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A novel generalized parametric directed divergence measure of intuitionistic fuzzy sets with its application

Harish Garg, Nikunj Aggarwal, Alka Tripathi

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Abstract. The theme of this work is to investigate a new generalized parametric directed divergence measure for intuitionistic fuzzy sets. For it, the entire paper is divided into two folds. Firstly, a new measure has been presented by incorporating the idea of convex linear combinations of the degree of their membership functions. Some desirable properties of the proposed measure have been also investigated. Secondly, divergence measure based method for solving the decision making problem has been presented. A ranking of the different attributes is based on the proposed generalized divergence measure and the sensitivity analysis on the ranking of the system has been done based on the decision-making parameters. An illustrative examples have been studied to show that the proposed function is more reasonable in the decision-making process than other existing functions.

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Corresponding Author: Harish Garg (harishg58iitr@gmail.com)

1. Introduction

Classical information theory has been widely used in the literature for representing the uncertainties in the data in the form of classical measure theory. But these measures are valid only for a precise data i.e., where the data related to the system are precisely known. But due to the various constraints in day-today life, decision makers may give their judgements under the uncertain and imprecise in nature. Thus, there is always a degree of hesitancy between the preferences of the decision making and hence, the analysis conducted under such circumstances are not ideal and hence does not tell the exact information to the system analyst. To handle this, fuzzy set (FS) theory, developed by Zadeh [30], has received much attention over the last decades due to its capability of handling the uncertainties in terms of their membership function. After it, Attanassov [1] proposed the concept of intuitionistic fuzzy set (IFS) which extend the theory of FS with the addition of degree of non-membership. As IFS theory has widely been used by the researchers in the different disciplines for handling the uncertainties in the data and hence their corresponding analysis is more meaningful than their crisp analysis.

The degree of distance, similarity, divergence measures have received a great deal of attention in the last decades for solving the decision-making, pattern recognition, medical diagnosis problems. For this, Szmidt and Kacprzyk [22] proposed the set of axioms for the entropy under the IFS environment. Later on, corresponding to Deluca and Termini [5] fuzzy entropy measure, Vlachos and Sergiadis [24] extended their measure in the IFS environment. Mitchell [19] presented the similarity measure for IFSs based on statistical point of view. Moreover, Xu [28] introduced a series of similarity measures for IFSs and applied them to multiple attribute decision making problem based on intuitionistic fuzzy information. Xu and Chen [29] introduced a series of distance and similarity measures, which are various combinations and generalizations of the weighted Hamming distance, the weighted Euclidean distance and the weighted Hausdorff distance. Szmidt and Kacprzyk [21] proposed a distance measure for measuring the two IFSs. Vlachos and Sergiadis [24] forwarded the notion of divergence from fuzzy set to IFS. Wei and Ye [26] discussed an extended version of [24] divergence measure by incorporating the idea of mid-value of the membership degrees of IFSs while Verma and Sharma [23] presented the generalized intuitionistic fuzzy divergence measure which is an extension of [26] measure. Apart from them, the various authors have incorporating the idea of IFS theory into the measure theory and applied in many practically uncertain situations such as decision making, pattern recognition, medical diagnosis by using similarity measures [3, 4, 6, 7, 16, 17, 19, 32], aggregation operators [8, 9, 10, 27], divergence measures [23, 24, 26], entropy measure [15, 25, 31] and many others [11, 12, 13, 14]. Thus, it has been concluded that the distance/similarity or divergence measures are of key importance in a number of theoretical and applied statistical inference and data processing problems. Furthermore, it has been deduced from the studies that the similarity, entropy and divergence measures could be induced by the normalized distance measure of IFS based on their axiomatic definitions.

But it has been observed from the above studies that all their measures do not incorporate the idea of the decision-maker preferences into the measure. Furthermore, the existing measure is in linear order, and hence it does not give the exact nature of the alternative. Therefore, keeping the criteria of flexibility and efficiency of IFS, this paper presents a new generalized parametric directed divergence measure for measuring the fuzziness degree of a set. For this, a divergence measure of order α and degree β has been presented which makes the decision makers more reliable and flexible for the different values of these parameters. These measures have been formulated by taking the convex linear combinations of the degree of membership functions between the two IFSs. Based on these representations, some desirable properties of these measures have been studied. It has been analyzed for the study that the existing divergence measures are the special cases of the proposed measure and hence concluded that the proposed one is more suitable and generalized. A numerical examples have been provided for demonstrating the performance of the proposed measure. The rest of the text has been summarized as follows. Section 2 presents some definition of the IFS, divergence measure and some existing measures on it. Section 3 proposed the generalized intuitionistic fuzzy directed and symmetric divergence measure of order α and degree β. Various properties of it have also been investigated in details. Section 4 describes an approach for solving the decision-making problems based on the proposed divergence measure followed by an illustrative examples given in section 5. Finally, concrete conclusion and discussion have been presented in section 6.

2. Preliminaries

In this section, some basic definitions related to IFS, divergence measures have been stated in briefly.

2.1. **Intuitionistic fuzzy set.** An intuitionistic fuzzy set (IFS) A defined on universal set X is given by [1]

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}, \tag{2.1}$$

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}, \tag{2.1}$$

where $\mu_A, \nu_A : X \longrightarrow [0,1]$ represent, respectively, the membership and non-membership degrees of the element x to the set A with the conditions $0 \leq \mu_A(x)$, $\nu_A(x) \leq 1$, and $\mu_A(x) + \nu_A(x) \leq 1$. For any $x \in X$, the intuitionistic index of x to A is defined as $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, the complementary set A^c of A is defined as $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$. Usually, the pair $\langle \mu_A(x), \nu_A(x)\rangle$ is called an intuitionistic fuzzy number (shorted by IFN), and it is often simplified as $\alpha = \langle \mu_A, \nu_A\rangle$ where $\mu_A \in [0,1], \nu_A \in [0,1], \mu_A + \nu_A \leq 1$. Let $\alpha = \langle \mu_\alpha, \nu_\alpha\rangle$ and $\beta = \langle \mu_\beta, \nu_\beta\rangle$ be the two IFNs defined on X then operations on IFNs are defined as follows:

- Union: $\alpha \cup \beta = \langle \max\{\mu_\alpha, \mu_\beta\}, \min\{\nu_\alpha, \nu_\beta\}\rangle$,
- Intersection: $\alpha \cap \beta = \langle \min\{\mu_\alpha, \mu_\beta\}, \max\{\nu_\alpha, \nu_\beta\}\rangle$,
- Complement: $\alpha^c = \langle \nu_\alpha, \mu_\alpha\rangle$,
- Containment: $\alpha \subseteq \beta$ iff $\mu_\alpha \leq \mu_\beta$, $\nu_\alpha \geq \nu_\beta$ and $\alpha \supseteq \beta$ iff $\mu_\alpha \geq \mu_\beta$, $\nu_\alpha \leq \nu_\beta$.

2.2. **Directed divergence measure.** The directed divergence measure is nothing but its a relative entropy measure and this provides a distance formula between the two discrete probability distributions. First, Kullback and Leibler [18] proposed the measure of directed divergence between the two distribution $P = (p_1, p_2, \ldots, p_n)$ $Q = (q_1, q_2, \ldots, q_n)$ as:

$$D(P|Q) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} \tag{2.2}$$

$$D(P|Q) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} \tag{2.2}$$

which satisfying the following conditions:

(i) $D(P \mid Q) \geq 0$,
(ii) $D(P \mid Q) = 0$ iff $P = Q$.

This idea of divergence measure was extended from probabilistic to fuzzy set theory by Bhandari and Pal [2] by giving a fuzzy information measure for discrimination of a fuzzy set B relative to some other fuzzy set A. Let A and B be two fuzzy sets defined in discrete universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$ having the membership values $\mu_A(x_i)$ and $\mu_B(x_i)$, $i = 1, 2, \ldots, n$ respectively, then Bhandari and Pal [2] defined the fuzzy divergence measure of fuzzy set B relative to A by

$$D(A \mid B) = \frac{1}{n}\sum_{i=1}^{n}\left[\mu_A(x_i)\log\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right) + (1-\mu_A(x_i))\log\left(\frac{1-\mu_A(x_i)}{1-\mu_B(x_i)}\right)\right]. \tag{2.3}$$

The above measure tends towards infinity if μ_B approaches either to 0 or 1 and hence they give an inaccurate result. To overcome this, Shang and Jiang [20] presented a modified measure of it given by

$$\begin{aligned} D(A \mid B) = \frac{1}{n}\sum_{i=1}^{n}\Bigg[&\mu_A(x_i)\log\left(\frac{2\mu_A(x_i)}{\mu_A(x_i)+\mu_B(x_i)}\right) \\ &+ (1-\mu_A(x_i))\log\left(\frac{2(1-\mu_A(x_i))}{2-\mu_A(x_i)-\mu_B(x_i)}\right)\Bigg]. \end{aligned} \tag{2.4}$$

In 2007, Vlachos and Sergiadis [24] forwarded the notion of divergence from fuzzy sets to intuitionistic fuzzy sets. Analogy to Shang and Jiang's fuzzy divergence measure in Eq. (2.4), they defined a measure of intuitionistic fuzzy divergence of IFS B relative of IFS A by

$$D(A \mid B) = \frac{1}{n}\sum_{i=1}^{n}\left[\mu_A(x_i)\log\left(\frac{2\mu_A(x_i)}{\mu_A(x_i)+\mu_B(x_i)}\right) + \nu_A(x_i)\log\left(\frac{2\nu_A(x_i)}{\nu_A(x_i)+\nu_B(x_i)}\right)\right]. \tag{2.5}$$

Afterward, Wei and Ye [26] extended Vlachos and Sergiadis [24] intuitionistic fuzzy divergence measure by

$$\begin{aligned} D(A \mid B) = \frac{1}{n}\sum_{i=1}^{n}\Bigg[&\mu_A(x_i)\log\left(\frac{2\mu_A(x_i)}{\mu_A(x_i)+\mu_B(x_i)}\right) \\ &+\nu_A(x_i)\log\left(\frac{2\nu_A(x_i)}{\nu_A(x_i)+\nu_B(x_i)}\right) \\ &+\pi_A(x_i)\log\left(\frac{2\pi_A(x_i)}{\pi_A(x_i)+\pi_B(x_i)}\right)\Bigg]. \end{aligned} \tag{2.6}$$

Later on, Verma and Sharma [23] presented an improved version of Wei and Ye [26] as

$$D(A \mid B) = \frac{1}{n}\sum_{i=1}^{n}\Bigg[\mu_A(x_i)\log\left(\frac{\mu_A(x_i)}{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)}\right) + \nu_A(x_i)\log\left(\frac{\nu_A(x_i)}{\lambda\nu_A(x_i)+(1-\lambda)\nu_B(x_i)}\right) + \pi_A(x_i)\log\left(\frac{\pi_A(x_i)}{\lambda\pi_A(x_i)+(1-\lambda)\pi_B(x_i)}\right)\Bigg]. \tag{2.7}$$

3. Proposed intuitionistic parametric divergence measure

In this section, we have proposed a flexible and generalized parametric divergence measure of order α and degree β denoted as class of (α, β). Some desirable properties of these are also being studied.

3.1. Parametric directed divergence measure for IFSs.

Definition 3.1. Let A and B be two IFSs defined on universal set $X = \{x_1, x_2, \ldots, x_n\}$, then a parametric directed divergence measure for IFSs based on the parameters α, β and $\lambda(0 \le \lambda \le 1)$ is denoted as $D^{\beta}_{\alpha\{\lambda\}}(A|B)$ and defined as follows:

$$D^{\beta}_{\alpha\{\lambda\}}(A|B) = \frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\Bigg[\mu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right) + \nu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right) + \pi_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg],$$

where μ, ν and π are the membership, non-membership and hesitancy functions respectively and it is valid for $\alpha, \beta > 0$ and except $\beta \neq 2$.

It is clearly seen from the definition that the $D^{\beta}_{\alpha\{\lambda\}}$ is not symmetric, so to imbue the measure with symmetry, a parametric Symmetric Divergence Measure for IFSs has been defined as follows.

3.2. Parametric symmetric divergence measure for IFSs.

Definition 3.2. A parametric symmetric divergence measure for two IFSs A and B based on α, β and λ is denoted as $D^{\beta}_{\alpha\{\lambda\}}(A;B)$ defined as follows:

$$D^{\beta}_{\alpha\{\lambda\}}(A;B) = D^{\beta}_{\alpha\{\lambda\}}(A|B) + D^{\beta}_{\alpha\{\lambda\}}(B|A)$$

$$= \frac{\alpha}{n(2-\beta)} \sum_{i=1}^{n} \Bigg[\mu_A^{\frac{\alpha}{2-\beta}}(x_i) \log \frac{\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda \mu_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}$$
$$+ \nu_A^{\frac{\alpha}{2-\beta}}(x_i) \log \frac{\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda \nu_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}$$
$$+ \pi_A^{\frac{\alpha}{2-\beta}}(x_i) \log \frac{\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda \pi_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\pi_B^{\frac{\alpha}{2-\beta}}(x_i)} \Bigg]$$
$$+ \frac{\alpha}{n(2-\beta)} \sum_{i=1}^{n} \Bigg[\mu_B^{\frac{\alpha}{2-\beta}}(x_i) \log \frac{\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda \mu_B^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}$$
$$+ \nu_B^{\frac{\alpha}{2-\beta}}(x_i) \log \frac{\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda \nu_B^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}$$
$$+ \pi_B^{\frac{\alpha}{2-\beta}}(x_i) \log \frac{\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda \pi_B^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\pi_A^{\frac{\alpha}{2-\beta}}(x_i)} \Bigg].$$

From the definition of $D^{\beta}_{\alpha\{\lambda\}}(A \mid B)$ and $D^{\beta}_{\alpha\{\lambda\}}(A;B)$, it has been observed that

(1) $D^{\beta}_{\alpha\{\lambda\}}(A \mid B) \geq 0$ and $D^{\beta}_{\alpha\{\lambda\}}(A;B) \geq 0$.

(2) When $\lambda = 1$ then $D^{\beta}_{\alpha\{\lambda\}}(A \mid B) = 0$ and $D^{\beta}_{\alpha\{\lambda\}}(A;B) = 0$.

(3) When $\lambda \neq 1$, then $D^{\beta}_{\alpha\{\lambda\}}(A \mid B) = D^{\beta}_{\alpha\{\lambda\}}(A;B)$ iff $A = B$, i.e., when $\mu_A(x_i) = \mu_B(x_i)$ and $\nu_A(x_i) = \nu_B(x_i)$.

Also it has been observed from the proposed divergence measure that some existing divergence measures [2, 20, 23, 26] are the particular cases of it and has been seen as below.

(1) When $\alpha = \beta = 1$ and $\lambda = 0$ with $\pi_A(x_i) = 0 = \pi_B(x_i)$, then the proposed measure reduces to [2] measures.

(2) When $\alpha = \beta = 1$ and $\lambda = \frac{1}{2}$ with $\pi_A(x_i) = 0 = \pi_B(x_i)$, then the proposed measure reduces to [20] measures.

(3) When $\alpha = 1$ and $\beta = 1$, then the proposed measure reduces to [23] measures.

(4) When $\alpha = \beta = 1$ and $\lambda = \frac{1}{2}$, then the proposed measure reduces to [26] measures.

Thus it has been observed that the proposed measure is the more generalized than the existing measures.

Divide the universe X into two parts X_1 and X_2, where

$X_1 = \{x_i : x_i \in X, A(x_i) \subseteq B(x_i)\}$, i.e.,
$\mu_A(x_i) \leq \mu_B(x_i), \nu_A(x_i) \geq \nu_B(x_i) \quad \forall x_i \in X_1$,
$X_2 = \{x_i : x_i \in X, A(x_i) \supseteq B(x_i)\}$, i.e.,
$\mu_A(x_i) \geq \mu_B(x_i), \nu_A(x_i) \leq \nu_B(x_i) \quad \forall x_i \in X_2$.

Now, we propose some properties based on the above considerations.

3.3. **Some properties of parametric symmetric divergence measure for IFSs.**

Property 3.1. *If A and B be the two IFSs defined on universal set X, such that they satisfy for any $x_i \in X$ either $A \subseteq B$ or $A \supseteq B$, then*

$$D^{\beta}_{\alpha\{\lambda\}}(A \cup B; A \cap B) = D^{\beta}_{\alpha\{\lambda\}}(A; B).$$

Proof. It is clear that

$$D^{\beta}_{\alpha\{\lambda\}}(A \cup B; A \cap B) = D^{\beta}_{\alpha\{\lambda\}}(A \cup B|A \cap B) + D^{\beta}_{\alpha\{\lambda\}}(A \cap B|A \cup B).$$

On the other hand,

$$\begin{aligned}
&D^{\beta}_{\alpha\{\lambda\}}(A \cup B|A \cap B)\\
&= \frac{\alpha}{n(2-\beta)} \sum_{i=1}^{n} \Bigg[\mu_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\mu_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda \mu_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\mu_{A\cap B}^{\frac{\alpha}{2-\beta}}(x_i)} \right)\\
&\qquad + \nu_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\nu_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda \nu_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\nu_{A\cap B}^{\frac{\alpha}{2-\beta}}(x_i)} \right)\\
&\qquad + \pi_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\pi_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda \pi_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\pi_{A\cap B}^{\frac{\alpha}{2-\beta}}(x_i)} \right) \Bigg]\\
&= \frac{\alpha}{n(2-\beta)} \sum_{x \in X_1} \Bigg[\mu_{B}^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\mu_{B}^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda \mu_{B}^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\mu_{A}^{\frac{\alpha}{2-\beta}}(x_i)} \right)\\
&\qquad + \nu_{B}^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\nu_{B}^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda \nu_{B}^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\nu_{A}^{\frac{\alpha}{2-\beta}}(x_i)} \right)\\
&\qquad + \pi_{B}^{\frac{\alpha}{2-\beta}} \log\left(\frac{\pi_{B}^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda \pi_{B}^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\pi_{A}^{\frac{\alpha}{2-\beta}}(x_i)} \right) \Bigg]\\
&\quad + \frac{\alpha}{n(2-\beta)} \sum_{x \in X_2} \Bigg[\mu_{A}^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\mu_{A}^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda \mu_{A}^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\mu_{B}^{\frac{\alpha}{2-\beta}}(x_i)} \right)\\
&\qquad + \nu_{A}^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\nu_{A}^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda \nu_{A}^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\nu_{B}^{\frac{\alpha}{2-\beta}}(x_i)} \right)\\
&\qquad + \pi_{A}^{\frac{\alpha}{2-\beta}} \log\left(\frac{\pi_{B}^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda \pi_{A}^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\pi_{B}^{\frac{\alpha}{2-\beta}}(x_i)} \right) \Bigg]
\end{aligned}$$

and

$$\begin{aligned}
&D^{\beta}_{\alpha\{\lambda\}}(A \cap B|A \cup B)\\
&= \frac{\alpha}{n(2-\beta)} \sum_{i=1}^{n} \Bigg[\mu_{A\cap B}^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\mu_{A\cap B}^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda \mu_{A\cap B}^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\mu_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)} \right)\\
&\qquad + \nu_{A\cap B}^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\nu_{A\cap B}^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda \nu_{A\cap B}^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\nu_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)} \right)
\end{aligned}$$

$$+ \pi_{A\cap B}^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_{A\cap B}^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_{A\cap B}^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg]$$

$$= \frac{\alpha}{n(2-\beta)}\sum_{x\in X_1}\Bigg[\mu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$+ \nu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$+ \pi_A^{\frac{\alpha}{2-\beta}}\log\left(\frac{\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg]$$

$$+ \frac{\alpha}{n(2-\beta)}\sum_{x\in X_2}\Bigg[\mu_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$+ \nu_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$+ \pi_B^{\frac{\alpha}{2-\beta}}\log\left(\frac{\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg].$$

Then by adding the above equations, we get,

$$D_{\alpha\{\lambda\}}^{\beta}(A\cup B|A\cap B)+D_{\alpha\{\lambda\}}^{\beta}(A\cap B|A\cup B)$$

$$= \frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\Bigg[\mu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$+ \nu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$+ \pi_A^{\frac{\alpha}{2-\beta}}(x_i)\log\frac{\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}\Bigg)\Bigg]$$

$$+ \frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\Bigg[\mu_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$+ \nu_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$+ \pi_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg]$$

$$= D_{\alpha\{\lambda\}}^{\beta}(A|B)+D_{\alpha\{\lambda\}}^{\beta}(B|A)$$

$$= D_{\alpha\{\lambda\}}^{\beta}(A;B).$$

Thus the result holds. □

Property 3.2. *For any two IFSs A and B, we have*
(1) $D^{\beta}_{\alpha\{\lambda\}}(A; A\cup B) = D^{\beta}_{\alpha\{\lambda\}}(B; A\cap B)$,
(2) $D^{\beta}_{\alpha\{\lambda\}}(A; A\cap B) = D^{\beta}_{\alpha\{\lambda\}}(B; A\cup B)$,
(3) $D^{\beta}_{\alpha\{\lambda\}}(A; A\cup B) + D^{\beta}_{\alpha\{\lambda\}}(A; A\cap B) = D^{\beta}_{\alpha\{\lambda\}}(A; B)$,
(4) $D^{\beta}_{\alpha\{\lambda\}}(B; A\cup B) + D^{\beta}_{\alpha\{\lambda\}}(B; A\cap B) = D^{\beta}_{\alpha\{\lambda\}}(A; B)$.

Proof. All can be proved similarly. So, we prove only the first one, i.e.,

$$\begin{aligned} &D^{\beta}_{\alpha\{\lambda\}}(A|A\cup B) + D^{\beta}_{\alpha\{\lambda\}}(A\cup B|A) \\ &= D^{\beta}_{\alpha\{\lambda\}}(B|A\cap B) + D^{\beta}_{\alpha\{\lambda\}}(A\cap B|B). \end{aligned} \tag{3.1}$$

By the definition of the divergence, we have

$$\begin{aligned} &D^{\beta}_{\alpha\{\lambda\}}(A|A\cup B) \\ &= \frac{\alpha}{n(2-\beta)} \sum_{x\in X_1}^{n} \left[\mu_A^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\mu_B^{\frac{\alpha}{2-\beta}}(x_i)} \right) \right. \\ &\qquad + \nu_A^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\nu_B^{\frac{\alpha}{2-\beta}}(x_i)} \right) \\ &\qquad \left. + \pi_A^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\pi_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)} \right) \right] \\ &= \frac{\alpha}{n(2-\beta)} \sum_{x\in X_1}^{n} \left[\mu_A^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\mu_B^{\frac{\alpha}{2-\beta}}(x_i)} \right) \right. \\ &\qquad + \nu_A^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\nu_B^{\frac{\alpha}{2-\beta}}(x_i)} \right) \\ &\qquad \left. + \pi_A^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\pi_B^{\frac{\alpha}{2-\beta}}(x_i)} \right] \right. \\ &\quad + \frac{\alpha}{n(2-\beta)} \sum_{x\in X_2}^{n} \left[\mu_A^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\mu_A^{\frac{\alpha}{2-\beta}}(x_i)} \right) \right. \\ &\qquad + \nu_A^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\nu_A^{\frac{\alpha}{2-\beta}}(x_i)} \right) \\ &\qquad \left. + \pi_A^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\pi_A^{\frac{\alpha}{2-\beta}}(x_i)} \right) \right] \end{aligned}$$

$$= \frac{\alpha}{n(2-\beta)} \sum_{x\in X_1}^{n} \left[\mu_A^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\mu_B^{\frac{\alpha}{2-\beta}}(x_i)} \right) + \nu_A^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\nu_B^{\frac{\alpha}{2-\beta}}(x_i)} \right) + \pi_A^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\pi_B^{\frac{\alpha}{2-\beta}}(x_i)} \right) \right]$$

and

$$D_{\alpha\{\lambda\}}^{\beta}(A \cup B|A)$$

$$= \frac{\alpha}{n(2-\beta)} \sum_{i=1}^{n} \left[\mu_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\mu_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_{(A\cup B)}^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\mu_A^{\frac{\alpha}{2-\beta}}(x_i)} \right) + \nu_{A\cap B}^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\nu_{A\cap B}^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_{A\cap B}^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\nu_A^{\frac{\alpha}{2-\beta}}(x_i)} \right) + \pi_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\pi_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_{A\cap B}^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\pi_A^{\frac{\alpha}{2-\beta}}(x_i)} \right) \right]$$

$$= \frac{\alpha}{n(2-\beta)} \sum_{x\in X_1} \left[\mu_B^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_B^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\mu_A^{\frac{\alpha}{2-\beta}}(x_i)} \right) + \nu_B^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_B^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\nu_A^{\frac{\alpha}{2-\beta}}(x_i)} \right) + \pi_B^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_B^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\pi_A^{\frac{\alpha}{2-\beta}}(x_i)} \right) \right]$$

$$+ \frac{\alpha}{n(2-\beta)} \sum_{x\in X_2} \left[\mu_A^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\mu_A^{\frac{\alpha}{2-\beta}}(x_i)} \right) + \nu_A^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\nu_A^{\frac{\alpha}{2-\beta}}(x_i)} \right) + \pi_A^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\pi_A^{\frac{\alpha}{2-\beta}}(x_i)} \right) \right]$$

$$= \frac{\alpha}{n(2-\beta)} \sum_{x\in X_1} \left[\mu_B^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_B^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\mu_A^{\frac{\alpha}{2-\beta}}(x_i)} \right) + \nu_B^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_B^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\nu_A^{\frac{\alpha}{2-\beta}}(x_i)} \right) \right.$$

$$+\pi_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg].$$

Similarly,

$$\begin{aligned}&D^{\beta}_{\alpha\{\lambda\}}(B|A\cap B)\\ &=\frac{\alpha}{n(2-\beta)}\sum_{x\in X_1}\Bigg[\mu_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\ &\qquad+\nu_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\ &\qquad+\pi_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg]\end{aligned}$$

and

$$\begin{aligned}&D^{\beta}_{\alpha\{\lambda\}}(A\cap B|B)\\ &=\frac{\alpha}{n(2-\beta)}\sum_{x\in X_1}\Bigg[\mu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\ &\qquad+\nu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\ &\qquad+\pi_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg].\end{aligned}$$

Clearly, by using Eq. (3.1), we can say that both sides of the expression are same. This proves the result. □

Property 3.3. *If A and B be the two IFSs defined on universal set X, then*
(1) $D^{\beta}_{\alpha\{\lambda\}}(A;C)+D^{\beta}_{\alpha\{\lambda\}}(B;C)-D^{\beta}_{\alpha\{\lambda\}}(A\cup B;C)\geq 0$,
(2) $D^{\beta}_{\alpha\{\lambda\}}(A;C)+D^{\beta}_{\alpha\{\lambda\}}(B;C)-D^{\beta}_{\alpha\{\lambda\}}(A\cap B;C)\geq 0$.

Proof. In this theorem, we prove only the first part because of having analogously similar proofs.

$$\begin{aligned}&D^{\beta}_{\alpha\{\lambda\}}(A;C)\\ &=\frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\Bigg[\mu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\ &\qquad+\nu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\ &\qquad+\pi_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg]\end{aligned}$$

$$+\frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\left[\mu_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\right.$$

$$+\frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\left[\mu_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\right.$$

$$\left.+\pi_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\right],$$

$$D^{\beta}_{\alpha\{\lambda\}}(B;C)$$

$$=\frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\left[\mu_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\right.$$

$$+\nu_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$\left.+\pi_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\right]$$

$$+\frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\left[\mu_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\right.$$

$$+\nu_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$\left.+\pi_C^{\frac{\alpha}{2-\beta}}(x_i)\log\frac{\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}\Bigg)\right]$$

and

$$D^{\beta}_{\alpha\{\lambda\}}(A\cup B;C)$$

$$=\frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\left[\mu_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\right.$$

$$+\nu_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$\left.+\pi_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\right]$$

$$+\frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\left[\mu_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)}\right)\right.$$

$$+\nu_C^{\frac{\alpha}{2-\beta}}(x_i)\log\frac{\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)}\Bigg)$$

$$+\pi_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_{A\cup B}^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg]$$

$$=\frac{\alpha}{n(2-\beta)}\sum_{x\in X_1}^{n}\Bigg[\mu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$+\nu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$+\pi_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$+\mu_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$+\nu_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$+\pi_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg]$$

$$+\frac{\alpha}{n(2-\beta)}\sum_{x\in X_2}^{n}\Bigg[\mu_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$+\nu_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$+\pi_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$+\mu_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$+\nu_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)$$

$$+\pi_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg].$$

Then,

$$
\begin{aligned}
&D^{\beta}_{\alpha\{\lambda\}}(A;C) + D^{\beta}_{\alpha\{\lambda\}}(B;C) - D^{\beta}_{\alpha\{\lambda\}}(A\cup B;C)\\
&= \frac{\alpha}{n(2-\beta)} \sum_{x\in X_1} \Bigg[\mu_B^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_B^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\quad + \nu_B^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_B^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\quad + \pi_B^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_B^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\quad + \mu_C^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_C^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\quad + \nu_C^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_C^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\quad + \pi_C^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_C^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg]\\
&+ \frac{\alpha}{n(2-\beta)} \sum_{x\in X_2} \Bigg[\mu_A^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\quad + \nu_A^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\quad + \pi_A^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_A^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\quad + \mu_C^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_C^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\quad + \nu_C^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_C^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\quad + \pi_C^{\frac{\alpha}{2-\beta}}(x_i) \log\left(\frac{\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_C^{\frac{\alpha}{2-\beta}}(x_i) + (1-\lambda)\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg].
\end{aligned}
$$

Since, $\mu(x_i), \nu(x_i), \pi(x_i) \in [0,1]$, $\forall\ \ x_i \in X$. This completes the proof. □

Property 3.4. *For any two IFSs A and B, we have*
$D^{\beta}_{\alpha\{\lambda\}}(A\cap B;C) + D^{\beta}_{\alpha\{\lambda\}}(A\cup B;C) = D^{\beta}_{\alpha\{\lambda\}}(A;C) + D^{\beta}_{\alpha\{\lambda\}}(B;C)$.

Proof.

$$
\begin{aligned}
&D^{\beta}_{\alpha\{\lambda\}}(A\cap B|C)\\
&= \frac{\alpha}{n(2-\beta)}\sum_{x\in X_1}\left[\mu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\right.\\
&\qquad + \nu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad \left. + \pi_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\right]\\
&+ \frac{\alpha}{n(2-\beta)}\sum_{x\in X_2}\left[\mu_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\right.\\
&\qquad + \nu_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad \left. + \pi_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
&D^{\beta}_{\alpha\{\lambda\}}(C|A\cap B)\\
&= \frac{\alpha}{n(2-\beta)}\sum_{x\in X_1}\left[\mu_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\right.\\
&\qquad + \nu_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad \left. + \pi_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\right]\\
&+ \frac{\alpha}{n(2-\beta)}\sum_{x\in X_2}\left[\mu_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\right.\\
&\qquad + \nu_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad \left. + \pi_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
&D^{\beta}_{\alpha\{\lambda\}}(A\cup B|C)\\
&= \frac{\alpha}{n(2-\beta)}\sum_{x\in X_2}\Bigg[\mu_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad + \nu_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad + \pi_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg]\\
&+ \frac{\alpha}{n(2-\beta)}\sum_{x\in X_2}\Bigg[\mu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad + \nu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad + \pi_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
&D^{\beta}_{\alpha\{\lambda\}}(C|A\cup B)\\
&= \frac{\alpha}{n(2-\beta)}\sum_{x\in X_1}\Bigg[\mu_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad + \nu_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad + \pi_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg]\\
&+ \frac{\alpha}{n(2-\beta)}\sum_{x\in X_2}\Bigg[\mu_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad + \nu_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad + \pi_C^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_C^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_C^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg].
\end{aligned}
$$

By adding all of the above equations, we get the required result and this completes the proof. □

Property 3.5. *If A and B be the two IFSs defined on universal set X, then*

(1) $D^{\beta}_{\alpha\{\lambda\}}(A;B) = D^{\beta}_{\alpha\{\lambda\}}(A^c;B^c)$,

(2) $D^{\beta}_{\alpha\{\lambda\}}(A;B^c) = D^{\beta}_{\alpha\{\lambda\}}(A^c;B)$,

(3) $D^{\beta}_{\alpha\{\lambda\}}(A;B)+D^{\beta}_{\alpha\{\lambda\}}(A^c;B)=D^{\beta}_{\alpha\{\lambda\}}(A^c;B^c)+D^{\beta}_{\alpha\{\lambda\}}(A;B^c)$.

Proof. First and second parts are similar and third part can be proved by adding first and second one. Then we prove only (1).

$$
\begin{aligned}
&D^{\beta}_{\alpha\{\lambda\}}(A;B))\\
&=\frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\Bigg[\mu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad+\nu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad+\pi_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg]\\
&+\frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\Bigg[\mu_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad+\nu_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad+\pi_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
&D^{\beta}_{\alpha\{\lambda\}}(A^c;B^c)\\
&=\frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\Bigg[\nu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad+\mu_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad+\pi_A^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_A^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg]\\
&+\frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\Bigg[\nu_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\nu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\nu_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\nu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad+\mu_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\mu_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\mu_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\mu_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\\
&\qquad+\pi_B^{\frac{\alpha}{2-\beta}}(x_i)\log\left(\frac{\pi_B^{\frac{\alpha}{2-\beta}}(x_i)}{\lambda\pi_B^{\frac{\alpha}{2-\beta}}(x_i)+(1-\lambda)\pi_A^{\frac{\alpha}{2-\beta}}(x_i)}\right)\Bigg].
\end{aligned}
$$

Then (1) holds. □

4. Decision-making method based on proposed directed divergence measure

In this section, we shall investigate the decision making problems based on the proposed divergence measure $D^{\beta}_{\alpha\{\lambda\}}$ in which the attribute values are evaluated by the expert which give their preferences in terms of IFNs. Assume that a set of 'm' alternatives $A = \{A_1, A_2, \ldots, A_m\}$ to be considered under the set of 'n' criteria $G = \{G_1, G_2, \ldots, G_n\}$. An expert have evaluated these 'm' alternative under each criterion and give their rating values in the form of IFNs. Then we have the following steps for computing the best alternative(s) based on the proposed measure.

Step 1: *Construction of decision-making matrix*: Suppose $D_{m\times n}(x_{ij}) = \langle \mu_{ij}, \nu_{ij}\rangle$ be the intuitionistic fuzzy decision matrix, where μ_{ij} represents the degree that the alternative A_i satisfies the criteria G_j and ν_{ij} indicates the degree that the alternative A_i doesn't satisfy the criteria G_j given by the decision maker such that $\mu_{ij} \in [0,1]$, $\nu_{ij} \in [0,1]$ such that $\mu_{ij} + \nu_{ij} \leq 1$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. So, the intuitionistic fuzzy decision matrix is constructed as follows:

$$D_{m\times n}(x_{ij}) = \begin{bmatrix} \langle \mu_{11}, \nu_{11}\rangle & \langle \mu_{12}, \nu_{12}\rangle & \ldots & \langle \mu_{1n}, \nu_{1n}\rangle \\ \langle \mu_{21}, \nu_{21}\rangle & \langle \mu_{22}, \nu_{22}\rangle & \ldots & \langle \mu_{2n}, \nu_{2n}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mu_{m1}, \nu_{m1}\rangle & \langle \mu_{m2}, \nu_{m2}\rangle & \ldots & \langle \mu_{mn}, \nu_{mn}\rangle \end{bmatrix}$$

Step 2: *Compute the ideal alternative*: Ideal alternative is denoted as A^* and given as:

$$A^* = \{\langle \mu_1^*, \nu_1^*\rangle, \langle \mu_2^*, \nu_2^*\rangle, \ldots, \langle \mu_n^*, \nu_n^*\rangle\}$$

where, $\mu_j^* = \max_{i=1}^{m}(\mu_{ij})$ and $\nu_j^* = \min_{i=1}^{m}(\nu_{ij})$

Step 3: *Evaluation of Proposed Symmetric Divergence Measure*: Now we calculate $D^{\beta}_{\alpha\{\lambda\}}(A_i; A^*)$, $i = 1, 2, \ldots, m$ by the given formula:

$$\begin{aligned} D^{\beta}_{\alpha\{\lambda\}}(A_i; A^*) = \frac{\alpha}{n(2-\beta)} \sum_{j=1}^{n} \Bigg[& \mu_{ij}^{\frac{\alpha}{2-\beta}}(x_{ij}) \log\left(\frac{\mu_{ij}^{\frac{\alpha}{2-\beta}}(x_{ij})}{\lambda\mu_{ij}^{\frac{\alpha}{2-\beta}}(x_{ij}) + (1-\lambda)\mu_{j^*}^{\frac{\alpha}{2-\beta}}(x_{ij})}\right) \\ & + \nu_{ij}^{\frac{\alpha}{2-\beta}}(x_{ij}) \log\left(\frac{\nu_{ij}^{\frac{\alpha}{2-\beta}}(x_{ij})}{\lambda\nu_{ij}^{\frac{\alpha}{2-\beta}}(x_{ij}) + (1-\lambda)\nu_{j^*}^{\frac{\alpha}{2-\beta}}(x_{ij})}\right) \\ & + \pi_{ij}^{\frac{\alpha}{2-\beta}}(x_{ij}) \log\left(\frac{\pi_{ij}^{\frac{\alpha}{2-\beta}}(x_{ij})}{\lambda\pi_{ij}^{\frac{\alpha}{2-\beta}}(x_{ij}) + (1-\lambda)\pi_{j^*}^{\frac{\alpha}{2-\beta}}(x_{ij})}\right)\Bigg] \end{aligned}$$

$$+\frac{\alpha}{n(2-\beta)}\sum_{j=1}^{n}\left[\mu_{j^*}^{\frac{\alpha}{2-\beta}}(x_{ij})\log\left(\frac{\mu_{j^*}^{\frac{\alpha}{2-\beta}}(x_{ij})}{\lambda\mu_{j^*}^{\frac{\alpha}{2-\beta}}(x_{ij})+(1-\lambda)\mu_{ij}^{\frac{\alpha}{2-\beta}}(x_{ij})}\right)\right.$$
$$+\nu_{j^*}^{\frac{\alpha}{2-\beta}}(x_{ij})\log\left(\frac{\nu_{j^*}^{\frac{\alpha}{2-\beta}}(x_{ij})}{\lambda\nu_{j^*}^{\frac{\alpha}{2-\beta}}(x_{ij})+(1-\lambda)\nu_{ij}^{\frac{\alpha}{2-\beta}}(x_{ij})}\right)$$
$$\left.+\pi_{j^*}^{\frac{\alpha}{2-\beta}}(x_{ij})\log\left(\frac{\pi_{j^*}^{\frac{\alpha}{2-\beta}}(x_{ij})}{\lambda\pi_{j^*}^{\frac{\alpha}{2-\beta}}(x_{ij})+(1-\lambda)\pi_{ij}^{\frac{\alpha}{2-\beta}}(x_{ij})}\right)\right]$$

Step 4: *Ranking the alternative*: Rank all the alternative according to indexing as obtained from $k = \arg\min_{1\le i\le m}\{D^{\beta}_{\alpha\{\lambda\}}(A_i; A^*)\}$.

Step 5: *Sensitivity analysis*: Do the sensitivity analysis on the parameter α, β and λ according to the decision makers' preferences.

5. Illustrative example

In this section, two illustrative examples, one from the field of decision-making and other from the pattern recognition, have been taken for demonstrating the proposed approach.

5.1. **Example 1: Decision-making problem.** Consider the field of investment, where a person wants to invest some sort of money. As in these days, more and more companies have attracted the customers by reducing price and giving some other kinds of benefits, so it is difficult for the investor to choose the best market for investment. In order to avoid the risk factor in the market and to make the decision more clear, they constitute a committee to invest the money in four major companies, namely retail, food, computer, petrochemical and a car company respectively denoted by A_1, A_2, A_3 and A_4. An expert has been hire which gave their preferences of each alternative under the set of four major analysis namely, the growth (G_1), the risk (G_2), the social-political impact (G_3) and the environmental impact (G_4). The rating value of each alternative $A_i (i = 1, 2, \ldots, 5)$ under each factor has been assessed in terms of IFNs $\alpha_{ij} = \langle\mu_{ij}, \nu_{ij}\rangle_{5\times4}$ and are summarized as below.

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array}\begin{array}{c} \begin{array}{cccc} G_1 & G_2 & G_3 & G_4 \end{array} \\ \begin{bmatrix} \langle 0.5, 0.4\rangle & \langle 0.6, 0.3\rangle & \langle 0.3, 0.6\rangle & \langle 0.2, 0.7\rangle \\ \langle 0.7, 0.3\rangle & \langle 0.7, 0.2\rangle & \langle 0.7, 0.2\rangle & \langle 0.4, 0.5\rangle \\ \langle 0.6, 0.4\rangle & \langle 0.5, 0.4\rangle & \langle 0.5, 0.3\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.8, 0.1\rangle & \langle 0.6, 0.3\rangle & \langle 0.3, 0.4\rangle & \langle 0.2, 0.6\rangle \\ \langle 0.6, 0.2\rangle & \langle 0.4, 0.3\rangle & \langle 0.7, 0.1\rangle & \langle 0.5, 0.3\rangle \end{bmatrix}\end{array}$$

By using these normalized data, the ideal value for all the criteria is given by

$$A^* = \{\langle 0.8, 0.1\rangle, \langle 0.7, 0.2\rangle, \langle 0.7, 0.1\rangle, \langle 0.6, 0.3\rangle\}.$$

Thus, based on it, the directed divergence measure from ideal alternative to each alternative is computed by taking $\alpha = 1$, $\beta = 0.5$, $\lambda = 0.3$ and their corresponding measures are summarized as below;

$$D^{\beta}_{\alpha\{\lambda\}}(A_1; A^*) = 0.1939; D^{\beta}_{\alpha\{\lambda\}}(A_2; A^*) = 0.1089; D^{\beta}_{\alpha\{\lambda\}}(A_3; A^*) = 0.1385$$
$$D^{\beta}_{\alpha\{\lambda\}}(A_4; A^*) = 0.1159; D^{\beta}_{\alpha\{\lambda\}}(A_5; A^*) = 0.0586.$$

So, the ranking of these alternatives is of order $A_5 \succ A_2 \succ A_4 \succ A_3 \succ A_1$ where "$\succ$" means preferred to". Hence, the best alternative is A_5, i.e. a person has to be invest car company.

The influence of the parameter α, β and λ on the decision making has been analyzed by using the different values of it in step 3 of the proposed approach. Their corresponding divergence values and their ranking order are summarized in Table 1. From this table, it has observed that if we fix value of λ say $\lambda = 0.3$ and by varying the value of α, β then the value of directed divergence measure corresponding to each alternative is increasing. On the other hand, if we fix the value of α and β and by varying the value of λ from 0.1 to 0.9 then the value of divergence measure of each alternative is decreased. The complete analysis of the variations of α, β and λ on the ranking of the alternatives $A_i (i = 1, 2, \ldots, 5)$ are observed and are summarized in Table 1. From this analysis, it has been concluded that the ranking of the given alternative is symmetric and found that the most desirable attribute is A_1 and A_5 is the least one for different values of λ's corresponding to different operators. For $\alpha = \beta = 1$ the generalized directed divergence measure reduces to existing measures. Clearly, the most preferable item/alternative among the five alternatives is X_5 and the most disfavored is X_1. Here, Ranking order differs due to change in α, β and λ and this shows the importance of parameters in our divergence measure.

In order to compare the performance of the proposed measure with the existing divergence measure, analysis has been conducted with the existing divergence measures [23, 24, 26] and their corresponding results are summarized in Table 2.

5.2. **Example 2: Pattern recognition.** Consider a three known IFSs pattern C_1, C_2 and C_3 in a given finite universe $X = \{x_1, x_2, x_3\}$ as

$$C_1 = \{(x_1, 1.0, 0.0), (x_2, 0.8, 0.0), (x_3, 0.7, 0.1)\},$$
$$C_2 = \{(x_1, 0.8, 0.1), (x_2, 1.0, 0.0), (x_3, 0.9, 0.1)\},$$
$$C_3 = \{(x_1, 0.6, 0.2), (x_2, 0.8, 0.0), (x_3, 1.0, 0.0)\}.$$

Consider an unknown IFS pattern P which will be recognized, where

$$P = \{(x_1, 0.5, 0.3), (x_2, 0.6, 0.2), (x_3, 0.8, 0.1)\}.$$

The target of this problem is to classify the pattern P in one of the classes C_1, C_2 and C_3. For it, the proposed divergence measure has been computed corresponding to $\alpha = 1$, $\beta = 0.5$ and $\lambda = 0.3$ from P to $C_k (k = 1, 2, 3)$ and their corresponding results are summarized as

$$D^{\beta}_{\alpha\{\lambda\}}(C_1, P) = 0.4762; D^{\beta}_{\alpha\{\lambda\}}(C_2, P) = 0.4097; D^{\beta}_{\alpha\{\lambda\}}(C_3, P) = 0.3071.$$

Thus, based on the recognition principle of the maximum degree of index, we observe that the pattern P should be classified in C_3.

On the other hand, if we apply [23] approach to the considered data, then their corresponding results are $D(C_1; P) = 0.5173$; $D(C_2; P) = 0.4371$ and $D(C_3; P) =$

Table 1. Effect of α, β and λ on each alternative: Example 1

(α,β)	λ	$D^{\beta}_{\alpha\{\lambda\}}(A_1;A^*)$	$D^{\beta}_{\alpha\{\lambda\}}(A_2;A^*)$	$D^{\beta}_{\alpha\{\lambda\}}(A_3;A^*)$	$D^{\beta}_{\alpha\{\lambda\}}(A_4;A^*)$	$D^{\beta}_{\alpha\{\lambda\}}(A_5;A^*)$	Ranking
	0.1	0.0157	0.1085	0.1026	0.0093	0.0054	(54132)
	0.3	0.0095	0.0571	0.0545	0.0056	0.0033	(54132)
(0.3,0.5)	0.5	0.0048	0.0326	0.0304	0.0029	0.0017	(54132)
	0.7	0.0017	0.0165	0.0145	0.0010	0.0006	(54132)
	0.9	0.0002	0.0047	0.0034	0.0001	0.0001	(54132)
	0.1	0.0884	0.1549	0.1681	0.0525	0.0288	(54123)
	0.3	0.0531	0.0828	0.0908	0.0316	0.0174	(54123)
(0.5, 0.7)	0.5	0.0272	0.0468	0.0508	0.0162	0.0089	(54123)
	0.7	0.0099	0.0230	0.0244	0.0059	0.0032	(54123)
	0.9	0.0011	0.0063	0.0064	0.0007	0.0004	(54123)
	0.1	0.5789	0.2249	0.3132	0.3434	0.1630	(52341)
	0.3	0.3397	0.1271	0.1787	0.2035	0.0974	(52341)
(0.7,1.2)	0.5	0.1761	0.0684	0.0952	0.1049	0.0500	(52341)
	0.7	0.0673	0.0295	0.0399	0.0393	0.0184	(52431)
	0.9	0.0084	0.0061	0.0074	0.0047	0.0021	(52431)
	0.1	0.7487	0.2421	0.3559	0.4440	0.2044	(52341)
	0.3	0.4363	0.1381	0.2041	0.2618	0.1219	(52341)
(1,1)	0.5	0.2271	0.0737	0.1081	0.1353	0.0626	(52341)
	0.7	0.0880	0.0310	0.0444	0.0513	0.0232	(52341)
	0.9	0.0114	0.0059	0.0077	0.0063	0.0027	(52431)
	0.1	2.0746	0.3610	0.6724	1.2165	0.5063	(25341)
	0.3	1.1572	0.2090	0.3845	0.6866	0.2956	(25341)
(1, 1.5)	0.5	0.6195	0.1094	0.2028	0.3642	0.1537	(25341)
	0.7	0.2635	0.0432	0.0822	0.1505	0.0594	(25341)
	0.9	0.0440	0.0061	0.0124	0.0227	0.0077	(25341)

Table 2. Comparison with the existing methodologies for Example 1

	Divergence measure values					Ranking
	$D(A_1,A^*)$	$D(A_2,A^*)$	$D(A_3,A^*)$	$D(A_4,A^*)$	$D(A_5,A^*)$	
Verma and Sharma [23]	0.4363	0.1381	0.2041	0.2618	0.1219	$A_5 \succ A_2 \succ A_3 \succ A_4 \succ A_1$
Vlachos and Sergiadis [24]	0.2209	0.0426	0.0831	0.1256	0.0315	$A_5 \succ A_2 \succ A_3 \succ A_4 \succ A_1$
Wei and Ye [26]	0.2271	0.0737	0.1081	0.1353	0.0626	$A_5 \succ A_2 \succ A_3 \succ A_4 \succ A_1$

0.2689. Again, if we apply [26] approach, then the measure corresponding to each pattern is observed as $D(C_1,P) = 0.2908$, $D(C_2,P) = 0.2425$ and $D(C_3,P) = 0.1522$ while if we utilize [24] approach than their corresponding values are $D(C_1,P) = 0.2160$, $D(C_2,P) = 0.1343$ and $D(C_3,P) = 0.1189$. Finally, if we utilize [20] approach to compute the corresponding divergence measure, then their values are $D(C_1,P) = 0.2372$, $D(C_2,P) = 0.2160$ and $D(C_3,P) = 0.1001$. Therefore, it has been concluded that pattern P belongs to the pattern C_3 to and the results is coincides with the existing divergence measures result.

Moreover, it is possible to analyze how the different parameter values of α, β and λ plays a role in the aggregation results, in this case, we consider different values of these parameters which are provided by the decision makers. The collective values of all the classifiers are summarized in Table 3. From this analysis, it has been observed that the best classifier for pattern P should be C_3 under all the cases. Also, it has been analyzed that the depending on the particular values of the parameter's value used, the orderings of the classifier is different, thus leading to different decisions. However, it seems that C_3 is the best choice while C_2 or C_1 are the worst choice in some cases.

Table 3. Effect of α, β and λ on the divergence measure: Example 2

(α, β)	λ	$D^{\beta}_{\alpha\{\lambda\}}(C_1; P)$	$D^{\beta}_{\alpha\{\lambda\}}(C_2; P)$	$D^{\beta}_{\alpha\{\lambda\}}(C_3; P)$	Ranking
(0.3, 0.5)	0.1	0.3129	0.3764	0.2681	$A_3 \succ A_1 \succ A_2$
	0.3	0.1728	0.2017	0.1474	$A_3 \succ A_1 \succ A_2$
	0.5	0.0929	0.1112	0.0796	$A_3 \succ A_1 \succ A_2$
	0.7	0.0400	0.0515	0.0347	$A_3 \succ A_1 \succ A_2$
	0.9	0.0066	0.0106	0.0060	$A_3 \succ A_1 \succ A_2$
(0.5, 0.7)	0.1	0.6924	0.6514	0.5211	$A_3 \succ A_2 \succ A_1$
	0.3	0.3649	0.3428	0.2758	$A_3 \succ A_2 \succ A_1$
	0.5	0.2054	0.1950	0.1548	$A_3 \succ A_2 \succ A_1$
	0.7	0.1003	0.0976	0.0750	$A_3 \succ A_2 \succ A_1$
	0.9	0.0246	0.0267	0.0182	$A_3 \succ A_1 \succ A_2$
(0.7, 1.2)	0.1	0.9465	0.7910	0.5384	$A_3 \succ A_2 \succ A_1$
	0.3	0.5053	0.4277	0.2851	$A_3 \succ A_2 \succ A_1$
	0.5	0.2857	0.2392	0.1621	$A_3 \succ A_2 \succ A_1$
	0.7	0.1408	0.1147	0.0810	$A_3 \succ A_2 \succ A_1$
	0.9	0.0387	0.0301	0.0226	$A_3 \succ A_2 \succ A_1$
(1, 1)	0.1	0.9624	0.8011	0.5047	$A_3 \succ A_2 \succ A_1$
	0.3	0.5173	0.4371	0.2689	$A_3 \succ A_2 \succ A_1$
	0.5	0.2908	0.2425	0.1522	$A_3 \succ A_2 \succ A_1$
	0.7	0.1412	0.1139	0.0753	$A_3 \succ A_2 \succ A_1$
	0.9	0.0379	0.0286	0.0208	$A_3 \succ A_2 \succ A_1$
(1, 1.5)	0.1	1.3290	1.1531	0.4438	$A_3 \succ A_2 \succ A_1$
	0.3	0.7519	0.6633	0.2562	$A_3 \succ A_2 \succ A_1$
	0.5	0.4021	0.3500	0.1356	$A_3 \succ A_2 \succ A_1$
	0.7	0.1702	0.1421	0.0555	$A_3 \succ A_2 \succ A_1$
	0.9	0.0316	0.0232	0.0101	$A_3 \succ A_2 \succ A_1$

Furthermore, in order to demonstrate the reasonability of the proposed divergence measure, we use different cases of the similarity as stated by the various researchers [3, 4, 6, 7, 16, 17, 19, 32] for IFSs A and B listed as below.

(i) $S_C(A,B) = 1 - \frac{\sum_{i=1}^{n} |\mu_A(x_i) - \nu_A(x_i) - \mu_B(x_i) + \nu_B(x_i)|}{2n}$.

(ii) $S_H(A,B) = 1 - \frac{\sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)|) + (|\nu_A(x_i) - \nu_B(x_i)|)}{2n}$.

(iii) $S_L(A,B) = 1 - \frac{\sum_{i=1}^{n} |S_A(x_i) - S_B(x_i)|}{4n} - \frac{\sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|}{4n}$.

(iv) $S_O(A,B) = 1 - \sqrt{\frac{\sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2}{2n}}$.

(v) $S_{DC}(A,B) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n} \left| \frac{\mu_A(x_i) - \nu_A(x_i) + \mu_B(x_i) - \nu_B(x_i)}{2} \right|^p}{n}}$.

(vi) $S_{HB}(A,B) = \frac{S_{DC}(\mu_A(x_i), \mu_B(x_i)) + S_{DC}(1 - \nu_A(x_i), 1 - \nu_B(x_i))}{2}$.

(vii) $S_e^p(A,B) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n}(\phi_\mu(x_i)+\phi_\nu(x_i))^p}{n}}$,
where $\phi_\mu(A,B) = \frac{|\mu_A(x_i)-\mu_B(x_i)|}{2}$; $\phi_\nu(x_i) = \frac{|\nu_B(x_i)-\nu_A(x_i)|}{2}$.

(viii) $S_s^p(A,B) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n}(\phi_{s1}(x_i)+\phi_{s2}(x_i))^p}{n}}$,
where $\phi_{s1}(x_i) = \frac{|m_{A1}(x_i)-m_{B1}(x_i)|}{2}$, $\phi_{s2}(x_i) = \frac{|m_{A2}(x_i)-m_{B2}(x_i)|}{2}$,
$m_{A1}(x_i) = \frac{\mu_A(x_i)+m_A(x_i)}{2}$, $m_{B1}(x_i) = \frac{\mu_B(x_i)+m_B(x_i)}{2}$,
$m_{A2}(x_i) = \frac{m_A(x_i)+1-\nu_A(x_i)}{2}$, $m_{B2}(x_i) = \frac{m_B(x_i)+1-\nu_B(x_i)}{2}$,
$m_A(x_i) = \frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}$ and $m_B(x_i) = \frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}$.

(ix) $S_{HY}^1 = 1 - d_H(A,B)$,
where $d_H(A,B) = \frac{1}{n}\sum_{i=1}^{n}\max(|\ \mu_A(x_i)-\mu_B(x_i)\ |,|\ \nu_A(x_i)-\nu_B(x_i)\ |)$.

(x) $S_{HY}^2 = (e^{-d_H(A,B)} - e^{-1})/(1-e^{-1})$, for the same $d_H(A,B)$

(xi) $S_{HY}^3 = (1-d_H(A,B))/(1+d_H(A,B))$, for the same $d_H(A,B)$.

The complete analysis has been given in Table 4 and hence concluded that the pattern P should be classifier with C_3 which is in accordance with the proposed technique ordering. Thus, we can say that the proposed divergence measure can suitably solve the problem in real-life situation and can be found as an alternative place than of the existing operators or measures.

Table 4. Similarity measure comparison for example Pattern recognition

	(P,Q_1)	(P,Q_2)	(P,Q_3)
S_C [3, 4]	0.7500	0.7667	0.9000
S_H [16]	0.7500	0.7667	0.9000
S_L [7]	0.7500	0.7667	0.9000
S_O [7]	0.7142	0.7551	0.8845
S_{DC} [6]	0.7500	0.7667	0.9000
S_{HB} [19]	0.7500	0.7667	0.9000
S_e^p [32]	0.7500	0.7667	0.9000
S_s^p [32]	0.7500	0.7667	0.9000
S_{HY}^1 [17]	0.7000	0.7333	0.8667
S_{HY}^2 [17]	0.5900	0.6297	0.8025
S_{HY}^3 [17]	0.5385	0.5789	0.7647

6. Conclusion

In this paper, a novel generalized directed divergence measure of order α and degree β has been presented under the IFS environment by considering the degrees of membership, non-membership and hesitation between the sets. Some desirable properties of the proposed measure have been discussed. It has been concluded from the proposed measure that the various existing divergence measures [2, 20, 23, 26] can deduce from it and hence the proposed measure is a more generalized one than others. Based on the proposed measure, a decision-making method has been proposed for finding the best alternative under the decision set. To demonstrate the proposed approach, two illustrative examples have been taken, one from finding the best alternative for investing a money and other one is from the pattern recognition field. Furthermore, in order to demonstrate the reasonability of the proposed measure, we use different cases of the similarity measures as stated by the various researchers and compared their results with the proposed measure. As the parameters of the measures provides the flexibility to the decision makers so a sensitivity analysis has also been addressed for showing the influence of these parameters on the performance of the decision making. Thus, we conclude that the proposed divergence measure can suitably solve the problem in real-life situation and can be found as an alternative place than of the existing operators or measures. In our further research will focus on adopting this approach to some more complicated applications in the field of medical diagnosis, fuzzy cluster analysis, uncertain programming and mathematical programming .

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Harish Garg (harishg58iitr@gmail.com)
School of Mathematics, Thapar University, Patiala-147004, Punjab, India,
URL: http://sites.google.com/site/harishg58iitr/

Nikunj Aggarwal (nikunj.aggarwal84@gmail.com)
Department of Mathematics, Jaypee Institute of Information and Technology Noida - 201307, U.P., India

Alka Tripathi (alka.choubey@jiit.ac.in)
Department of Mathematics, Jaypee Institute of Information and Technology Noida - 201307, U.P., India