu) = (\,\varphi_{1,2}.\mathrm{int}\,\mu)(x), \tag{2.5}$$

for all $\mu \in L^X$ is fuzzy filter, called $\varphi_{1,2}$-fuzzy neighborhood filter at x [1]. If $\varnothing \neq \mathrm{F} \in P(X)$, then the $\varphi_{1,2}$- fuzzy neighborhood filter at F will be denoted by $\mathcal{N}_{\varphi_{1,2}}(\mathrm{F})$ and it defined by

$$\mathcal{N}_{\varphi_{1,2}}(\mathrm{F}) = \bigvee_{x \in \mathrm{F}} \mathcal{N}_{\varphi_{1,2}}(x).$$

Since $\mathcal{N}_{\varphi_{1,2}}(x)$ is fuzzy filter, for all $x \in X$, $\mathcal{N}_{\varphi_{1,2}}(\mathrm{F})$ is also fuzzy filter on X. Moreover, because of $[\chi_{\mathrm{F}}] = \bigvee\limits_{x \in \mathrm{F}} \dot{x}$, we have $\mathcal{N}_{\varphi_{1,2}}(\mathrm{F}) \geq [\chi_{\mathrm{F}}]$ holds. Furthermore, the fuzzy filter $\dot{\mathrm{F}}$ is defined by $\dot{\mathrm{F}} = \bigvee\limits_{x \in \mathrm{F}} \dot{x}$ and we easily have that $\dot{\mathrm{F}} \leq \mathcal{N}_{\varphi_{1,2}}(\mathrm{F})$ holds, for all $\mathrm{F} \in P(X)$. Recall that the $\varphi_{1,2}$-fuzzy neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(\mu)$ at the fuzzy set μ of X is defied as follows:

$$\mathcal{N}_{\varphi_{1,2}}(\mu)(\eta) = \big(\bigvee_{0<\mu(x)} \mathcal{N}_{\varphi_{1,2}}(x)\big)(\eta), \tag{2.6}$$

for all $\eta \in L^X$. Obviously, $\dot{\mu} \leq \mathcal{N}_{\varphi_{1,2}}(\mu)$, for all $\mu \in L^X$. If the related $\varphi_{1,2}$-interior operator fulfill the axioms (I1) and (I2) only, then the mapping $\mathcal{N}_{\varphi_{1,2}}(x) : L^X \to L$, defined by (2.2) is a fuzzy stack ([21]), called $\varphi_{1,2}$-fuzzy neighborhood stack at x. Moreover, if the $\varphi_{1,2}$-interior operator fulfill the axioms (I1), (I2) and (I4) such that in (I4) instead of $\rho \in L^X$, we take $\bar{\alpha}$, then the mapping $\mathcal{N}_{\varphi_{1,2}}(x) : L^X \to L$, is a fuzzy stack with the cutting property, called $\varphi_{1,2}$-fuzzy neighborhood stack with the cutting property at x. The $\varphi_{1,2}$-fuzzy neighborhood filters fulfill the following conditions:

(N1) $\dot{x} \leq \mathcal{N}_{\varphi_{1,2}}(x)$ holds for all $x \in X$,
(N2) $\mathcal{N}_{\varphi_{1,2}}(x)(\mu) \leq \mathcal{N}_{\varphi_{1,2}}(x)(\rho)$ holds, for all $\mu, \rho \in L^X$ and $\mu \leq \rho$,
(N3) $\mathcal{N}_{\varphi_{1,2}}(x)\big(y \mapsto \mathcal{N}_{\varphi_{1,2}}(y)(\mu)\big) = \mathcal{N}_{\varphi_{1,2}}(x)(\mu)$, for all $x \in X$ and $\mu \in L^X$.

Clearly, $y \mapsto \mathcal{N}_{\varphi_{1,2}}(y)(\mu) = \varphi_{1,2}.\mathrm{int}\,\mu$. The characterized fuzzy space $(X, \varphi_{1,2}.\mathrm{int})$ of all $\varphi_{1,2}$-open fuzzy subsets of a set X is characterized as a filter fuzzy pre topology ([1]), that is, as the mapping $\mathcal{N}_{\varphi_{1,2}} : X \to \mathcal{F}_L X$ such that (N1) to (N3) are fulfilled.

$\varphi_{1,2}\psi_{1,2}$**-fuzzy continuity.** Let the fuzzy topological spaces (X, τ_1) and (Y, τ_2) be fixed, $\varphi_1, \varphi_2 \in O_{(L^X, \tau_1)}$ and $\psi_1, \psi_2 \in O_{(L^Y, \tau_2)}$. Then the mapping $f : (X, \varphi_{1,2}.\mathrm{int}) \to (Y, \psi_{1,2}.\mathrm{int})$ is said to be $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous, if

$$(\psi_{1,2}.\mathrm{int}\,\eta) \circ f \leq \varphi_{1,2}.\mathrm{int}\,(\eta \circ f), \tag{2.7}$$

for all $\eta \in L^Y$ [5]. If an order reversing involution $'$ of L is given, then we have that f is a $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous if and only if $\varphi_{1,2}.\mathrm{cl}\,(\eta \circ f) \leq (\psi_{1,2}.\mathrm{cl}\,\eta) \circ f$, for all $\eta \in L^Y$. Note that $\varphi_{1,2}.\mathrm{cl}$ and $\psi_{1,2}.\mathrm{cl}$, means that the closure operators related to $\varphi_{1,2}.\mathrm{int}$ and $\psi_{1,2}.\mathrm{int}$, respectively which are defined by $\varphi_{1,2}.\mathrm{cl}\,\mu = co\,(\varphi_{1,2}.\mathrm{int}\,co\mu)$, for all $\mu \in L^X$. Obviously, if f is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous and the inverse f^{-1} of f exists, then $f^{-1} : (Y, \psi_{1,2}.\mathrm{int}) \to (X, \varphi_{1,2}.\mathrm{int})$ is $\psi_{1,2}\varphi_{1,2}$-fuzzy continuous, that is, $(\varphi_{1,2}.\mathrm{int}\,\mu) \circ f^{-1} \leq \psi_{1,2}.\mathrm{int}\,(\mu \circ f^{-1})$, for all $\mu \in L^X$. By means of characterizing, the $\varphi_{1,2}$-fuzzy neighborhoods $\mathcal{N}_{\varphi_{1,2}}(x)$ of $\varphi_{1,2}.\mathrm{int}$ and $\mathcal{N}_{\psi_{1,2}}(x)$ of $\psi_{1,2}.\mathrm{int}$ which are defined by (2.5), the $\varphi_{1,2}\psi_{1,2}$-fuzzy continuity of f can also be characterized as follows:

The mapping $f : (X, \varphi_{1,2}.\mathrm{int}) \to (Y, \psi_{1,2}.\mathrm{int})$ is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous if and only if $\mathcal{N}_{\psi_{1,2}}(f(x)) \geq \mathcal{F}_L f(\mathcal{N}_{\varphi_{1,2}}(x))$ holds, for each $x \in X$.

Obviously, in case of $L = \{0, 1\}$, $\varphi_1 = \psi_1 = \mathrm{int}$, $\varphi_2 = 1_{L^X}$ and $\psi_2 = 1_{L^Y}$, the $\varphi_{1,2}\psi_{1,2}$-fuzzy continuity coincides with the usual fuzzy continuity.

Characterized fuzzy proximity spaces. Let a fuzzy topological space (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then each fuzzy proximity δ on X is associated a set of all $\varphi_{1,2}$-open fuzzy subsets of X with respect to δ denoted by $\varphi_{1,2}OF(X)_\delta$. The triple $(X, \varphi_{1,2}OF(X)_\delta$ as will as $(X, \varphi_{1,2}.\mathrm{int}_\delta)$ is said to be characterized fuzzy proximity space [5]. The related $\varphi_{1,2}$-interior and the $\varphi_{1,2}$-closure operators $\varphi_{1,2}.\mathrm{int}_\delta$ and $\varphi_{1,2}.\mathrm{cl}_\delta$ are given by:

$$\varphi_{1,2}.\mathrm{int}_\delta\, \mu = \bigvee_{\mu'\,\bar{\delta}\rho} \rho \text{ and } \varphi_{1,2}.\mathrm{cl}_\delta\, \mu = \bigwedge_{\rho'\,\bar{\delta}\mu} \rho,$$

respectively, for all $\mu \in L^X$. Obviously, from the definition of the complementarily symmetric fuzzy topogeneous structure, there is an identification between the fuzzy proximity δ and the complementarily symmetric fuzzy topogeneous structure $\ll$ on the same set X given by:

$$\mu \ll \eta' \iff \mu\bar{\delta}\eta, \tag{2.8}$$

for all $\mu, \eta \in L^X$.

Proposition 2.1 ([5]). *Let (X, τ) be a fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then the binary relation δ on L^X which is defined by:*

$$\mu\bar{\delta}\rho \text{ if and only if } \mathcal{N}_{\varphi_{1,2}}(\rho) \leq \dot{\mu}',$$

for all $\mu, \rho \in L^X$ is a fuzzy proximity on X.

Proposition 2.2 ([8]). *Let $(X, \varphi_{1,2}.\mathrm{int}_\delta)$ be a characterized fuzzy proximity space and $F, G \in P(X)$ such that $\chi_F\bar{\delta}\chi_G$. If Φ is the family of all $\varphi_{1,2}\psi_{1,2}\delta$-fuzzy continuous mappings $f : (X, \varphi_{1,2}.\mathrm{int}_\delta) \to (I_L, \psi_{1,2}.\mathrm{int}_{\delta^*})$ for which $x \in X$ implies $\bar{0} \leq f(x) \leq \bar{1}$, then χ_F and χ_G are Φ-separable.*

Proposition 2.3 ([8]). *Let a fuzzy topological space (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. If $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ is characterized $FR_{2\frac{1}{2}}$-space and Φ is a fuzzy function family of all $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mappings, then the binary relation δ on L^X which is defied by: $\mu\bar{\delta}\rho \iff \mu$ and ρ are Φ-separated for all $\mu, \rho \in L^X$, is fuzzy proximity on X compatible with the family of all $\varphi_{1,2}$-open fuzzy set $\varphi_{1,2}OF(X)$, that is, $(X, \varphi_{1,2}.\mathrm{int}_\tau) = (X, \varphi_{1,2}.\mathrm{int}_\delta)$.*

Fuzzy function family. Let X be non-empty set. By the fuzzy function family Φ on X, we mean the set of all fuzzy real functions $f : X \to I_L$.

Consider $\mu, \eta \in L^X$. Then the fuzzy real functions $f : X \to I_L$ is said to be separate μ and η, if it fulfilled the following conditions:

(i) $\bar{0} \leq f(x) \leq \bar{1}$ holds, for all $x \in X$,

(ii) if $x_1, y_1 \in S(X)$ such that $x_1 \leq \mu$ and $y_1 \leq \eta$, then $f(x) = \bar{1}$ and $f(y) = \bar{0}$, for all $x, y \in X$.

Moreover, if Φ is a fuzzy function family on X, then the fuzzy subsets $\mu, \eta \in L^X$ are called Φ-separable or Φ-separated, if there exists a fuzzy real function $h \in \Phi$ separating them.

Characterized fuzzy T_s- and fuzzy $\varphi_{1,2}$-T_s spaces. The notions of characterized fuzzy T_s- spaces and of fuzzy $\varphi_{1,2}$-T_s spaces are investigated and studied in [2, 3, 8], for all $s \in \{0, 1, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4\}$. These spaces depend only on the usual points and the operation defined on the class of all fuzzy subsets of X endowed with a fuzzy topological space (X, τ). The characterized fuzzy T_s- spaces and of fuzzy $\varphi_{1,2}$-T_s spaces will be denoted by characterized FT_s- spaces and $F\varphi_{1,2}$-T_s spaces, respectively for shorts. Let a fuzzy topological space (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then

(i) the characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$ is said to be characterized FT_1-space (resp. characterized FT_2-space), if for all $x, y \in X$ such that $x \neq y$, there exist $\mu, \rho \in L^X$ and $\alpha, \beta \in L_0$ such that

$\mu(x) < \alpha \leq (\varphi_{1,2}.\text{int}\mu)(y)$ and $\rho(y) < \beta \leq (\varphi_{1,2}.\text{int}\rho)(x)$

(resp. the infimum $\mathcal{N}_{\varphi_{1,2}}(x) \wedge \mathcal{N}_{\varphi_{1,2}}(y)$ does not exists),

(ii) the related fuzzy topological space (X, τ) is said to be $F\varphi_{1,2}$-T_1 (resp. $F\varphi_{1,2}$-$T_2F\varphi_{1,2}$-T_2), if for all $x, y \in X$ such that $x \neq y$, we have $\dot{x} \not\leq \mathcal{N}_{\varphi_{1,2}}(y)$ and $\dot{y} \not\leq \mathcal{N}_{\varphi_{1,2}}(x)$.

Characterized fuzzy R_k- and fuzzy $\varphi_{1,2}$-R_k spaces. The notions of characterized fuzzy R_k- and fuzzy $\varphi_{1,2}$-R_k spaces are introduced and studied in [3, 4, 8], for all $k \in \{0, 1, 2, 2\frac{1}{2}, 3\}$, by means of the notion of $\varphi_{1,2}$-fuzzy neighborhood filter at a point x and at the ordinary subset of the characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$, and also by the notion of $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous. However, the notions of fuzzy $\varphi_{1,2}$-R_k spaces are also given by means of the $\varphi_{1,2}$-fuzzy convergence at a point x and at the ordinary set in the space. The characterized fuzzy R_k-spaces and fuzzy $\varphi_{1,2}$-R_k spaces will be denoted by characterized FR_k- and $F\varphi_{1,2}$-R_k spaces, respectively for shorts. Let a fuzzy topological space (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then the characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$ is said to be characterized $FR_{2\frac{1}{2}}$-space, if for all $x \in X$, $F \in \varphi_{1,2}C(X)$ such that $x \notin F$, there exists a $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping $f : (X, \varphi_{1,2}.\text{int}) \to (I_L, \psi_{1,2}.\text{int}_{\Im})$ such that $f(x) = \bar{1}$ and $f(y) = \bar{0}$, for all $y \in F$. Moreover, $(X, \varphi_{1,2}.\text{int})$ is said to be characterized FT_s-space, if it is characterized FR_k and characterized FT_1 for $k \in \{2\frac{1}{2}, 3\}$ and $s \in \{3\frac{1}{2}, 4\}$. The related fuzzy topological space (X, τ) is said to be $F\varphi_{1,2}$-T_s, if it is $F\varphi_{1,2}$-R_k and $F\varphi_{1,2}$-T_1.

Proposition 2.4 ([2, 3]). *Let a fuzzy topological space (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then*

(1) *every characterized FT_i-space $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized FT_{i-1}-space for each $i \in \{1, 2, 3, 4\}$,*

(2) *the initial and final characterized fuzzy spaces of a family of characterized FT_i-spaces are also characterized FT_i -spaces for each $i \in \{0, 1, 2, 3, 4\}$,*

(3) *the characterized fuzzy subspace and the characterized fuzzy product space of a family of characterized FT_i-spaces are also characterized FT_i-spaces for each* $i \in \{0, 1, 2, 3, 4\}$.

3. Characterized global fuzzy neighborhood spaces and pre spaces

By the global fuzzy neighborhood structure on a set X [18], we mean the mapping $h : \mathcal{F}_L X \to \mathcal{F}_L X$ such that the following axioms are fulfilled:

(N1) $\mathcal{M} \leq h(\mathcal{M})$ holds, for all $\mathcal{M} \in \mathcal{F}_L X$,
(N2) $h(\mathcal{L} \vee \mathcal{M}) = h(\mathcal{L}) \vee h(\mathcal{M})$, for all $\mathcal{L}, \mathcal{M} \in \mathcal{F}_L X$,
(N3) $h \circ h = h$.
(N4) $\mu_X \circ \mathcal{F}_L h \circ \mathcal{F}_L \eta_X \leq h$ holds,

where η and μ are the natural transformations appearing in the fuzzy filter monad. If h and k are global fuzzy neighborhood structures on a set X, then h is said to be finer than k and will be denoted by $h \leq k$, if the fuzzy filter $h(\mathcal{M})$ is finer than the fuzzy filter $k(\mathcal{M})$, for all $\mathcal{M} \in \mathcal{F}_L X$. As shown in [18, 19], the fuzzy topogeneous structures are characterized by means of the global fuzzy neighborhood structure. However, the following weakening of the notion of the global fuzzy neighborhood structure is used to characterizations of the fuzzy topogenuous orders.

By the global fuzzy neighborhood pre structure on a set X [13, 20], we mean the mapping $h : \mathcal{F}_L X \to \mathcal{F}_L X$ which fulfills the axioms (N1), (N2) and (N4). Each global fuzzy neighborhood structure h associated canonically a fuzzy pre topology τ_h on X. Recall that the fuzzy pre topology is usually defined as a fuzzy interior operator, that is, as a mapping $\mathrm{int}_h : L^X \to L^X$ such that $\mathrm{int}_h \, \bar{1} = \bar{1}$, $\mathrm{int}_h \mu \leq \mu$ and $\mathrm{int}_h(\mu \wedge \eta) = \mathrm{int}_h \mu \wedge \mathrm{int}_h \eta$ for all $\mu, \eta \in L^X$. The global fuzzy neighborhood space and the global homogenous fuzzy neighborhood space are sets equipped with global fuzzy neighborhood structure h and global homogenous fuzzy neighborhood structure k and will be denoted by (X, h) and (X, k), respectively. Consider $\varphi_1, \varphi_2 \in O_{(L^X, \tau_h)}$, then the associated characterized fuzzy pre topology generated by $\varphi_{1,2}.\mathrm{int}_{\tau_h}$ of the global fuzzy neighborhood pre structure $h : \mathcal{F}_L X \to \mathcal{F}_L X$ is defined by:

$$(\varphi_{1,2}.\mathrm{int}_{\tau_h} \mu)(x) \;=\; h(\dot{x})(\mu), \tag{3.1}$$

for all $x \in X$ and $\mu \in L^X$. For technical reason condition (N4) of the global fuzzy neighborhood structure will also be applied in a more simple formulation given in the following proposition.

Proposition 3.1. *Condition (N4) of the global fuzzy neighborhood structure is equivalent to the condition:*

$$h(\mathcal{M})(\rho) \leq \mathcal{M}(\varphi_{1,2}.\mathrm{int}_{\tau_h} \rho) \text{ holds for all } \mathcal{M} \in \mathcal{F}_L X \text{ and } \rho \in L^X,$$

where $\varphi_{1,2}.\mathrm{int}_{\tau_h} \rho$ *is the fuzzy set* $x \mapsto h(\dot{x})(\rho)$ *of* X.

Proof. From the definitions of the natural transformations μ_X and η_X, it follows that for all $\mathcal{M} \in \mathcal{F}_L X$ and $\rho \in L^X$, we have

$$\begin{aligned}
(\mu_X \circ \mathcal{F}_L h \circ \mathcal{F}_L \eta_X)(\mathcal{M})(\rho) &= \mu_X(\mathcal{F}_L(h \circ \eta_X)(\mathcal{M}))(\rho) \\
&= (\mathcal{F}_L(h \circ \eta_X)(\mathcal{M}) \circ e_X)(\rho) \\
&= \mathcal{F}_L(h \circ \eta_X)(\mathcal{M})(e_X(\rho)) \\
&= \mathcal{M}(e_X(\rho) \circ h \circ \eta_X)
\end{aligned}$$

$$= \mathcal{M}\big(x \mapsto h(\dot{x})(\rho)\big)$$
$$= \mathcal{M}(\varphi_{1,2}.\mathrm{int}_{\tau_h}\rho).$$

This shows that both the conditions are equivalent. $\square$

Let h be global fuzzy neighborhood pre structure on a set X and $\mathcal{M} \in \mathcal{F}_L X$. Then, $h(\mathcal{M})$ is said to be fuzzy neighborhood of $\mathcal{M}$ with respect to h and the mapping $h \circ \varphi_{1,2}.\mathrm{int}_{\tau_h}$, which coincides with $(\mu_X \circ \mathcal{F}_L h \circ \mathcal{F}_L \eta_X)(\mathcal{M})$ is said to be $\varphi_{1,2}$-fuzzy neighborhood of $\mathcal{M}$ with respect to the characterized fuzzy pre topology generated by $\varphi_{1,2}.\mathrm{int}_{\tau_h}$. Note that, $\mu_X \circ \mathcal{F}_L h \circ \mathcal{F}_L \eta_X$ is itself a global fuzzy neighborhood pre structure and it can be identified with the characterized fuzzy pre topology generated by $\varphi_{1,2}.\mathrm{int}_{\tau_h}$. In case of h is global fuzzy neighborhood structure, then $\mu_X \circ \mathcal{F}_L h \circ \mathcal{F}_L \eta_X$ also is. In this case, $\varphi_{1,2}.\mathrm{int}_{\tau_h}$ is $\varphi_{1,2}$-interior operator of a characterized fuzzy space on X denoted by $(X, \varphi_{1,2}.\mathrm{int}_{\tau_h})$, that is, $\mu_X \circ \mathcal{F}_L h \circ \mathcal{F}_L \eta_X$ is identified with this associated characterized fuzzy space. This associated characterized fuzzy space will be called characterized global fuzzy neighborhood space associated by the global fuzzy neighborhood structures h.

If the fuzzy filter monad $(\mathcal{F}_L, \eta, \mu)$ is replaced by the homogeneous fuzzy filter monad $(\mathsf{F}_L, \eta^{'}, \mu^{'})$, then we have the global homogeneous fuzzy neighborhood structure instead of the global fuzzy neighborhood structure, that is, the global homogeneous fuzzy neighborhood structure on a set X is the mapping $h : \mathsf{F}_L X \to \mathsf{F}_L X$ such that the following axioms are fulfilled:

(N1$^{'}$) $\mathcal{M} \leq h(\mathcal{M})$, for all $\mathcal{M} \in \mathsf{F}_L X$,
(N2$^{'}$) $h(\mathcal{L} \vee \mathcal{M}) = h(\mathcal{L}) \vee h(\mathcal{M})$, for all $\mathcal{L}, \mathcal{M} \in \mathsf{F}_L X$,
(N3$^{'}$) $h \circ h = h$,
(N4$^{'}$) $\mu^{'}_X \circ \mathsf{F}_L h \circ \mathsf{F}_L \eta^{'}_X \leq h$.

If the axioms (N1$^{'}$), (N2$^{'}$) and (N4$^{'}$) are only fulfilled, then h will be called global homogeneous fuzzy neighborhood per structure.

Each global homogenous fuzzy neighborhood pre structure h on a set X is associated canonically with a stratified fuzzy pre topology τ^s_h on X. Consider $\varphi_1, \varphi_2 \in O_{(L^X, \tau^s_h)}$. Then the associated stratified characterized fuzzy pre topology generated by $\varphi_{1,2}.\mathrm{int}_{\tau^s_h}$ of the global homogenous fuzzy neighborhood pre structure $h : \mathsf{F}_L X \to \mathsf{F}_L X$ is defined by $(\varphi_{1,2}.\mathrm{int}_{\tau^s_h} \mu)(x) = h(\dot{x})(\mu)$, for all $x \in X$ and $\mu \in L^X$. The ordered pair $(X, \varphi_{1,2}.\mathrm{int}_{\tau^s_h})$ will be called characterize global fuzzy neighborhood pre space. The associated characterized global fuzzy neighborhood pre topology generated by $\varphi_{1,2}.\mathrm{int}_{\tau^s_h}$ is stratified, that is, $\varphi_{1,2}.\mathrm{int}_{\tau^s_h}(\bar{\alpha}) = \bar{\alpha}$, for all $\alpha \in L$. In case of h is global homogeneous fuzzy neighborhood structure, then $\mu^{'}_X \circ \mathsf{F}_L h \circ \mathsf{F}_L \eta^{'}_X$ also is and thus $\varphi_{1,2}.\mathrm{int}_{\tau^s_h}$ is $\varphi_{1,2}$-interior operator of a stratified characterized fuzzy space on X denoted also by $(X, \varphi_{1,2}.\mathrm{int}_{\tau^s_h})$, that is, $\mu^{'}_X \circ \mathsf{F}_L h \circ \mathsf{F}_L \eta^{'}_X$ is identified with this stratified characterized fuzzy space. This stratified characterized fuzzy space will be called associated stratified characterized global fuzzy neighborhood space, by the global homogeneous fuzzy neighborhood structures h.

(h, k)**-Continuity.** The mapping f between the global fuzzy neighborhood space (X, h) into the global fuzzy neighborhood space (Y, k) is said to be (h, k)-continuous, provided

$$\mathcal{F}_L f \circ h \leq k \circ \mathcal{F}_L f. \tag{3.2}$$

Analogously, the mapping $g : (X^*, h) \to (Y^*, k)$ between global homogenous fuzzy neighborhood spaces is said to be (h, k)-continuous, provided $\mathsf{F}_L g \circ h \leq k \circ \mathsf{F}_L g$.

Proposition 3.2. *Let $h : \mathcal{F}_L X \to \mathcal{F}_L X$ be global fuzzy neighborhood structure and let (X, τ_h) be the associated global fuzzy neighborhood space. If $\varphi_1, \varphi_2 \in O_{(L^X, \tau_h)}$, then h defines a characterized global fuzzy neighborhood space generated by $\varphi_{1,2}.\mathrm{int}_{\tau_h}$ on X given by:*

$$\mu \in \varphi_{1,2} OF(X)_{\tau_h} \iff \mu(x) = h(\dot{x})(\mu) \ \text{ for each } \ x \in X.$$

Proof. Condition (F1) of the fuzzy filter implies that $\overline{0}$ and $\overline{1}$ are members of $\varphi_{1,2} OF(X)_{\tau_h}$. If $\mu, \eta \in \varphi_{1,2} OF(X)_{\tau_h}$, then by the condition (F2), we have

$$h(\dot{x})(\mu \wedge \eta) = (\mu \wedge \eta)(x).$$

Thus $\mu \wedge \eta \in \varphi_{1,2} OF(X)_{\tau_h}$.

Let $\{ \mu_i \}_{i \in I}$ be a family of members of $\varphi_{1,2} OF(X)_{\tau_h}$. Since h is hull operator, it follows that $h(\dot{x})(\bigvee_{i \in I} \mu_i) \leq (\bigvee_{i \in I} \mu_i)(x)$. Since h is isotone, we have

$$h(\dot{x})(\bigvee_{i \in I} \mu_i) \geq \bigvee_{i \in I} h(\dot{x})(\mu_i) = (\bigvee_{i \in I} \mu_i)(x).$$

So, $h(\dot{x})(\bigvee_{i \in I} \mu_i) = (\bigvee_{i \in I} \mu_i)(x)$, for each $x \in X$. Hence $\bigvee_{i \in I} \mu_i \in \varphi_{1,2} OF(X)_{\tau_h}$, i.e., $\varphi_{1,2} OF(X)_{\tau_h}$ is $\varphi_{1,2}.\mathrm{int}_{\tau_h}$ operator for a characterized fuzzy space on X. Therefore $(X, \varphi_{1,2}.\mathrm{int}_{\tau_h})$ is characterized global fuzzy neighborhood space. $\square$

Proposition 3.3. *Let $f : (X, h) \to (Y, k)$ be a mapping between global fuzzy neighborhood pre spaces, let τ_h and τ_k be the fuzzy pre topologies associated to h and k, respectively. Consider $\varphi_1, \varphi_2 \in O_{(L^X, \tau_h)}$ and $\psi_1, \psi_2 \in O_{(L^Y, \tau_k)}$. If $f : (X, h) \to (Y, k)$ is (h, k)-continuous, then $f : (X, \varphi_{1,2}.\mathrm{int}_{\tau_h}) \to (Y, \psi_{1,2}.\mathrm{int}_{\tau_k})$ between the associated characterized global fuzzy neighborhood pre spaces is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous. In case, h and k are coincide up to identifications with $\varphi_{1,2}.\mathrm{int}_{\tau_h}$ and $\psi_{1,2}.\mathrm{int}_{\tau_k}$, respectively, then $f : (X, h) \to (Y, k)$ is (h, k)-continuous if and only if $f : (X, \varphi_{1,2}.\mathrm{int}_{\tau_h}) \to (Y, \psi_{1,2}.\mathrm{int}_{\tau_k})$ is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous.*

Proof. Assume at first that f is (h, k)-continuous. For all $\eta \in L^Y$ and $x \in X$, (3.1) and (3.2) imply that

$$\begin{aligned} \varphi_{1,2}.\mathrm{int}_{\tau_h}(\eta \circ f)(x) &= h(\dot{x})(\eta \circ f) \\ &= \mathcal{F}_L f(h(\dot{x}))(\eta) \geq k(\mathcal{F}_L f(\dot{x}))(\eta) \\ &= k(\dot{y})(\eta) = (\psi_{1,2}.\mathrm{int}_{\tau_k} \eta)(f(x)), \end{aligned}$$

where $y = f(x)$ and $\varphi_{1,2}.\mathrm{int}_{\tau_h}$ and $\psi_{1,2}.\mathrm{int}_{\tau_k}$ are the $\varphi_{1,2}$-interior and $\psi_{1,2}$-interior operators with respect to the associated characterized global fuzzy neighborhood pre topologies, respectively. Then, $(\psi_{1,2}.\mathrm{int}_{\tau_k} \eta) \circ f \leq \varphi_{1,2}.\mathrm{int}_{\tau_h}(\eta \circ f)$, for all $\eta \in L^Y$, which of course means from (2.7) that f is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous. As in case of [15, 17], under the assumption that h and k coincide up to identifications with $\varphi_{1,2}.\mathrm{int}_{\tau_h}$ and $\psi_{1,2}.\mathrm{int}_{\tau_k}$,respectively the $\varphi_{1,2}\psi_{1,2}$-fuzzy continuity of $f : (X, \varphi_{1,2}.\text{int}_{\tau_h}) \to (Y, \psi_{1,2}.\text{int}_{\tau_k})$ is equivalent to the (h, k)-continuity of $f : (X, h) \to (Y, k)$. □

Corollary 3.4. *If $f : (X, h) \to (Y, k)$ is (h, k)-continuous mapping between global homogenous neighborhood fuzzy pre spaces, then $f : (X, \varphi_{1,2}.\text{int}_{\tau^s_h}) \to (Y, \psi_{1,2}.\text{int}_{\tau^s_k})$ between the associated stratified characterized global fuzzy neighborhood pre spaces is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous.*

Proof. Immediate from Proposition 3.3. □

4. Characterized fuzzy uniform spaces, characterized fuzzy perfect topoeneous structures and characterized $FR_{2\frac{1}{2}}$-spaces

In this section, we are going to investigate and study the notions of characterized fuzzy uniform spaces and of characterized fuzzy perfect topoeneous structure as a generalization of all the weaker and stronger forms of the notions of fuzzy uniform spaces presented in [20] and of the fuzzy perfect topoeneous structure presented in [19, 26], respectively. For this, we applied the notion of homogeneous fuzzy filter at the point and at the fuzzy set which is defined by (2.1), the superior principal fuzzy filter $[\mu]$ generated by $\mu \in L^X$ and the $\varphi_{1,2}$-fuzzy neighborhoods at the fuzzy set μ which is defined by (2.6) in the characterized fuzzy space $(X, \varphi_{1,2}.\text{int}_\tau)$. Moreover, the relation between the separated fuzzy uniform spaces, the associated characterized uniform FT_1-spaces, the associated characterized uniform FT_3 and the $F\varphi_{1,2}T_i$-spaces which introduced in [2] are investigated for all $i \in \{1, 3\}$.

By a fuzzy relation on a set X, we mean the mapping $R : X \times X \to L$, that is, a fuzzy subset of $X \times X$. For each fuzzy relation R on X, the inverse R^{-1} of R is the fuzzy relation on X defined by $R^{-1}(x, y) = R(y, x)$, for all $x, y \in X$. Let $\mathcal{U}$ be a fuzzy filer on $X \times X$. The inverse $\mathcal{U}^{-1}$ of $\mathcal{U}$ is a fuzzy filter on $X \times X$ defined by $\mathcal{U}^{-1}(R) = \mathcal{U}(R^{-1})$, for all $R \in L^{X\times X}$. The composition $R_1 \circ R_2$ of two fuzzy relations R_1 and R_2 on a set X is fuzzy relation on X defined by:

$$(R_1 \circ R_2)(x, y) = \bigvee_{z\in X} \Big(R_2(x, z) \wedge R_1(z, y)\Big),$$

for all $x, y \in X$. For each pair (x, y) of elements x and y of X, the mapping $(x, y)^{\cdot} : L^{X\times X} \to L$ defined by $(x, y)^{\cdot}(R) = R(x, y)$, for all $R \in L^{X\times X}$ is a homogeneous fuzzy filter on $X \times X$.

Let $\mathcal{U}$ and $\mathcal{V}$ are fuzzy filers on $X \times X$ such that $(x, y)^{\cdot} \le \mathcal{U}$ and $(y, z)^{\cdot} \le \mathcal{V}$ hold, for some $x, y, z \in X$. Then the composition $\mathcal{V} \circ \mathcal{U}$ of $\mathcal{V}$ and $\mathcal{U}$ is a fuzzy filter ([20]) on $X \times X$ defined by:

$$(\mathcal{V} \circ \mathcal{U})(R) = \bigvee_{R_2\circ R_1\le R} \Big(\mathcal{U}(R_1) \wedge \mathcal{V}(R_2)\Big),$$

for all $R \in L^{X\times X}$. The fuzzy uniform structure $\mathcal{U}$ ([20]) on a set X is said to be separated, if for all $x, y \in X$ with $x \neq y$ there is $R \in L^{X\times X}$ such that $\mathcal{U}(R) = 1$ and $R(x, y) = 0$. In this case, the fuzzy uniform space $(X, \mathcal{U})$ is called a separated fuzzy uniform space. Let $\mathcal{U}$ be a fuzzy uniform structure on a set X such that $(x, y)^{\cdot} \le \mathcal{U}$

holds, for all $x \in X$ and let $\mathcal{M} \in \mathcal{F}_L X$. Then the mapping $\mathcal{U}[\mathcal{M}] : L^X \to L$ which is defined by:

$$\mathcal{U}[\mathcal{M}](\mu) = \bigvee_{R(\eta) \leq \mu} \big(\mathcal{U}(R) \wedge \mathcal{M}(\eta)\big),$$

for all $\mu \in L^X$ is fuzzy filter on X, called the image of $\mathcal{M}$ with respect to the fuzzy uniform structure $\mathcal{U}$ ([20]), where $\eta, R[\eta] \in L^X$ such that

$$R[\eta](x) = \bigvee_{y \in X} (\eta(y), R(y,x)).$$

To each fuzzy uniform structure $\mathcal{U}$ on a set X is associated with a stratified fuzzy topology $\tau_{\mathcal{U}}$ on X. Consider $\varphi_1, \varphi_2 \in O_{(L^X, \tau_{\mathcal{U}})}$. Then the set of all $\varphi_{1,2}$-open fuzzy subsets of X related to $\tau_{\mathcal{U}}$ forms a base for a characterized stratified fuzzy space on X generated by the $\varphi_{1,2}$-interior operator with respect to $\tau_{\mathcal{U}}$ denoted by $\varphi_{1,2}.\mathrm{int}_{\mathcal{U}}$ and $(X, \varphi_{1,2}.\mathrm{int}_{\mathcal{U}})$ is a characterized stratified fuzzy space. In this case, $(X, \varphi_{1,2}.\mathrm{int}_{\mathcal{U}})$ will be called the associated characterized fuzzy uniform space which is stratified. The related $\varphi_{1,2}$-interior operator $\varphi_{1,2}.\mathrm{int}_{\mathcal{U}}$ is given by:

$$(\varphi_{1,2}.\mathrm{int}_{\mathcal{U}}\mu)(x) = \mathcal{U}[\dot{x}](\mu), \tag{4.1}$$

for all $x \in X$ and $\mu \in L^X$. The fuzzy set μ is said to be $\varphi_{1,2}\mathcal{U}$-fuzzy neighborhood of $x \in X$ provided $\mathcal{U}[\dot{x}] \leq \dot{\mu}$. Because of (2.1), (2.6) and (4.1), we have that

$$\mathcal{U}[\dot{x}] = \mathcal{N}_{\varphi_{1,2}}(x) \text{ and } \mathcal{U}[\dot{\nu}] = \mathcal{N}_{\varphi_{1,2}}(\mu), \tag{4.2}$$

for all $x \in X$ and $\mu \in L^X$. In this case, $\mathcal{N}_{\varphi_{1,2}}(x)$ and $\mathcal{N}_{\varphi_{1,2}}(\mu)$ are fuzzy neighborhood filters of the associated characterized fuzzy uniform space $(X, \varphi_{1,2}.\mathrm{int}_{\mathcal{U}})$ at x and μ, respectively.

Proposition 4.1. *Let X be a non-empty set, let $\mathcal{U}$ be a fuzzy uniform structure on X and $\varphi_1, \varphi_2 \in O_{(L^X, \tau_{\mathcal{U}})}$. Then the fuzzy uniform space $(X, \mathcal{U})$ is separated if and only if the associated characterized fuzzy uniform space $(X, \varphi_{1,2}.\mathrm{int}_{\mathcal{U}})$ is characterized FT_1-space.*

Proof. Let $(X, \mathcal{U})$ is separated and let $x, y \in X$ such that $x \neq y$. Then, there exists $R_1, R_2 \in L^{X \times X}$ such that $\mathcal{U}(R_i) = 1$ and $R_i(x, y) = 0$, for all $i \in \{1.2\}$. Consider $\mu = R[y_1]$ and $\eta = R[x_1]$. Then we have

$$\mu(x) = R_1[y_1](x) = \bigvee_{z \in X} \big(R_1(z, x) \wedge y_1(z)\big) = 0$$

and

$$\eta(y) = R_2[x_1](y) = \bigvee_{z \in X} \big(R_2(z, y) \wedge x_1(z)\big) = 0.$$

Moreover,

$$(\varphi_{1,2}.\mathrm{int}_{\mathcal{U}}\, \mu)(y) = \mathcal{U}[\dot{y}](\mu) = \bigvee_{R_1(\rho) \leq \mu} \big(\mathcal{U}(R_1) \wedge \rho(y)\big) = 1$$

and

$$(\varphi_{1,2}.\mathrm{int}_{\mathcal{U}}\, \eta)(x) = \mathcal{U}[\dot{x}](\eta) = \bigvee_{R_1(\rho) \leq \eta} \big(\mathcal{U}(R_2) \wedge \rho(x)\big) = 1,$$

for all $\rho \in L^X$. Thus, there exists $\mu, \eta \in L^X$ and $\alpha, \beta \in L_0$ such that

$$\mu(x) < \alpha \leq (\varphi_{1,2}.\mathrm{int}_{\mathcal{U}}\, \mu)(y) \text{ and } \eta(y) < \beta \leq (\varphi_{1,2}.\mathrm{int}_{\mathcal{U}}\, \eta)(x).$$

So, $(X, \varphi_{1,2}.\mathrm{int}_{\mathcal{U}})$ is characterized FT_1-space.

Conversely, let $(X, \varphi_{1,2}.\text{int}_{\mathcal{U}})$ be characterized FT_1-space and let $x \neq y$ in X. Then, there exists $\mu, \eta \in L^X$ and $\alpha, \beta \in L_0$ such that

$\mu(x) < \alpha \leq (\varphi_{1,2}.\text{int}_{\mathcal{U}}\, \mu)(y)$ and $\eta(y) < \beta \leq (\varphi_{1,2}.\text{int}_{\mathcal{U}}\, \eta)(x)$.

This means that for all $\rho \in L^X$,

$$\bigvee_{R_1(\rho)\leq\mu} \big(\mathcal{U}(R_1) \wedge \rho(y)\big) > \mu(x) \text{ and } \bigvee_{R_2(\rho)\leq\eta} \big(\mathcal{U}(R_2) \wedge \rho(x)\big) > \eta(y).$$

Thus, there is $R_1, R_2 \in L^{X\times X}$ such that $R_1(x, y) = \varphi_{1,2}.\text{int}_{\mathcal{U}}\, \mu(y)$, if $x = y$ and $R_1(x, y) = \mu(x)$, if $x \neq y$ such that $R_1(x, y) = 0$ and $\mathcal{U}(R_1) = 1$ and $R_2(x, y) = \varphi_{1,2}.\text{int}_{\mathcal{U}}\, \eta(x)$, if $x = y$ and $R_2(x, y) = \eta(y)$ if $x \neq y$ such that $R_2(x, y) = 0$ and $\mathcal{U}(R_2) = 1$. So, in every case, $(X, \mathcal{U})$ is separated. □

Corollary 4.2. *Let X be a non-empty set, let $\mathcal{U}$ be a fuzzy uniform structure on X and $\varphi_1, \varphi_2 \in O_{(X,\tau_{\mathcal{U}})}$. Then the fuzzy uniform space $(X, \mathcal{U})$ is separated if and only if the associated stratified fuzzy topological space $(X, \tau_{\mathcal{U}})$ is $F\varphi_{1,2}$-T_1-space.*

Proof. Immediate from Proposition 4.1 and Theorem 2.2 in [2]. □

Proposition 4.3 ([19])**.** *For each fuzzy uniform structure $\mathcal{U}$ on a set X, the mapping $h : \mathsf{F}_L X \to \mathsf{F}_L X$ which is defined by*

$$h(\mathcal{M}) \ = \ \mathcal{U}[\mathcal{M}], \tag{4.3}$$

for all $\mathcal{M} \in \mathsf{F}_L X$, is a global homogeneous fuzzy neighborhood structure.

The mapping h will be called global homogeneous fuzzy neighborhood structure associated to the fuzzy uniform structure $\mathcal{U}$ and will be denoted by $h_{\mathcal{U}}$. The global fuzzy neighborhood structure h on a set X is said to be symmetric [13, 20], provided that $h(\mathcal{L}) \wedge \mathcal{M}$ exists if and only if $\mathcal{L} \wedge h(\mathcal{M})$ exists for all $\mathcal{M}, \mathcal{L} \in \mathcal{F}_L X$. As shown in [20], for each fuzzy uniform structure $\mathcal{U}$, the associated homogenous fuzzy neighborhood structure $h_{\mathcal{U}}$ is symmetric and both the global homogenous fuzzy neighborhood structures associated to the fuzzy uniform structures $\mathcal{U}$ and its homogenization $\mathcal{U}^*$ are coincide.

Proposition 4.4 ([20])**.** *Let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be uniformly fuzzy continuous mapping between fuzzy uniform spaces. Then the mapping $f : (X, h_{\mathcal{U}}) \to (Y, h_{\mathcal{V}})$ between the associated global homogeneous fuzzy neighborhood spaces is $(h_{\mathcal{U}}, h_{\mathcal{V}})$-fuzzy continuous.*

Proposition 4.5. *Let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be uniformly fuzzy continuous mapping between fuzzy uniform spaces, $\varphi_1, \varphi_2 \in O_{(L^X,\tau_{\mathcal{U}})}$ and $\psi_1, \psi_2 \in O_{(L^Y,\tau_{\mathcal{V}})}$. Then the mapping $f : (X, \varphi_{1,2}.\text{int}_{\mathcal{U}}) \to (Y, \psi_{1,2}.\text{int}_{\mathcal{V}})$ between the associated characterized fuzzy uniform spaces is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous.*

Proof. Immediate from Propositions 3.3 and 4.4. □

In the following proposition, we prove that for each fuzzy uniform structure on a set X there is induced stratified fuzzy proximity on L^X. Moreover, both the fuzzy uniform structure and this induced stratified fuzzy proximity are associated with the same stratified characterized fuzzy uniform space.

Proposition 4.6. *Let X be a non-empty set and let $\mathcal{U}$ be a fuzzy uniform structure on X and $\varphi_1, \varphi_2 \in O_{(X,\tau_{\mathcal{U}})}$. Then the binary relation $\delta_{\mathcal{U}}$ on L^X defined by:*

$$\mu \bar{\delta}_{\mathcal{U}} \rho \text{ if and only if } \mathcal{U}[\dot{\mu}] \leq \dot{\rho}^{'}, \tag{4.4}$$

for all $\mu, \rho \in L^X$ is stratified fuzzy proximity on X. Moreover, both the fuzzy uniform structure $\mathcal{U}$ and the induced stratified fuzzy proximity $\delta_{\mathcal{U}}$ are associated with the same stratified characterized fuzzy uniform space, that is, $(X, \varphi_{1,2}.\text{int}_{\mathcal{U}}) = (X, \varphi_{1,2}.\text{int}_{\delta_{\mathcal{U}}})$.

Proof. By (4.2) and Proposition 2.1, we get $\delta_{\mathcal{U}}$ which is defined by (4.4) is fuzzy proximity on X. Since $(X, \varphi_{1,2}.\text{int}_{\mathcal{U}})$ is stratified, $\bar{\alpha}$ is $\varphi_{1,2}$-open in $(X, \varphi_{1,2}.\text{int}_{\mathcal{U}})$, for all $\alpha \in L$, i.e., $\mathcal{U}[\dot{\bar{\alpha}}] \leq \dot{\bar{\alpha}}$, for all $\alpha \in L$ and then $\bar{\alpha} \bar{\delta}_{\mathcal{U}} \bar{\alpha}^{'}$, for all $\alpha \in L$. Thus, $\delta_{\mathcal{U}}$ is stratified fuzzy proximity on X. By (4.4), we have $x_1 \bar{\delta}_{\mathcal{U}} \mu^{'}$ if and only if $\mathcal{U}[\dot{x}] \leq \dot{\mu}$, i.e., μ is $\varphi_{1,2}\delta_{\mathcal{U}}$-fuzzy neighborhood of x if and only if it is $\varphi_{1,2}\mathcal{U}$-fuzzy neighborhood of x. So, both the fuzzy uniform structure $\mathcal{U}$ and the induced stratified fuzzy proximity $\delta_{\mathcal{U}}$ are associated with the same stratified characterized fuzzy uniform space, i.e., $(X, \varphi_{1,2}.\text{int}_{\mathcal{U}}) = (X, \varphi_{1,2}.\text{int}_{\delta_{\mathcal{U}}})$. □

Corollary 4.7. *Let $(X, \mathcal{U})$, $(Y, \mathcal{V})$ be two fuzzy uniform spaces and let $\varphi_1, \varphi_2 \in O_{(X,\tau_{\delta_{\mathcal{U}}})}$ and $\psi_1, \psi_2 \in O_{(Y,\tau_{\delta_{\mathcal{V}}})}$. Then $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is uniformly fuzzy continuous mapping between fuzzy uniform spaces if and only if the mapping $f : (X, \varphi_{1,2}.\text{int}_{\delta_{\mathcal{U}}}) \to (Y, \psi_{1,2}.\text{int}_{\delta_{\mathcal{V}}})$ between the associated stratified fuzzy proximity spaces is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous.*

Proof. Immediate from Propositions 4.5 and 4.6. □

By Propositions 2.2 and 4.6 and Corollary 4.7, we can deduce the following result:

Proposition 4.8. *Let $(X, \mathcal{U})$ be a fuzzy uniform space and let $F, G \in P(X)$ such that $\mathcal{U}[\dot{F}] = \mathcal{U}[\dot{\chi}_F] \leq \dot{\chi}_{G'} = \dot{G}^{'}$ and $\varphi_1, \varphi_2 \in O_{(X,\tau_{\delta_{\mathcal{U}}})}$. If Φ is the family of all uniformly fuzzy continuous functions $f : (X, \mathcal{U}) \to (I_L, \mathcal{U}^*)$ for which $x \in X$ implies $\bar{0} \leq f(x) \leq \bar{1}$, then χ_F and χ_G are Φ-separable.*

Proof. By Proposition 4.6, we have $\chi_F \bar{\delta}_{\mathcal{U}} \chi_G$. From Proposition 2.2, we get χ_F and χ_G are Φ-separated by the $\varphi_{1,2}\psi_{1,2}\delta_{\mathcal{U}}$-fuzzy continuous mapping $f : (X, \varphi_{1,2}.\text{int}_{\delta_{\mathcal{U}}}) \to (Y, \psi_{1,2}.\text{int}_{\delta_{\mathcal{V}}})$ between the associated stratified characterized fuzzy proximity spaces. Corollary 4.7, implies that $f : (X, \mathcal{U}) \to (I_L, \mathcal{U}^*)$ is uniformly fuzzy continuous and therefore χ_F and χ_G are Φ-separable. □

Now, we shall prove that the stratified characterized fuzzy uniform space which associated with a fuzzy uniform structure is characterized $FR_{2\frac{1}{2}}$-space in sense of [8].

Proposition 4.9. *Let X be a non-empty set and let $\mathcal{U}$ be a fuzzy uniform structure on X and $\varphi_1, \varphi_2 \in O_{(X,\tau_{\mathcal{U}})}$. Then the associated stratified characterized fuzzy uniform space $(X, \varphi_{1,2}.\text{int}_{\mathcal{U}})$ with the fuzzy uniform structure $\mathcal{U}$ is characterized $FR_{2\frac{1}{2}}$-space.*

Proof. Let $x \in X$, $F \in \varphi_{1,2}C(X)$ such that $x \notin F$. Since $\chi_{F'}$ is $\varphi_{1,2}\mathcal{U}$-fuzzy neighborhood of x, $\mathcal{U}[\dot{x}] = \mathcal{N}_{\varphi_{1,2}}(x) \leq \dot{F}^{'}$. On account of Proposition 4.8, we get that x_1 and χ_F are Φ-separated by the uniformly fuzzy continuous function $f : (X, \mathcal{U}) \to$

$(I_L,\mathcal{U}^*)$. By Proposition 4.5, the function $f:(X,\varphi_{1,2}.\text{int}_{\mathcal{U}})\to(I_L,\psi_{1,2}.\text{int}_{\mathcal{U}^*})$ is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous. Consequently, $(X,\varphi_{1,2}.\text{int}_{\mathcal{U}})$ is characterized $FR_{2\frac{1}{2}}$-space. □

Corollary 4.10. *Let $(X,\mathcal{U})$ be a separated fuzzy uniform space and $\varphi_1,\varphi_2\in O_{(X,\tau_{\mathcal{U}})}$. Then the associated stratified characterized fuzzy uniform space $(X,\varphi_{1,2}.\text{int}_{\mathcal{U}})$ with the fuzzy uniform structure $\mathcal{U}$ is characterized $FT_{3\frac{1}{2}}$-space.*

Proof. Immediate from Propositions 4.1 and 4.9.

□

Example of stratified characterized fuzzy uniform $FR_{2\frac{1}{2}}$-space. In the following, we give an example of a homogeneous fuzzy uniform structure and we show that the associated stratified characterized fuzzy uniform space is characterized uniform $FR_{2\frac{1}{2}}$-space.

Example 4.11. The fuzzy metric in sense of S. Gähler and W. Gähler [14] canonically generate homogeneous fuzzy structure as follows: Consider X is non-empty set and d is fuzzy metric on X, then the mapping $\mathcal{U}_{\text{d}}:L^{X\times X}\to L$ which is defined by:

$$\mathcal{U}_{\text{d}}(R)=\bigvee_{0<\delta,\varepsilon_{\alpha,\delta}\circ\text{d}\le R}\alpha,$$

for all R $\in L^{X\times X}$ is homogeneous fuzzy uniform structures on X. Moreover, the associated stratified characterized fuzzy uniform space $(X,\varphi_{1,2}.\text{int}_{\mathcal{U}_{\text{d}}})$ is identical with the associated characterized fuzzy metrizable space $(X,\varphi_{1,2}.\text{int}_{\tau_{\text{d}}})$, that is, $(X,\varphi_{1,2}.\text{int}_{\mathcal{U}_{\text{d}}})=(X,\varphi_{1,2}.\text{int}_{\tau_{\text{d}}})$. By Proposition 3.1 in [9], $(X,\varphi_{1,2}.\text{int}_{\tau_{\text{d}}})$ is characterized FT_4-space and thus $(X,\varphi_{1,2}.\text{int}_{\mathcal{U}_{\text{d}}})$ is characterized FT_4-space. From Proposition 4.6 in [8], we get $(X,\varphi_{1,2}.\text{int}_{\mathcal{U}_{\text{d}}})$ is characterized $FR_{2\frac{1}{2}}$-space.

Remark 4.12. In Example 4.11, if we choose $\varphi_1=\text{int}_\tau$, $\varphi_2=1_{L^X}$, $\psi_1=\text{int}_{\Im}$ and $\psi_2=1_{L^I}$, the associated stratified characterized fuzzy uniform space $(X,\varphi_{1,2}.\text{int}_{\mathcal{U}_{\text{d}}})$ and the associated characterized fuzzy metrizable space $(X,\varphi_{1,2}.\text{int}_{\tau_{\text{d}}})$ are identical with the associated stratified fuzzy topological uniform space $(X,\tau_{\mathcal{U}_{\text{d}}})$ and the associated fuzzy metrizable topological space (X,τ_{d}), respectively.

Some special characterization of fuzzy pre filters. The fuzzy topogeneous order (resp. structure) $\ll$ will be called perfect [26], if for each family $(\mu_i)_{i\in I}$ of fuzzy subsets of X such that $\mu_i\ll\eta$, for all $i\in I$, it follows that $\bigvee\limits_{i\in I}\mu_i\le\eta$ holds, for some $\eta\in L^X$. As in [26], there is a one-to-one correspondence between the fuzzy perfect topoeneous structures $\ll$ on a set X and the characterized fuzzy spaces generated by the $\varphi_{1,2}$-interior operators $\varphi_{1,2}.\text{int}$ on X. This correspondence is given by:

$$\mu\ll\eta \text{ if and only if } \mu\le\rho\le\eta \text{ for some } \rho\in\varphi_{1,2}OF(X),$$

for all $\mu,\eta\in L^X$ and $\varphi_{1,2}OF(X)=\{\mu\in L^X:\mu\ll\mu\}$. The fuzzy perfect topoeneous structure $\ll$ in this case will be called characterized fuzzy perfect topoeneous structure.

Specially, let (X,τ) be a stratified fuzzy topological space, $\varphi_1,\varphi_2\in O_{(X,\tau)}$ and $\ll$ is characterized completely symmetric fuzzy perfect topoeneous structure on X

identified with the stratified characterized fuzzy space $(X, \varphi_{1,2}.\text{int}_\tau)$. Then for each $\alpha \in L_0$, let $R_\alpha : X \times X \to L$ be the mapping satisfied that:

$$(4.5)\quad R_\alpha(x,x) = 1 \text{ and } R_\alpha[\mu] = \begin{cases} \{\mu\} & \text{if } \mu \ll (\eta \wedge \bar{\alpha}) \text{for some} \eta \in \varphi_{1,2}OF(X), \\ \{\,\bar{1}\,\} & \text{otherwise,} \end{cases}$$

for all $x \in X$ and $\mu \in L^X$. As easily seen the mapping R_α fulfilled the following properties for all $\alpha \in L_0$:

(1) if $\varphi_2 \geq 1_{L^X}$, then $\mu \leq R_\alpha[\mu]$ holds, for all $\mu \in L^X$,

(2) $R_\alpha[\bar{\alpha}] = \bar{\alpha}$.

(3) if φ_2 is isotone and idempotent operation, then $R_\alpha \circ R_\alpha = R_\alpha$,

(4) if $\varphi_2 \geq 1_{L^X}$ is isotone and φ_1 is wfip with respect to $\varphi_1 OF(X)$, then $R_\alpha[\mu] = \mu$, for all $\mu \in \varphi_{1,2}OF(X)$.

Proposition 4.13. *Let $\mathcal{U}_\alpha$ be the set of all mappings R_α which is defined by (4.5) for all $\alpha \in L_0$. Then the family $\big(\mathcal{U}_\alpha\big)_{\alpha \in L_0}$ is a family of fuzzy pre filters on $X \times X$ satisfied the following axioms :*

(R1) *if $\alpha, \beta \in L_0$ with $\beta \geq \alpha$, then $\mathcal{U}_\alpha \subseteq \mathcal{U}_\beta$,*

(R2) *for each $\alpha \in L_0$ such that $\alpha = \bigvee\limits_{0<\beta<\alpha} \beta$, we have $\mathcal{U}_\alpha = \bigcap\limits_{0<\beta<\alpha} \mathcal{U}_\beta$,*

(R3) *for all $\alpha \in L_0$, $R_\alpha \in \mathcal{U}_\alpha$ and $x \in X$, we have $R_\alpha(x,x) \geq \alpha$,*

(R4) *if $R_\alpha \in \mathcal{U}_\alpha$, then $R_\alpha^{-1} \in \mathcal{U}_\alpha$,*

(R5) *for each $\alpha \in L_0$ and $R_\alpha \in \mathcal{U}_\alpha$, we have $\bigvee\limits_{R_\beta \circ R_\beta \leq R_\alpha, R_\beta \in \mathcal{U}_\beta} \beta \geq \alpha$.*

Proof. At first, we prove that $\mathcal{U}_\alpha$ is a fuzzy pre filter on $X \times X$, for all $\alpha \in L_0$. Consider $\tilde{o} : X \times X \to L$ is the mapping defined by : $\tilde{o}(x,y) = 0$, for all $x, y \in X$. Then, $\tilde{o}[\mu](x) = \bigvee\limits_{y \in X} \big(\mu(y) \wedge \tilde{o}(y,x)\big) = 0$, for all $\mu \in L^X$ and $x \in X$ and even that $\tilde{o}(x,x) = 0 \neq 1$. Thus, $\tilde{o} \notin \mathcal{U}_\alpha$.

Now, let $R_\alpha \in \mathcal{U}_\alpha$ and $R_\alpha \leq R_\beta$. Then, $R_\beta(x,x) = 1$, for all $x \in X$ and also $R_\alpha[\mu] \leq R_\beta[\mu]$ holds, for all $\mu \in L^X$. In case of $\mu \ll (\eta \wedge \bar{\alpha})$ for some $\eta \in \varphi_{1,2}OF(X)$, we have $\mu \leq \bar{\alpha}$ and $R_\beta(x,x) \geq \alpha$. So $R_\beta[\mu](x) = \bigvee\limits_{y \in X} \big(\mu(y) \wedge R_\beta(y,x)\big) = \mu(x)$, for all $x \in X$, that is, $R_\beta[\mu] = \mu$. Otherwise, if $\mu \ll (\eta \wedge \bar{\alpha})$ does not holds, for all $\eta \in \varphi_{1,2}OF(X)$, we get $R_\beta[\mu] \geq R_\alpha[\mu] = \bar{1}$ holds, for all $\mu \in L^X$. Hence $R_\beta \in \mathcal{U}_\alpha$.

Consider $R_\alpha, R_\beta \in \mathcal{U}_\alpha$. Then $(R_\alpha \wedge R_\beta)(x,x) = R_\alpha(x,x) \wedge R_\beta(x,x) = 1$, for all $x \in X$. Since

$$\begin{aligned} (R_\alpha \wedge R_\beta)[\mu](x) &= \bigvee_{y \in X} \Big(\mu(y) \wedge (R_\alpha \wedge R_\beta)(y,x)\Big) \\ &= \bigvee_{y \in X} \Big(\mu(y) \wedge R_\alpha(y,x)\Big) \wedge \bigvee_{y \in X} \Big(\mu(y) \wedge R_\beta(y,x)\Big) \\ &= R_\alpha[\mu](x) \wedge R_\beta[\mu](x), \end{aligned}$$

for all $\mu \in L^X$ and $x \in X$, $(R_\alpha \wedge R_\beta)[\mu] = R_\alpha[\mu] \wedge R_\beta[\mu]$ for all $\mu \in L^X$. In case of $\mu \ll (\eta \wedge \bar{\alpha})$ for some $\eta \in \varphi_{1,2}OF(X)$, we have $(R_\alpha \wedge R_\beta)[\mu] = \mu$. Otherwise, $(R_\alpha \wedge R_\beta)[\mu] = \bar{1}$. Thus, $(R_\alpha \wedge R_\beta) \in \mathcal{U}_\alpha$. So, $\mathcal{U}_\alpha$ is a fuzzy pre filter on $X \times X$. Hence the family $\big(\mathcal{U}_\alpha\big)_{\alpha \in L_0}$ is a family of fuzzy pre filters on $X \times X$.

Now, consider $\alpha, \beta \in L_0$ such that $\beta \geq \alpha$. Then in case of $\mu \ll (\eta \wedge \bar{\alpha})$ for some $\eta \in \varphi_{1,2}OF(X)$, we have $R_\alpha[\mu] = \mu$ and $\mu \ll (\eta \wedge \bar{\alpha}) \leq (\eta \wedge \bar{\beta})$ implies $\mu \ll (\eta \wedge \bar{\beta})$ and thus $R_\beta[\mu] = \mu = R_\alpha[\mu]$. The other case of $\mu \ll (\eta \wedge \bar{\beta})$ for some $\eta \in \varphi_{1,2}OF(X)$, we get $R_\beta[\mu] = \mu \leq R_\alpha[\mu]$. So $R_\beta[\mu] \leq R_\alpha[\mu]$ holds, for all $\mu \in L^X$. In every cases of $\mu \in L^X$, it easily seen that also $R_\beta[\mu] \leq R_\alpha[\mu]$ holds. Hence, $\mathcal{U}_\alpha \subseteq \mathcal{U}_\beta$. Therefore (R1) is fulfilled.

To fulfills (R2), let $\alpha \in L_0$ and $\alpha = \bigvee_{0<\beta<\alpha} \beta$. Because of (R1), we get $\bigcap_{0<\beta\alpha} \mathcal{U}_\beta \subseteq \mathcal{U}_\alpha$. In case of $\mu \ll (\eta \wedge \bar{\alpha})$ for some $\eta \in \varphi_{1,2}OF(X)$, we have $R_\alpha[\mu] = \mu \leq R_\beta[\mu]$ holds, for all $0 < \beta < \alpha$ and $\alpha = \bigvee_{0<\beta<\alpha} \beta$. In case of $\mu \ll (\eta \wedge \bar{\alpha})$ does not holds for all $\eta \in \varphi_{1,2}OF(X)$, we get that $R_\alpha[\mu] = \bar{1} \leq \bigcap_{0<\beta\alpha} R_\beta[\mu] = \bar{1}$, that is, $\bigcap_{0<\beta\alpha} R_\beta[\mu] \geq R_\alpha[\mu]$, for all $\mu \in L^X$. Then $\mathcal{U}_\alpha \subseteq \bigcap_{0<\beta\alpha} \mathcal{U}_\beta$. Thus $\mathcal{U}_\alpha = \bigcap_{0<\beta\alpha} \mathcal{U}_\beta$. So, (R2) is fulfilled.

To prove (R3), from the fact that $R_\alpha[\bar{1}] = \bar{1}$, for all $\alpha \in L_0$, we get

$$R_\alpha[\bar{1}](x) = \bigvee_{y \in X} \Big(\bar{1}(y) \wedge R_\alpha(y,x)\Big) = \bigvee_{y \in X} R_\alpha(y,x) = R_\alpha(x,x) = 1,$$

for all $x \in X$, that is, $R_\alpha(x,x) \geq \alpha$, for all $\alpha \in L_0$, $R_\alpha \in \mathcal{U}_\alpha$ and $x \in X$. Then, (R3) is fulfilled.

Now, let $R_\alpha \in \mathcal{U}_\alpha$. Since $\mu \ll (\eta \wedge \bar{\alpha})$ holds for some $\eta \in \varphi_{1,2}OF(X)$, $\mu \leq \bar{\alpha}$ holds for all $\alpha \in L_0$. Then we get that $R_\alpha[\mu](x) = \bigvee_{y \in X} \Big(\mu(y) \wedge R_\alpha(y,x)\Big) = \mu(x)$, because $\mu(x) \leq \alpha$ and $R_\alpha(x,x) \geq \alpha$ for all $x \in X$. In case of $R_\alpha[\mu] = \bar{1}$, we get $R_\alpha[\mu](x) = \bigvee_{y \in X} \Big(\mu(y) \wedge R_\alpha(y,x)\Big) = R_\alpha(x,x) = 1$, for all $x \in X$. Since

$$R_\alpha^{-1}[\mu](x) = \bigvee_{y \in X} \Big(\mu(y) \wedge R_\alpha^{-1}(y,x)\Big) = \bigvee_{y \in X} \Big(\mu(y) \wedge R_\alpha(x,y)\Big),$$

$R_\alpha^{-1}[\mu](x) = \mu(x)$, for all $x \in X$ in case of $\mu \ll (\eta \wedge \bar{\alpha})$ for some $\eta \in \varphi_{1,2}OF(X)$. Otherwise, $R_\alpha^{-1}[\mu](x) = R_\alpha(x,x) = 1$, for all $x \in X$. Thus, $R_\alpha^{-1} \in \mathcal{U}_\alpha$, for all $R_\alpha \in \mathcal{U}_\alpha$. So (R4) is also fulfilled.

Finally, since $R_\alpha \circ R_\alpha = R_\alpha$, for all $\alpha \in L_0$ and $R_\alpha \in \mathcal{U}_\alpha$,

$$\alpha \leq \bigvee_{R_\beta \in \mathcal{U}_\beta, R_\beta \leq R_\alpha} \beta = \bigvee_{R_\beta \in \mathcal{U}_\beta, R_\beta \circ R_\beta \leq R_\alpha} \beta.$$

Then (R5) is fulfilled.

Consequently, $\big(\mathcal{U}_\alpha\big)_{\alpha \in L_0}$ is a family of fuzzy pre filters on $X \times X$ fulfilled the axioms (R1) to (R5). □

Now, we have the following important result which show that the associated stratified characterized fuzzy uniform spaces with the fuzzy uniform structures are compatible with the stratified characterized $FR_{2\frac{1}{2}}$-spaces.

Proposition 4.14. *Let (X, τ) be a fuzzy topological space, $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$, $\psi_1, \psi_2 \in O_{(L^I,\Im)}$ and let Φ be the fuzzy function family of all $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous functions on X. If the characterized fuzzy space $(X, \varphi_{1,2}.\text{int}_\tau)$ is stratified characterized*

$FR_{2\frac{1}{2}}$-space, then the mapping $\mathcal{U} : L^{X\times X} \to L$ which is defined by:

$$\mathcal{U}(R_\alpha) = \bigvee_{R_\beta \in \mathcal{U}_\alpha, R_\beta \leq R_\alpha} \alpha \quad \text{for all } R_\alpha \in L^{X\times X},$$

where $\mathcal{U}_\alpha$ is the set of all mappings R_α defined by (4.5) is a fuzzy uniform structure on X and the associated stratified characterized fuzzy uniform space $(X, \varphi_{1,2}.\text{int}_{\tau_\mathcal{U}})$ with $\mathcal{U}$ is compatible with $(X, \varphi_{1,2}.\text{int}_\tau)$.

Proof. Because of Proposition 4.13, $\mathcal{U}$ is fuzzy uniform structure on X. Now let μ is $\varphi_{1,2}$-open with respect to $\varphi_{1,2}.\text{int}_{\tau_\mathcal{U}}$ such that $\mu \neq \bar{1}$ and $\mu(x) = 1$. Then, $\varphi_{1,2}.\text{int}_{\tau_\mathcal{U}} \mu(x) = \bigvee_{R_\alpha[\eta]\leq\mu} \big(\mathcal{U}_\alpha(R_\alpha, \eta(x)\big) = 1$, that is, there is some $R_{\alpha_0} \in \mathcal{U}_\alpha$ with $\mathcal{U}(R_{\alpha_0}) = 1 \geq \alpha$ such that $R_{\alpha_0}[\eta] = \eta \leq \mu$ and $\eta \ll (\rho \wedge \bar{\alpha})$ for some $\rho \in \varphi_{1,2}OF(X)$. This means that $\rho = (\mu \wedge \bar{\alpha})$ is $\varphi_{1,2}$-open with respect to $\varphi_{1,2}.\text{int}_\tau$ and fulfilled that $\eta \leq \rho \leq \mu$, $\eta(x) = 1$ and $\rho \in \varphi_{1,2}OF(X)$, that is, $\rho(x) = 1$, $\rho \leq \mu$ and $\rho \in \varphi_{1,2}OF(X)$. Thus, μ is $\varphi_{1,2}$-open with respect to $\varphi_{1,2}.\text{int}_\tau$. So, $\varphi_{1,2}.\text{int}_{\tau_\mathcal{U}}(\mu) \geq \varphi_{1,2}.\text{int}_\tau(\mu)$ holds for all $\mu \in L^X$. Hence $\varphi_{1,2}.\text{int}_{\tau_\mathcal{U}} \leq \varphi_{1,2}.\text{int}_\tau$.

Conversely, let μ is $\varphi_{1,2}$-open with respect to $\varphi_{1,2}.\text{int}_\tau$ such that $\mu \neq \bar{1}$ and $\varphi_{1,2}.\text{int}_{\tau_\mathcal{U}}(\mu) \neq \mu$. Then there is $x \in X$ such that $\varphi_{1,2}.\text{int}_{\tau_\mathcal{U}} \mu(x) = 0$ and $\mu(x) > 0$. Since $x \in F^{'} = S_0\mu \in \varphi_{1,2}O(X)$ and $(X, \varphi_{1,2}.\text{int}_\tau)$ is stratified characterized $FR_{2\frac{1}{2}}$-space, there exists a $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous function $f : (X, \varphi_{1,2}.\text{int}_\tau) \to (I_L, \psi_{1,2}.\text{int}_\Im)$ such that $f(x) = \bar{1}$ and $f(y) = \bar{0}$ for all $y \in F$. Consider $\sigma \in L^X$ defined by:

$$\sigma(y) = \big(R^1(f(y))\big)^{'} = \bigvee_{\alpha\geq 1} f(y)(\alpha) \text{ for all} y \in X.$$

Then $\sigma(y) \leq R_0(f(y)) \leq \mu(y)$, for all $y \in X$. This means that $\bigvee_{\alpha\geq 1} f(x)(\alpha) = 1$, for all $x_1 \leq \sigma$ and $\bigvee_{\beta\geq 0} f(y)(\beta) = 0$, for all $y_1 \leq \mu^{'}$, that is, $f(x) = \bar{1}$, for all $x_1 \leq \sigma$ and $f(y) = \bar{0}$, for all $y_1 \leq \mu^{'}$. Thus, x_1 and $\mu^{'}$ are Φ-separated, for all $x_1 \leq \sigma$. So σ and $\mu^{'}$ are Φ-separated. Because of (2.8) and Proposition 2.3, we get $\sigma \ll \mu$ and $\sigma(x) = 1$. Now,

$$\varphi_{1,2}.\text{int}_{\tau_\mathcal{U}} \mu(x) = \bigvee_{R_\alpha[\eta]\leq\mu} \big(\mathcal{U}_\alpha(R_\alpha, \eta(x)\big) \geq \bigvee_{R_\alpha[\eta]\leq\mu} \eta(x),$$

for some $R_\alpha \in \mathcal{U}_\alpha$ with $\mathcal{U}(R_\alpha) = 1 \geq \alpha$, which means that

$$\varphi_{1,2}.\text{int}_{\tau_\mathcal{U}} \mu(x) \geq \bigvee_{\eta\ll\mu, \mu\in\varphi_{1,2}OF(X)} \eta(x)$$

and is also fulfilled when replacing η by σ, that is,

$$\varphi_{1,2}.\text{int}_{\tau_\mathcal{U}} \mu(x) \geq \bigvee_{\eta\ll\mu, \mu\in\varphi_{1,2}OF(X)} \eta(x) \geq \sigma(x) = 1.$$

So $\varphi_{1,2}.\text{int}_{\tau_\mathcal{U}} \mu(x) = 1 > 0$, which is a contradiction and thus $\varphi_{1,2}.\text{int}_{\tau_\mathcal{U}} \mu = \mu$. Hence, $\varphi_{1,2}.\text{int}_\tau \mu \geq \varphi_{1,2}.\text{int}_{\tau_\mathcal{U}} \mu$ holds, for all $\mu \in L^X$. Therefore, $\varphi_{1,2}.\text{int}_\tau \leq \varphi_{1,2}.\text{int}_{\tau_\mathcal{U}}$. Consequently, $(X, \varphi_{1,2}.\text{int}_{\tau_\mathcal{U}}) = (X, \varphi_{1,2}.\text{int}_\tau)$. $\square$

5. New representations for the characterized fuzzy compact spaces by characterized $FT_{3\frac{1}{2}}$-space

The notion of $\varphi_{1,2}$-fuzzy compactness of the fuzzy filters and of the fuzzy topological spaces are introduced in [7] by means of the $\varphi_{1,2}$-fuzzy convergence in the characterized fuzzy spaces. Moreover, the fuzzy compactness in the characterized fuzzy spaces is also introduced by means of the $\varphi_{1,2}$-fuzzy compactness of the fuzzy filters and therefore it will be suitable to study here the relation between the characterized fuzzy compact spaces and some of our classes of fuzzy separation axioms in the characterized fuzzy spaces.

Let (X, τ) be a fuzzy topological space, $F \subseteq X$ and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then $x \in X$ is said to be $\varphi_{1,2}$-adherence point for the fuzzy filter $\mathcal{M}$ on X [7], if the infimum $\mathcal{M} \wedge \mathcal{N}_{\varphi_{1,2}}(x)$ exists for all $\varphi_{1,2}$-fuzzy neighborhood filters $\mathcal{N}_{\varphi_{1,2}}(x)$ at $x \in X$. As shown in [7], the point $x \in X$ is said to be $\varphi_{1,2}$-adherence point for the fuzzy filter $\mathcal{M}$ on X, if there exists a fuzzy filter $\mathcal{K} \in \mathcal{F}_L X$ finer than $\mathcal{M}$ and $\mathcal{K} \xrightarrow[\varphi_{1,2}.\text{int}]{} x$, that is, $\mathcal{K} \leq \mathcal{M}$ and $\mathcal{K} \leq \mathcal{N}_{\varphi_{1,2}}(x)$ are hold for some $\mathcal{K} \in \mathcal{F}_L X$. The ordinary subset F is said to be $\varphi_{1,2}$-closed with respect to $\varphi_{1,2}$.int, if $\mathcal{M} \leq \mathcal{N}_{\varphi_{1,2}}(x)$ implies $x \in F$ for some $\mathcal{M} \in \mathcal{F}_L F$. The subset F is said to be $\varphi_{1,2}$-fuzzy compact subset [6], if every fuzzy filter on F has a finer $\varphi_{1,2}$-fuzzy converging fuzzy filter, that is, every fuzzy filter on F has $\varphi_{1,2}$-adherence point in F. Moreover, the fuzzy topological space (X, τ) is said to be $\varphi_{1,2}$-fuzzy compact, if X is $\varphi_{1,2}$-fuzzy compact. Generally, the characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$ is said to be characterized fuzzy compact space, if the related fuzzy topological space (X, τ) is $\varphi_{1,2}$-fuzzy compact.

Proposition 5.1 ([5])**.** *Let a fuzzy topological space (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(X,\tau)}$. Then every $\varphi_{1,2}$-fuzzy compact subset of a characterized FT_2-space $(X, \varphi_{1,2}.\text{int})$ is $\varphi_{1,2}$-fuzzy closed and every characterized compact FT_2-space $(X, \varphi_{1,2}.\text{int})$ is characterized FT_4-space. Moreover, every $\varphi_{1,2}$-fuzzy closed subset of a characterized fuzzy compact space $(X, \varphi_{1,2}.\text{int})$ is $\varphi_{1,2}$-fuzzy compact.*

In the following at first we shall benefit from these facts. Consider the fuzzy unit interval topological space $(I_L, \Im)$ be given and $\psi_1, \psi_2 \in O_{(I_L,\Im)}$. Then:

(1) the usual topological space (I, T_I) and the ordinary characterized usual space $(I, \psi_{1,2}.\text{int}_{T_I})$ on the closed unite interval $I = [0, 1]$ are compact $\psi_{1,2}T_2$-space and characterized compact T_2-space, respectively in the classical sense,

(2) the closed unite interval I is identified with the fuzzy number $[0, 1]^\sim$ in [14] defined by $[0, 1]^\sim(\alpha) = 0$, for all $\alpha \in I$ and $[0, 1]^\sim(\alpha) = 0$, for all $\alpha \notin I$,

(3) the characterized fuzzy unite space $(I_L, \psi_{1,2}.\text{int}_\Im)$ is up to a identification the characterized usual space $(I, \psi_{1,2}.\text{int}_{T_I})$ in the classical sense.

Proposition 5.2. *Let $(I_L, \Im)$ be a fuzzy unit interval topological space and $\psi_1, \psi_2 \in O_{(L^{I_L},\Im)}$. Then the characterized fuzzy unit interval space $(I_L, \psi_{1,2}.\text{int}_\Im)$ is characterized fuzzy compact FT_2-space.*

Proof. Let $(I, \psi_{1,2}.\text{int}_{T_I})$ be an ordinary characterized usual space. Then, $(I, \psi_{1,2}.\text{int}_{T_I})$ is characterized compact space in the classical sense, that is, every filter on I has $\psi_{1,2}$-adherence point. Consider the mapping $f : (I, \psi_{1,2}.\text{int}_{T_I}) \to (I_L, \psi_{1,2}.\text{int}_\Im)$ defined by: $f(\alpha) = \tilde{\alpha}$, for all $\alpha \in I$. Then it is easily to seen that f is $\psi_{1,2}\psi_{1,2}$-fuzzy homeomorphism between $(I, \psi_{1,2}.\mathrm{int}_{T_I})$ and $(I_L, \psi_{1,2}.\mathrm{int}_{\Im})$. Thus, $(I_L, \psi_{1,2}.\mathrm{int}_{\Im})$ is characterized fuzzy compact space. Since (I, T_I) is $\psi_{1,2}T_2$-space, $(I, \psi_{1,2}.\mathrm{int}_{T_I})$ is characterized T_2-space. So by using the same $\psi_{1,2}\psi_{1,2}$-fuzzy homeomorphism, we have for all $\tilde{\alpha}, \tilde{\beta} \in I_L$ such that $\tilde{\alpha} \neq \tilde{\beta}$, the infimum $\mathcal{N}_{\psi_{1,2}}(\tilde{\alpha}) \wedge \mathcal{N}_{\psi_{1,2}}(\tilde{\beta})$ does not exists. Hence, $(I_L, \psi_{1,2}.\mathrm{int}_{\Im})$ is characterized FT_2-space. Therefore $(I_L, \psi_{1,2}.\mathrm{int}_{\Im})$ is characterized fuzzy compact FT_2-space. □

For more generally we have the following result:

Proposition 5.3. *Let $(I_L, \Im)$ be a fuzzy unit interval topological space and $\psi_1, \psi_2 \in O_{(L^{I_L}, \Im)}$. Then the characterized fuzzy unit interval space $(I_L, \psi_{1,2}.\mathrm{int}_{\Im})$ is characterized $FT_{3\frac{1}{2}}$-space.*

Proof. Because of Proposition 5.2, the characterized fuzzy unit interval space $(I_L, \psi_{1,2}.\mathrm{int}_{\Im})$ is characterized fuzzy compact FT_2-space. Then from Proposition 5.1, we get $(I_L, \psi_{1,2}.\mathrm{int}_{\Im})$ is characterized FT_4-space. Thus, Proposition 4.6 in [8] gives us that, $(I_L, \psi_{1,2}.\mathrm{int}_{\Im})$ is characterized $FT_{3\frac{1}{2}}$-space. □

Because of the $\varphi_{1,2}$-fuzzy compactness in the characterized fuzzy spaces the Generalized Tychonoff Theorem is fulfilled ([8]) and from (3) in Proposition 2.4, the characterized fuzzy product space of the characterized FT_2-spaces is also characterized FT_2-space. Then, by means of Propositions 5.1 and 5.2, the following result goes clear.

Proposition 5.4. *Let $(I_L, \Im)$ be a fuzzy unit interval topological space and $\psi_1, \psi_2 \in O_{(L^{I_L}, \Im)}$. Then the characterized fuzzy cube is characterized FT_2-space and it is characterized FT_4-space.*

Proof. Since the characterized fuzzy cube is product of copies of $(I_L, \psi_{1,2}.\mathrm{int}_{\Im})$ and by means of Proposition 5.2, $(I_L, \psi_{1,2}.\mathrm{int}_{\Im})$ is characterized fuzzy compact FT_2-space. Because of Proposition 2.4 part (3) and Generalized Tychonoff Theorem in [8], it follows that, the characterized fuzzy cube is characterized FT_2-space. Moreover, Proposition 5.1 implies that the characterized fuzzy cube is characterized FT_4-space. □

Proposition 5.5. *Let (X, τ) be a fuzzy topological spaces, $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$ and $\psi_1, \psi_2 \in O_{(L^{I_L}, \Im)}$. Consider Φ is the family of all $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous functions $f : (X, \varphi_{1,2}.\mathrm{int}_\tau) \to (I_L, \psi_{1,2}.\mathrm{int}_{\Im})$ and for each $f \in \Phi$, let Y_f denote the characterized fuzzy unit interval space and $Y = \prod\limits_{f \in \Phi} Y_f$ with the characterized fuzzy product space generated by $\psi_{1,2}.\mathrm{int}_{\Im_Y}$ on it. If $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ is characterized $FT_{3\frac{1}{2}}$-space, then X is $\varphi_{1,2}\psi_{1,2}$-fuzzy homeomorphic to a characterized fuzzy subspace of Y. More precisely, the mapping $e : X \to Y$, $e(x) = \hat{x} = \prod\limits_{f \in \Phi} x_f$, $x_f(x) = f(x)$ is a $\varphi_{1,2}\psi_{1,2}$-fuzzy homeomorphism from X into $e(X)$, when $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ is characterized $FT_{3\frac{1}{2}}$-space.*

Proof. Let $(X, \varphi_{1,2}.\mathrm{int}_\tau)$ be characterized $FT_{3\frac{1}{2}}$-space and consider the evaluation mapping $e : X \to Y$ defined by: $x \mapsto \big(f(x)\big)_{f \in \Phi} = \hat{x}$ for $x \in X$. Because of Corollary 5.1 in [8], e is injective. Since $f \in \Phi$ and the projection mapping $p_f : Y \hookrightarrow Y_f$ is

$\psi_{1,2}\psi_{1,2}$-fuzzy continuous, $p_f \circ e : x \mapsto f(x)$ is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous. Consider $e(X) = Z$. Then $e : (X, \varphi_{1,2}.\text{int}_\tau) \to (Z, \psi_{1,2}.\text{int}_{\Im_Z})$ is bijective $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping.

Now, we show that e is $\varphi_{1,2}\psi_{1,2}$-fuzzy open mapping. As in the proof of Proposition 5.2 in [8], the family

$$\Omega = \{f^{-1}(\mu) \,:\, f \in \Phi \text{ and } \mu = \psi_{1,2}.\text{int}_\Im \mu\}$$

is a base for the characterized fuzzy space $(X, \varphi_{1,2}.\text{int}_\tau)$. Since for a family $(\mu_j)_{j\in J}$ of fuzzy sets in X, we have

$$e(\bigvee_{j\in J} \mu_j) = \bigvee_{j\in J} e(\mu_j) \text{ and } e(\mu_1 \wedge \ldots \wedge \mu_n) = e(\mu_1) \wedge \ldots \wedge e(\mu_n).$$

To show that e is $\varphi_{1,2}\psi_{1,2}$-fuzzy open mapping, it is sufficient to show that $e(\rho)$ is $\psi_{1,2}$-open in $(Z, \psi_{1,2}.\text{int}_{\Im_Z})$, for all $\rho \in \Omega$. Let $f \in \Phi$, $\mu = \psi_{1,2}.\text{int}_\Im \mu$ in Y_f and $\rho = f^{-1}(\mu) = \mu \circ f$ with $\rho \in \Omega$. Then,

$$e(\rho)(\hat{x}) = \bigvee_{x \in e^{-1}(\hat{x})} \rho(x) = \rho(x) = \mu(f(x)) = \bigvee_{\hat{x} \in p_f^{-1}(f(x))} \mu(f(x)) = p_f^{-1}(\mu)(\hat{x}),$$

for all $\hat{x} = e(x) \in Z$. Since $p_f^{-1}(\mu)|_Z = e(\rho)$ and p_f is $\psi_{1,2}\psi_{1,2}$-fuzzy continuous, $p_f^{-1}(\mu)|_Z$ is $\psi_{1,2}$-open in $(Z, \psi_{1,2}.\text{int}_{\Im_Z})$, that is, $e(\rho)$ is $\psi_{1,2}$-open in $(Z, \psi_{1,2}.\text{int}_{\Im_Z})$. Thus, e is $\varphi_{1,2}\psi_{1,2}$-fuzzy open mapping. So $e : (X, \varphi_{1,2}.\text{int}_\tau) \to (Z, \psi_{1,2}.\text{int}_{\Im_Z})$ is $\varphi_{1,2}\psi_{1,2}$-fuzzy homeomorphism. Hence, $(X, \varphi_{1,2}.\text{int}_\tau)$ is $\varphi_{1,2}\psi_{1,2}$-fuzzy homeomorphic to a characterized fuzzy subspace of $Y = \prod_{f\in\Phi} Y_f$. □

Proposition 5.6. *Let (X, τ) be a fuzzy topological spaces, $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$ and $\psi_1, \psi_2 \in O_{(L^{I_L},\Im)}$. Then $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized $FT_{3\frac{1}{2}}$-space if and only if X is $\varphi_{1,2}\psi_{1,2}$-fuzzy homeomorphic to a characterized fuzzy subspace of the characterized fuzzy cub.*

Proof. The necessary of the condition follows from Proposition 5.5. For the sufficiency, because of Proposition 5.3, $(I_L, \psi_{1,2}.\text{int}_\Im)$ is characterized $FT_{3\frac{1}{2}}$-space. Because of Corollary 4.2 in [9], the characterized fuzzy product space of a characterized $FT_{3\frac{1}{2}}$-space is characterized $FT_{3\frac{1}{2}}$-space. Then, $(X, \varphi_{1,2}.\text{int}_\tau)$ itself characterized $FT_{3\frac{1}{2}}$-space. □

Proposition 5.7. *Let (X, τ) be a fuzzy topological spaces and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then every characterized fuzzy compact space $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized FT_2-space if and only if it is characterized $FT_{3\frac{1}{2}}$-space.*

Proof. Let $(X, \varphi_{1,2}.\text{int}_\tau)$ be characterized fuzzy compact FT_2-space. Then by Proposition 5.1, $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized FT_4-space. Thus by Proposition 4.6 in [8], $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized $FT_{3\frac{1}{2}}$-space.

Conversely, let $(X, \varphi_{1,2}.\text{int}_\tau)$ be characterized $FT_{3\frac{1}{2}}$-space. then by Propositions 3.2 in [8] and 2.4 part (1), it follows that $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized fuzzy compact FT_2-space. □

Lemma 5.8 (5)**.** *Let (X, τ) and (X, σ) be two fuzzy topological spaces such that τ is finer than σ, $\varphi_1, \varphi_2 \in O_{(X,\tau)}$ and $\psi_1, \psi_2 \in O_{(X,\sigma)}$. If $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized fuzzy compact space, then $(X, \psi_{1,2}.\text{int}_\sigma)$ is also characterized fuzzy compact space.*

From Lemma 5.8 and Corollary 3.3 in [9], we can prove the following result.

Proposition 5.9. *Let (X,τ) and (X,σ) be two fuzzy topological spaces such that τ is finer than σ, $\varphi_1,\varphi_2 \in O_{(X,\tau)}$ and $\psi_1,\psi_2 \in O_{(X,\sigma)}$. If $(X,\varphi_{1,2}.\mathrm{int}_\tau)$ is characterized fuzzy compact space and $(X,\psi_{1,2}.\mathrm{int}_\sigma)$ is characterized $FT_{3\frac{1}{2}}$-space, then $(X,\varphi_{1,2}.\mathrm{int}_\tau)$ and $(X,\psi_{1,2}.\mathrm{int}_\sigma)$ are equivalent.*

Proof. Because of Corollary 3.3 in [9], we get $(X,\varphi_{1,2}.\mathrm{int}_\tau)$ is characterized $FT_{3\frac{1}{2}}$-space. By Lemma 5.8, we have $(X,\psi_{1,2}.\mathrm{int}_\sigma)$ is also characterized fuzzy compact space. Then, the identity mapping $\mathrm{id}_X : (X,\varphi_{1,2}.\mathrm{int}_\tau) \to (X,\psi_{1,2}.\mathrm{int}_\sigma)$ is bijective $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous and $\varphi_{1,2}\psi_{1,2}$-fuzzy open, that is, id_X is $\varphi_{1,2}\psi_{1,2}$-fuzzy homeomorphism. Thus, $(X,\varphi_{1,2}.\mathrm{int}_\tau)$ and $(X,\psi_{1,2}.\mathrm{int}_\sigma)$ are equivalent. □

From Propositions 2.4 and 5.6, we have the following important characterization for the characterized $FT_{3\frac{1}{2}}$-spaces.

Theorem 5.10. *Let (X,τ) be a fuzzy topological space and $\varphi_1,\varphi_2 \in O_{(X,\tau)}$. Then, the following axioms are equivalent:*

(1) *$(X,\varphi_{1,2}.\mathrm{int}_\tau)$ is characterized $FT_{3\frac{1}{2}}$-space,*

(2) *$(X,\varphi_{1,2}.\mathrm{int}_\tau)$ is $\varphi_{1,2}\psi_{1,2}$-fuzzy homeomorphic to a characterized fuzzy subspace of characterized fuzzy cub,*

(3) *$(X,\varphi_{1,2}.\mathrm{int}_\tau)$ is $\varphi_{1,2}\psi_{1,2}$-fuzzy homeomorphic to a characterized fuzzy subspace of characterized fuzzy compact FT_2-space,*

(4) *$(X,\varphi_{1,2}.\mathrm{int}_\tau)$ is $\varphi_{1,2}\psi_{1,2}$-fuzzy homeomorphic to a characterized subspace of characterized FT_4-space.*

Proof. Let $(X,\varphi_{1,2}.\mathrm{int}_\tau)$ be characterized $FT_{3\frac{1}{2}}$-space. Then by Proposition 5.6, it follows that (2) is fulfilled. Thus, (1) implies (2). Consider (2) is fulfilled. since every characterized fuzzy cub is characterized fuzzy compact FT_2-space, (3) is fulfilled. So, (2) implies (3). Obviously, (3) implies (4), because every characterized fuzzy compact FT_2-space is characterized FT_4-space.

Finally, let (4) be fulfilled. Then by Proposition 2.4, every characterized fuzzy subspace of a characterized FT_4-space is characterized FT_4. Thus Proposition 4.6 in [8], $(X,\varphi_{1,2}.\mathrm{int}_\tau)$ is characterized $FT_{3\frac{1}{2}}$-space. So, (1) is fulfilled. Hence (4) implies (1). □

6. Conclusions

In this research work, we introduced and studied four new notions. The notions are named characterized global fuzzy neighborhood space, characterized global fuzzy neighborhood pre space, characterized fuzzy uniform space and characterized fuzzy perfect topoeneous structure. The properties of such characterized fuzzy spaces were deeply studied. Some sorts of relationship were introduced among such characterized fuzzy spaces and other published characterized fuzzy spaces presented by the authors. Each global fuzzy neighborhood structure introduced a characterized global fuzzy neighborhood space, however each global fuzzy neighborhood pre structure is identified with a characterized global fuzzy neighborhood pre space. In case of the homogenous global fuzzy neighborhood structures and of the homogenous global fuzzy neighborhood pre structures the stratified characterized global fuzzy neighborhood spaces and the stratified characterized global fuzzy neighborhood pre spaces were introduced. We proved that the mappings between the characterized fuzzy pre spaces are $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous if the related mappings between the global fuzzy neighborhood pre spaces are (h, k)-continuous. The vise versa is true when h and k are coincide up to identifications with $\varphi_{1,2}.\mathrm{int}_{\tau_h}$ and $\psi_{1,2}.\mathrm{int}_{\tau_k}$, respectively. The fuzzy uniform spaces are separated if and only if the associated characterized fuzzy uniform spaces are characterized FT_1-spaces. The mappings between the associated characterized fuzzy uniform spaces are $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous if the related mappings between the fuzzy uniform spaces are fuzzy uniform continuous. For each fuzzy uniform structure on a set X, there is an induced stratified fuzzy proximity on L^X and both the fuzzy uniform structure and this induced stratified fuzzy proximity are associated with the same stratified characterized fuzzy uniform space. The associated stratified characterized fuzzy uniform space with the fuzzy uniform structure is characterized $FR_{2\frac{1}{2}}$-space and in case of the fuzzy uniform space is separated, then it is characterized $FT_{3\frac{1}{2}}$-space. The relation between characterized fuzzy compact spaces which introduced in [7] and some our of characterized FT_s-spaces for $s \in \{2, 3\frac{1}{2}, 4\}$ are introduced by means of the characterized fuzzy unit interval spaces, the characterized fuzzy FT_2 and the characterized FT_4 fuzzy cubes. Finally, we showed that the characterized fuzzy compact spaces and the characterized $FT_{3\frac{1}{2}}$-spaces are equivalent. Many new special classes from the characterized fuzzy perfect topoeneous structures, characterized global fuzzy neighborhood spaces, characterized fuzzy proximity spaces, characterized fuzzy compact spaces and characterized fuzzy uniform spaces are listed in Table 1.

	Operations	Character. Fuzzy Perfect topogeneous Structure	Character. Global Fuzzy Neighborhood Space	Character. Fuzzy Proximity Space	Character. Fuzzy Compact Space	Character. Fuzzy Uniform Space
1	$\varphi_1 = \mathrm{int}$ $\varphi_2 = 1_{L^X}$	Fuzzy perfect topogeneous Str.,[19]	Global fuzzy neighborhood space[18]	Fuzzy Proximity space [10]	Fuzzy Compact space [10]	Fuzzy Uniform space [10]
2	$\varphi_1 = \mathrm{int}$ $\varphi_2 = \mathrm{cl}$	Fuzzy perfect θ- topogeneous str.	Global fuzzy θ- neighborhood space	Fuzzy proximity θ-space	Fuzzy compact θ-space	Fuzzy uniform θ-space
3	$\varphi_1 = \mathrm{int}$ $\varphi_2 = \mathrm{int} \circ \mathrm{cl}$	Fuzzy perfect δ- topogeneous str.	Global fuzzy δ- neighborhood space	Fuzzy proximity δ-space	Fuzzy compact δ-space	Fuzzy uniform δ-space
4	$\varphi_1 = \mathrm{cl} \circ \mathrm{int}$ $\varphi_2 = 1_{L^X}$	Fuzzy perfect semi topogeneous str.	Global fuzzy Semi neighborhood space	Fuzzy proximity semi space	Fuzzy compact semi space	Fuzzy uniform semi space
5	$\varphi_1 = \mathrm{cl} \circ \mathrm{int}$ $\varphi_2 = \mathrm{cl}$	Fuzzy perfect θ- semi topogeneous str.	Global fuzzy θ- semi neighborhood space	Fuzzy proximity θ- semi space	Fuzzy compact θ- semi space	Fuzzy uniform θ- semi space
6	$\varphi_1 = \mathrm{cl} \circ \mathrm{int}$ $\varphi_2 = \mathrm{int} \circ \mathrm{cl}$	Fuzzy perfect δ-semi topogeneous str.	Global fuzzy δ-semi neighborhood space	Fuzzy proximity δ-semi space	Fuzzy compact δ-semi space	Fuzzy uniform δ-semi space
7	$\varphi_1 = \mathrm{int} \circ \mathrm{cl}$ $\varphi_2 = 1_{L^X}$	Fuzzy perfect pre- topogeneous str.	Global fuzzy pre- neighborhood space	Fuzzy proximity pre-space	Fuzzy compact pre-space	Fuzzy uniform pre-space
8	$\varphi_1 = \mathrm{cl} \circ \mathrm{int}$ $\varphi_2 = S.\mathrm{cl}$	Fuzzy perfect semi θ-topogeneous str.	Global fuzzy Semi θ-neighborhood space	Fuzzy proximity semi θ-space	Fuzzy compact semi θ-space	Fuzzy uniform semi θ-space
9	$\varphi_1 = \mathrm{cl} \circ \mathrm{int}$ $\varphi_2 = S.\mathrm{int} \circ S.\mathrm{cl}$	Fuzzy perfect semi δ-topogeneous str.	Global fuzzy Semi δ-neighborhood space	Fuzzy proximity semi δ-space	Fuzzy compact semi δ-space	Fuzzy uniform semi δ-space
10	$\varphi_1 = \mathrm{cl} \circ \mathrm{int} \circ \mathrm{cl}$ $\varphi_2 = 1_{L^X}$	Fuzzy perfect β- β-topogeneous str.	Global fuzzy β- neighborhood space	Fuzzy proximity β-space	Fuzzy compact β-space	Fuzzy uniform β-space
11	$\varphi_1 = \mathrm{int} \circ \mathrm{cl} \circ \mathrm{int}$ $\varphi_2 = 1_{L^X}$	Fuzzy perfect λ- topogeneous str.	Global fuzzy λ- neighborhood space	Fuzzy proximity λ-space	Fuzzy compact λ-space	Fuzzy uniform λ- space
12	$\varphi_1 = S.\mathrm{cl} \circ \mathrm{int}$ $\varphi_2 = 1_{L^X}$	Fuzzy perfect feebly topogeneous str.	Global fuzzy feebly neighborhood space	Fuzzy proximity feebly space	Fuzzy compact feebly space	Fuzzy uniform feebly space

Table 1: Some special classes of character. fuzzy perfect topog. struc., character. global fuzzy neighbor. spaces, character. fuzzy proximity space, character. fuzzy compact spaces and character. fuzzy uniform spaces.

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