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Fuzzy sets in UP-algebras

Jaruwat Somjanta, Natthanicha Thuekaew, Praprisri Kumpeangkeaw, Aiyared Iampan

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Abstract. The aim of this paper is to introduce and study fuzzy UP-subalgebras, fuzzy UP-ideals and fuzzy UP-filters of UP-algebras and investigate some of its properties. The notions of upper t-(strong) level subsets and lower t-(strong) level subsets are introduced from some fuzzy sets, and its characterizations are given.

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Keywords: UP-algebra, Fuzzy UP-subalgebra, Fuzzy UP-ideal, Fuzzy UP-filter, Upper t-(strong) level subset, Lower t-(strong) level subset.

Corresponding Author: Aiyared Iampan (aiyared.ia@up.ac.th)

1. Introduction and preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [6], BCI-algebras [7], BCH-algebras [4], KU-algebras [18], SU-algebras [14] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [7] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [6, 7] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

A fuzzy subset f of a set S is a function from S to a closed interval $[0, 1]$. The concept of a fuzzy subset of a set was first considered by Zadeh [25] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.

After the introduction of the concept of fuzzy sets by Zadeh [25], several researches were conducted on the generalizations of the notion of fuzzy set and application to many logical algebras such as: In 2000, Jun [8] introduced the notion of M-BCK/BCI-algebras and M-fuzzy subalgebras for a set M, and investigated some of their properties. In 2001, Lele, Wu, Weke, Mamadou and Njock [15] studied fuzzy ideals and weak ideals in BCK-algebras. Jun [9] introduced the notion of Q-fuzzy subalgebras of BCK/BCI-algebras, and provided some appropriate examples. In 2002, Jun [10] studied the fuzzification of sub-implicative ideals in BCI-algebras, and investigated some related properties. Yonglin and Xiaohong [24] introduced the notion of fuzzy a-ideals in BCI-algebras, and investigated its properties. In 2003, Ahn and Bang [1] classified the subalgebras by their family of level subalgebras in B-algebras. In 2004, Jun [11] introduced the concept of (α, β)-fuzzy ideals of BCK/BCI-algebras. In 2005, Akram and Dar [2] introduced the notions of T-fuzzy subalgebras and T-fuzzy H-ideals in BCI-algebras and investigated some of their properties. Jun [13] introduced the notion of (α, β)-fuzzy subalgebras of BCK/BCI-algebras. In 2007, Jun [12] introduced the notion of fuzzy subalgebras with thresholds of BCK/BCI-algebras, and obtained its characterizations. Akram and Dar [3] introduced the notion of fuzzy ideals in K-algebras. In 2008, Saeid and Jun [20] introduced the concept of anti fuzzy subalgebras of BCK/BCI-algebras by using the notion of anti fuzzy points. In 2009, Saeid and Rezvani [21] introduced the notion of fuzzy BF-algebra and fuzzy topological BF-algebras. In 2010, Song, Jun and Lee [22] introduced the notion of fuzzy ideals in BE-algebras, and investigated related properties. In 2011, Mostafa, Abd-Elnaby and Yousef introduced the notion of fuzzy KU-ideals in KU-algebras. In 2012, Mostafa, Abdel-Naby and Elgendy [16] introduced the notion of Q-ideals and fuzzy Q-ideals in Q-algebras, and studied their properties. In 2013, Rao [19] introduced the notion of fuzzy filters in BE-algebras. In 2014, Yamini and Kailasavalli [23] introduced the notion of B-ideals and fuzzy B-ideals in B-algebras. Mostafa and Kareem [17] introduced the concept of fuzzy n-fold KU-ideals in KU-algebras.

Iampan [5] now introduced a new algebraic structure, called a UP-algebra and a concept of UP-ideals and UP-subalgebras of UP-algebras. The notions of fuzzy subalgebras, fuzzy ideals and fuzzy filters play an important role in studying the many logical algebras. In this paper, we introduce the notions of fuzzy UP-subalgebras, fuzzy UP-ideals and fuzzy UP-filters of UP-algebras, and their properties are investigated.

Before we begin our study, we will introduce the definition of a UP-algebra.

Definition 1.1 ([5])**.** An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra, if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1): $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,
(UP-2): $0 \cdot x = x$,
(UP-3): $x \cdot 0 = 0$, and
(UP-4): $x \cdot y = y \cdot x = 0$ implies $x = y$.

Example 1.2 ([5])**.** Let X be a set. Define a binary operation $\cdot$ on the power set of X by putting $A \cdot B = B \cap A'$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X); \cdot, \varnothing)$ is a UP-algebra.

Example 1.3 ([5])**.** Let $A = \{0, a, b, c\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

(1.1)

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	a	0	c
c	0	a	b	0

Then $(A; \cdot, 0)$ is a UP-algebra.

In what follows, let A denote a UP-algebra unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 1.4 ([5])**.** *In a UP-algebra A, the following properties hold for any $x, y \in A$,*

(1) $x \cdot x = 0$,
(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,
(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,
(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,
(5) $x \cdot (y \cdot x) = 0$,
(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*
(7) $x \cdot (y \cdot y) = 0$.

Theorem 1.5 ([5])**.** *An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ is a UP-algebra if and only if it satisfies the following conditions: for all $x, y, z \in A$,*

(1) UP-1: $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,
(2) $(y \cdot 0) \cdot x = x$, *and*
(3) UP-4: $x \cdot y = y \cdot x = 0$ *implies* $x = y$.

Definition 1.6 ([5])**.** A nonempty subset B of A is called a UP-ideal of A, if it satisfies the following properties:
(i) the constant 0 of A is in B,
(ii) for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.
Clearly, A and $\{0\}$ are UP-ideals of A.

Theorem 1.7 ([5])**.** *Let A be a UP-algebra and $\{B_i\}_{i \in I}$ a family of UP-ideals of A. Then $\bigcap_{i \in I} B_i$ is a UP-ideal of A.*

Definition 1.8 ([5])**.** A subset S of A is called a UP-subalgebra of A, if it constant 0 of A is in S, and $(S; \cdot, 0)$ itself forms a UP-algebra.
Clearly, A and $\{0\}$ are UP-subalgebras of A.

Applying Proposition 1.4 (1), we can then easily prove the following Proposition.

Proposition 1.9 ([5])**.** *A nonempty subset S of a UP-algebra $A = (A; \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.*

Theorem 1.10 ([5])**.** *Let A be a UP-algebra and $\{B_i\}_{i \in I}$ a family of UP-subalgebras of A. Then $\bigcap_{i \in I} B_i$ is a UP-subalgebra of A.*

Definition 1.11. A nonempty subset F of A is called a UP-filter of A, if it satisfies the following properties:
(i) the constant 0 of A is in F,
(ii) for any $x, y \in A, x \in F$ and $x \cdot y \in F$ implies $y \in F$.

We can easily show the following example.

Example 1.12 ([5])**.** Let $A = \{0, a, b, c, d\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

(1.2)

$\cdot$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	d
b	0	0	0	c	d
c	0	0	b	0	d
d	0	0	0	0	0

It can be easily verified that $(A; \cdot, 0)$ is a UP-algebra, $\{0, a, c\}$ and $\{0, a, b\}$ are UP-ideals of A, and $\{0, a, c\}$ is a UP-filter of A. By Proposition 1.9, we can check that the set $\{0, a, b, c\}$ is a UP-subalgebra of A.

2. Main results

In this section, we first introduce the notions of a fuzzy UP-subalgebra, a fuzzy UP-ideal and a fuzzy UP-filter of a UP-algebra and study some of their basic properties. Finally, Upper t-(strong) level subsets and lower t-(strong) level subsets are derived from some fuzzy sets.

Definition 2.1 ([25])**.** A fuzzy set in a nonempty set X (or a fuzzy subset of X) is an arbitrary function $f \colon X \to [0, 1]$, where $[0, 1]$ is the unit segment of the real line.

If $A \subseteq X$, the characteristic function f_A of X is a function of X into $\{0, 1\}$ defined as follows:

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

By the definition of characteristic function, f_A is a function of X into $\{0, 1\} \subset [0, 1]$. Then f_A is a fuzzy set in X.

Definition 2.2. Let f be a fuzzy set in A. The fuzzy set $\overline{f}$ defined by $\overline{f}(x) = 1 - f(x)$ for all $x \in A$ is called the complement of f in A.

Definition 2.3. A fuzzy set f in A is called a fuzzy UP-subalgebra of A, if for any $x, y \in A$,

$$f(x \cdot y) \geq \min\{f(x), f(y)\}. \tag{2.1}$$

Example 2.4. By Example 1.12, we get $\{0, a, b, c\}$ is a UP-subalgebra of A. Then it can be easily verified that

$$f(x) = \begin{cases} 1 & \text{if } x \in \{0, a, b, c\}, \\ 0 & \text{if } x \in \{d\} \end{cases}$$

is a fuzzy UP-subalgebra of A.

Theorem 2.5. *Let B be a nonempty subset of A. Then B is a UP-subalgebra of A if and only if the characteristic function f_B is a fuzzy UP-subalgebra of A.*

Proof. Assume that B is a UP-subalgebra of A. Let $x, y \in A$.

Case 1: Suppose $x, y \in B$. Then $f_B(x) = 1$ and $f_B(y) = 1$. Thus

$$\min\{f_B(x), f_B(y)\} = \min\{1, 1\} = 1.$$

Since B is a UP-subalgebra of A, we have $x \cdot y \in B$. So $f_B(x \cdot y) = 1$. Hence $f_B(x \cdot y) = 1 \geq 1 = \min\{f_B(x), f_B(y)\}$.

Case 2: Suppose $x \notin B$ or $y \notin B$. Then $f_B(x) = 0$ or $f_B(y) = 0$. Thus $\min\{f_B(x), f_B(y)\} = 0$. So $f_B(x \cdot y) \geq 0 = \min\{f_B(x), f_B(y)\}$. Hence f_B is a fuzzy UP-subalgebra of A.

Conversely, assume that f_B is a fuzzy UP-subalgebra of A. Let $x, y \in B$. Then $f_B(x) = 1$ and $f_B(y) = 1$. Thus $\min\{f_B(x), f_B(y)\} = 1$. Since f_B is a fuzzy UP-subalgebra of A, we have $f_B(x \cdot y) \geq \min\{f_B(x), f_B(y)\} = 1$. So $f_B(x \cdot y) = 1$ and thus $x \cdot y \in B$. Hence B is a UP-subalgebra of A. □

Definition 2.6. A fuzzy set f in A is called a fuzzy UP-ideal of A, if it satisfies the following properties: for any $x, y, z \in A$,

(i) $f(0) \geq f(x)$,

(ii) $f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\}$.

Example 2.7. By Example 1.12, we get $\{0, a, b\}$ is a UP-ideal of A. Then it can be easily verified that

$$f(x) = \begin{cases} 1 & \text{if } x \in \{0, a, b\}, \\ 0 & \text{if } x \in \{c, d\} \end{cases}$$

is a fuzzy UP-ideal of A.

Lemma 2.8. *Let B be a nonempty subset of A. Then the constant 0 of A is in B if and only if $f_B(0) \geq f_B(x)$ for all $x \in A$.*

Proof. If $0 \in B$, then $f_B(0) = 1$. Thus $f_B(0) = 1 \geq f_B(x)$ for all $x \in A$.

Conversely, assume that $f_B(0) \geq f_B(x)$ for all $x \in A$. Since B is a nonempty subset of A, we have $a \in B$ for some $a \in A$. Then $f_B(0) \geq f_B(a) = 1$. Thus $f_B(0) = 1$. So $0 \in B$. □

Theorem 2.9. *Let B be a nonempty subset of A. Then B is a UP-ideal of A if and only if the characteristic function f_B is a fuzzy UP-ideal of A.*

Proof. Assume that B is a UP-ideal of A. Since $0 \in B$, it follows from Lemma 2.8 that $f_B(0) \geq f_B(x)$ for all $x \in A$. Next, let $x, y, z \in A$.

Case 1: Suppose $x \cdot (y \cdot z) \in B$ and $y \in B$. Then $f_B(x \cdot (y \cdot z)) = 1$ and $f_B(y) = 1$. Thus $\min\{f_B(x \cdot (y \cdot z)), f_B(y)\} = \min\{1, 1\} = 1$. Since $x \cdot (y \cdot z) \in B$ and $y \in B$, we have $x \cdot z \in B$. So $f_B(x \cdot z) = 1$. Hence $f_B(x \cdot z) = 1 \geq 1 = \min\{f_B(x \cdot (y \cdot z)), f_B(y)\}$.

Case 2: Suppose $x \cdot (y \cdot z) \notin B$ or $y \notin B$. Then $f_B(x \cdot (y \cdot z)) = 0$ or $f_B(y) = 0$. Thus $\min\{f_B(x \cdot (y \cdot z)), f_B(y)\} = 0$. So $f_B(x \cdot z) \geq 0 = \min\{f_B(x \cdot (y \cdot z)), f_B(y)\}$. Hence f_B is a fuzzy UP-ideal of A.

Conversely, assume that f_B is a fuzzy UP-ideal of A. Since $f_B(0) \geq f_B(x)$ for all $x \in A$, it follows from Lemma 2.8 that $0 \in B$. Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in B$ and $y \in B$. To show that $x \cdot z \in B$, assume that $x \cdot z \notin B$. Then $f_B(x \cdot z) =$

0, so $0 = f_B(x \cdot z) \geq \min\{f_B(x \cdot (y \cdot z)), f_B(y)\}$. Thus $\min\{f_B(x \cdot (y \cdot z)), f_B(y)\} = 0$. This implies that $f_B(x \cdot (y \cdot z)) = 0$ or $f_B(y) = 0$. So $x \cdot (y \cdot z) \notin B$ or $y \notin B$, a contradiction. Hence $x \cdot z \in B$ and thus B is a UP-ideal of A. □

Definition 2.10. A fuzzy set f in A is called a fuzzy UP-filter of A, if it satisfies the following properties: for any $x, y \in A$,
(i) $f(0) \geq f(x)$,
(ii) $f(y) \geq \min\{f(x), f(x \cdot y)\}$.

Example 2.11. By Example 1.12, we get $\{0, a, c\}$ is a UP-filter of A. Then it can be easily verified that

$$f(x) = \begin{cases} 1 & \text{if } x \in \{0, a, c\}, \\ 0 & \text{if } x \in \{b, d\} \end{cases}$$

is a fuzzy UP-filter of A.

Theorem 2.12. *Let F be a nonempty subset of A. Then F is a UP-filter of A if and only if the characteristic function f_F is a fuzzy UP-filter of A.*

Proof. Assume that F is a UP-filter of A. Since $0 \in F$, it follows from Lemma 2.8 that $f_F(0) \geq f_F(x)$, for all $x \in A$. Next, let $x, y \in A$.

Case 1: Suppose $x, y \in F$. Then $f_F(x) = 1$ and $f_F(y) = 1$. Thus

$$f_F(y) = 1 \geq f_F(x \cdot y) = \min\{1, f_F(x \cdot y)\} = \min\{f_F(x), f_F(x \cdot y)\}.$$

Case 2: Suppose $x \notin F$ or $y \notin F$. Then $f_F(x) = 0$ or $f_B(y) = 0$.
Case 2.1: If $x \notin F$, then $f_F(x) = 0$. Thus

$$f_F(y) \geq 0 = \min\{0, f_F(x \cdot y)\} = \min\{f_F(x), f_F(x \cdot y)\}.$$

Case 2.2: If $y \notin F$, then $f_F(y) = 0$. Since F is a UP-filter of A, we have $x \notin F$ or $x \cdot y \notin F$. Thus $f_F(x) = 0$ or $f_F(x \cdot y) = 0$. So $f_F(y) = 0 = \min\{f_F(x), f_F(x \cdot y)\}$. Hence f_F is a fuzzy UP-filter of A.

Conversely, assume that f_F is a fuzzy UP-filter of A. Since $f_F(0) \geq f_F(x)$ for all $x \in A$, it follows from Lemma 2.8 that $0 \in F$. Next, let $x, y \in A$ be such that $x \in F$ and $x \cdot y \in F$. Then $f_F(x) = 1$ and $f_F(x \cdot y) = 1$. To show that $y \in F$, assume that $y \notin F$. Then $f_F(y) = 0$. Thus $0 = f_F(y) \geq \min\{f_F(x), f_F(x \cdot y)\} = \min\{1, 1\} = 1$, a contradiction. So $y \in F$. Hence F is a UP-filter of A. □

Definition 2.13. A nonempty subset B of A is called a prime subset of A, if for any $x, y \in A$,

$$x \cdot y \in B \text{ implies } x \in B \text{ or } y \in B. \tag{2.2}$$

Definition 2.14. A UP-subalgebra (resp. UP-ideal, UP-filter) B of A is called a prime UP-subalgebra (resp. prime UP-ideal, prime UP-filter) of A, if B is a prime subset of A.

Definition 2.15. A fuzzy set f in A is called a prime fuzzy set in A, if for any $x, y \in A$,

$$f(x \cdot y) \leq \max\{f(x), f(y)\}. \tag{2.3}$$

Definition 2.16. A fuzzy UP-subalgebra (resp. fuzzy UP-ideal, fuzzy UP-filter) f of A is called a prime fuzzy UP-subalgebra (resp. prime fuzzy UP-ideal, prime fuzzy UP-filter) of A, if f is a prime fuzzy set in A.

Theorem 2.17. *Let B be a nonempty subset of A. Then B is a prime subset of A if and only if the characteristic function f_B is a prime fuzzy set in A.*

Proof. Assume that B is a prime subset of A and let $x, y \in A$.

Case 1: Suppose $x \cdot y \in B$. Since B is a prime subset of A, we have $x \in B$ or $y \in B$. Then $f_B(x) = 1$ or $f_B(y) = 1$. Thus $\max\{f_B(x), f_B(y)\} = 1$. So $f_B(x \cdot y) \leq 1 = \max\{f_B(x), f_B(y)\}$.

Case 2: Suppose $x \cdot y \notin B$. Then $f_B(x \cdot y) = 0 \leq \max\{f_B(x), f_B(y)\}$. Thus f_B is a prime fuzzy set in A.

Conversely, assume that f_B is a prime fuzzy set in A. Let $x, y \in A$ be such that $x \cdot y \in B$. Then $f_B(x \cdot y) = 1$. Thus $1 = f_B(x \cdot y) \leq \max\{f_B(x), f_B(y)\}$. So $\max\{f_B(x), f_B(y)\} = 1$. Hence $f_B(x) = 1$ or $f_B(y) = 1$. Therefore $x \in B$ or $y \in B$ and thus B is a prime subset of A. □

Theorem 2.18. *Let B be a nonempty subset of A. Then B is a prime UP-subalgebra of A if and only if the characteristic function f_B is a prime fuzzy UP-subalgebra of A.*

Proof. It is straightforward by Theorem 2.5 and 2.17. □

Theorem 2.19. *Let B be a nonempty subset of A. Then B is a prime UP-ideal of A if and only if the characteristic function f_B is a prime fuzzy UP-ideal of A.*

Proof. It is straightforward by Theorem 2.9 and 2.17. □

Theorem 2.20. *Let F be a nonempty subset of A. Then F is a prime UP-filter of A if and only if the characteristic function f_F is a prime fuzzy UP-filter of A.*

Proof. It is straightforward by Theorem 2.12 and 2.17. □

Definition 2.21. Let f be a fuzzy set in A. For any $t \in [0,1]$, the set

$$U(f;t) = \{x \in A \mid f(x) \geq t\} \text{ and } U^+(f;t) = \{x \in A \mid f(x) > t\}$$

are called an upper t-level subset and an upper t-strong level subset of f, respectively. The set

$$L(f;t) = \{x \in A \mid f(x) \leq t\} \text{ and } L^-(f;t) = \{x \in A \mid f(x) < t\}$$

are called a lower t-level subset and a lower t-strong level subset of f, respectively.

Theorem 2.22. *Let f be a fuzzy set in A. Then f is a fuzzy UP-subalgebra of A if and only if for all $t \in [0,1], U(f;t)$ is a UP-subalgebra of A, if $U(f;t)$ is nonempty.*

Proof. Assume that f is a fuzzy UP-subalgebra of A. Let $t \in [0,1]$ be such that $U(f;t) \neq \varnothing$ and let $x, y \in U(f;t)$. Then $f(x) \geq t$ and $f(y) \geq t$, so t is a lower bound of $\{f(x), f(y)\}$. Since f is a fuzzy UP-subalgebra of A, we have $f(x \cdot y) \geq \min\{f(x), f(y)\} \geq t$. Thus $x \cdot y \in U(f;t)$. So $U(f;t)$ is a UP-subalgebra of A.

Conversely, assume that for all $t \in [0,1], U(f;t)$ is a UP-subalgebra of A, if $U(f;t)$ is nonempty. Let $x, y \in A$. Then $f(x), f(y) \in [0,1]$. Choose $t = \min\{f(x), f(y)\}$. Then $f(x) \geq t$ and $f(y) \geq t$. Thus $x, y \in U(f;t) \neq \varnothing$. By assumption, we have $U(f;t)$ is a UP-subalgebra of A. So $x \cdot y \in U(f;t)$. Hence $f(x \cdot y) \geq t = \min\{f(x), f(y)\}$. Therefore f is a fuzzy UP-subalgebra of A. $\square$

Theorem 2.23. *Let f be a fuzzy set in A. Then f is a fuzzy UP-ideal of A if and only if for all $t \in [0,1], U(f;t)$ is a UP-ideal of A, if $U(f;t)$ is nonempty.*

Proof. Assume that f is a fuzzy UP-ideal of A. Let $t \in [0,1]$ be such that $U(f;t) \neq \varnothing$ and let $a \in U(f;t)$. Then $f(a) \geq t$. Since f is a fuzzy UP-ideal of A, we have $f(0) \geq f(a) \geq t$. Thus $0 \in U(f;t)$.

Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U(f;t)$ and $y \in U(f;t)$. Then $f(x \cdot (y \cdot z)) \geq t$ and $f(y) \geq t$. Thus t is a lower bound of $\{f(x \cdot (y \cdot z)), f(y)\}$. Since f is a fuzzy UP-ideal of A, we have $f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\} \geq t$. So $x \cdot z \in U(f;t)$. Hence $U(f;t)$ is a UP-ideal of A.

Conversely, assume that for all $t \in [0,1], U(f;t)$ is a UP-ideal of A, if $U(f;t)$ is nonempty. Let $x \in A$. Then $f(x) \in [0,1]$. Choose $t = f(x)$. Then $f(x) \geq t$. Thus $x \in U(f;t) \neq \varnothing$. By assumption, we have $U(f;t)$ is a UP-ideal of A. So $0 \in U(f;t)$. Hence $f(0) \geq t = f(x)$.

Next, let $x, y, z \in A$. Then $f(x \cdot (y \cdot z)), f(y) \in [0,1]$. Choose $t = \min\{f(x \cdot (y \cdot z)), f(y)\}$. Then $f(x \cdot (y \cdot z)) \geq t$ and $f(y) \geq t$. Thus $x \cdot (y \cdot z), y \in U(f;t) \neq \varnothing$. By assumption, we have $U(f;t)$ is a UP-ideal of A. So $x \cdot z \in U(f;t)$. Hence $f(x \cdot z) \geq t = \min\{f(x \cdot (y \cdot z)), f(y)\}$. Therefore f is a fuzzy UP-ideal of A. $\square$

Theorem 2.24. *Let f be a fuzzy set in A. Then f is a fuzzy UP-filter of A if and only if for all $t \in [0,1], U(f;t)$ is a UP-filter of A if $U(f;t)$ is nonempty.*

Proof. Assume that f is a fuzzy UP-filter of A. Let $t \in [0,1]$ be such that $U(f;t) \neq \varnothing$ and let $a \in U(f;t)$. Then $f(a) \geq t$. Since f is a fuzzy UP-filter of A, we have $f(0) \geq f(a) \geq t$. Thus $0 \in U(f;t)$.

Next, let $x, y \in A$ be such that $x \in U(f;t)$ and $x \cdot y \in U(f;t)$. Then $f(x) \geq t$ and $f(x \cdot y) \geq t$. Thus t is a lower bound of $\{f(x), f(x \cdot y)\}$. Since f is a fuzzy UP-filter of A, we have $f(y) \geq \min\{f(x), f(x \cdot y)\} \geq t$. So $y \in U(f;t)$. Hence $U(f;t)$ is a UP-filter of A.

Conversely, assume that for all $t \in [0,1], U(f;t)$ is a UP-filter of A, if $U(f;t)$ is nonempty. Let $x \in A$. Then $f(x) \in [0,1]$. Choose $t = f(x)$. Then $f(x) \geq t$. Thus $x \in U(f;t) \neq \varnothing$. By assumption, we have $U(f;t)$ is a UP-filter of A. So $0 \in U(f;t)$. Hence $f(0) \geq t = f(x)$.

Next, let $x, y \in A$. Then $f(x), f(x \cdot y) \in [0,1]$. Choose $t = \min\{f(x), f(x \cdot y)\}$. Then $f(x) \geq t$ and $f(x \cdot y) \geq t$. Thus $x, x \cdot y \in U(f;t) \neq \varnothing$. By assumption, we have $U(f;t)$ is a UP-filter of A. So $y \in U(f;t)$. Hence $f(y) \geq t = \min\{f(x), f(x \cdot y)\}$. Therefore f is a fuzzy UP-filter of A. $\square$

Theorem 2.25. *Let f be a fuzzy set in A. Then f is a prime fuzzy set in A if and only if for all $t \in [0,1], U(f;t)$ is a prime subset of A, if $U(f;t)$ is nonempty.*

Proof. Assume that f is a prime fuzzy set in A. Let $t \in [0,1]$ be such that $U(f;t) \neq \varnothing$. Let $x, y \in A$ be such that $x \cdot y \in U(f;t)$. Assume that $x \notin U(f;t)$ and $y \notin U(f;t)$. Then $f(x) < t$ and $f(y) < t$. Thus t is an upper bound of $\{f(x), f(y)\}$. Since f is a prime fuzzy set in A, we have $f(x \cdot y) \leq \max\{f(x), f(y)\} < t$. So$x \cdot y \notin U(f;t)$, a contradiction. Hence $x \in U(f;t)$ or $y \in U(f;t)$. Therefore $U(f;t)$ is a prime subset of A.

Conversely, assume that for all $t \in [0,1], U(f;t)$ is a prime subset of A if $U(f;t)$ is nonempty. Let $x, y \in A$. Then $f(x \cdot y) \in [0,1]$. Choose $t = f(x \cdot y)$. Then $f(x \cdot y) \geq t$. Thus $x \cdot y \in U(f;t) \neq \varnothing$. By assumption, we have $U(f;t)$ is a prime subset of A. So $x \in U(f;t)$ or $y \in U(f;t)$. Hence $t \leq f(x)$ or $t \leq f(y)$, so $f(x \cdot y) = t \leq \max\{f(x), f(y)\}$. Therefore f is a prime fuzzy set in A. □

Theorem 2.26. *Let f be a fuzzy set in A. Then f is a prime fuzzy UP-subalgebra of A if and only if for all $t \in [0,1], U(f;t)$ is a prime UP-subalgebra of A, if $U(f;t)$ is nonempty.*

Proof. It is straightforward by Theorem 2.22 and 2.25. □

Theorem 2.27. *Let f be a fuzzy set in A. Then f is a prime fuzzy UP-ideal of A if and only if for all $t \in [0,1], U(f;t)$ is a prime UP-ideal of A, if $U(f;t)$ is nonempty.*

Proof. It is straightforward by Theorem 2.23 and 2.25. □

Theorem 2.28. *Let f be a fuzzy set in A. Then f is a prime fuzzy UP-filter of A if and only if for all $t \in [0,1], U(f;t)$ is a prime UP-filter of A, if $U(f;t)$ is nonempty.*

Proof. It is straightforward by Theorem 2.24 and 2.25. □

Theorem 2.29. *Let f be a fuzzy set in A. Then f is a fuzzy UP-subalgebra of A if and only if for all $t \in [0,1], U^+(f;t)$ is a UP-subalgebra of A, if $U^+(f;t)$ is nonempty.*

Proof. Assume that f is a fuzzy UP-subalgebra of A. Let $t \in [0,1]$ be such that $U^+(f;t) \neq \varnothing$ and let $x, y \in U^+(f;t)$. Then $f(x) > t$ and $f(y) > t$. Thus t is a lower bound of $\{f(x), f(y)\}$. Since f is a fuzzy UP-subalgebra of A, we have $f(x \cdot y) \geq \min\{f(x), f(y)\} > t$. So $x \cdot y \in U^+(f;t)$. Hence $U^+(f;t)$ is a UP-subalgebra of A.

Conversely, assume that for all $t \in [0,1], U^+(f;t)$ is a UP-subalgebra of A, if $U^+(f;t)$ is nonempty. Assume that there exist $x, y \in A$ such that $f(x \cdot y) < \min\{f(x), f(y)\}$. Then $f(x \cdot y) \in [0,1]$. Choose $t = f(x \cdot y)$. Then $f(x) > t$ and $f(y) > t$. Thus $x, y \in U^+(f;t) \neq \varnothing$. By assumption, we have $U^+(f;t)$ is a UP-subalgebra of A and thus $x \cdot y \in U^+(f;t)$. So $f(x \cdot y) > t = f(x \cdot y)$, a contradiction. Hence $f(x \cdot y) \geq \min\{f(x), f(y)\}$, for all $x, y \in A$. Therefore f is a fuzzy UP-subalgebra of A. □

Theorem 2.30. *Let f be a fuzzy set in A. Then f is a fuzzy UP-ideal of A if and only if for all $t \in [0,1], U^+(f;t)$ is a UP-ideal of A, if $U^+(f;t)$ is nonempty.*

Proof. Assume that f is a fuzzy UP-ideal of A. Let $t \in [0,1]$ be such that $U^+(f;t) \neq \varnothing$ and let $a \in U^+(f;t)$. Then $f(a) > t$. Since f is a fuzzy UP-ideal of A, we have $f(0) \geq f(a) > t$. Thus $0 \in U^+(f;t)$.

Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U^+(f;t)$ and $y \in U^+(f;t)$. Then $f(x \cdot (y \cdot z)) > t$ and $f(y) > t$. Thus t is a lower bound of $\{f(x \cdot (y \cdot z)), f(y)\}$. Since f is a fuzzy UP-ideal of A, we have $f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\} > t$. So $x \cdot z \in U^+(f;t)$. Hence $U^+(f;t)$ is a UP-ideal of A.

Conversely, assume that for all $t \in [0,1], U^+(f;t)$ is a UP-ideal of A, if $U^+(f;t)$ is nonempty. Assume that there exists $x \in A$ such that $f(0) < f(x)$. Then $f(0) \in [0,1]$. Choose $t = f(0)$. Then $f(x) > t$. Thus $x \in U^+(f;t) \neq \varnothing$. By assumption, we have $U^+(f;t)$ is a UP-ideal of A and thus $0 \in U^+(f;t)$. So $f(0) > t = f(0)$, a contradiction. Hence $f(0) \geq f(x)$, for all $x \in A$.

Assume that there exist $x, y, z \in A$ such that $f(x \cdot z) < \min\{f(x \cdot (y \cdot z)), f(y)\}$. Then $f(x \cdot z) \in [0,1]$. Choose $t = f(x \cdot z)$. Then $f(x \cdot (y \cdot z)) > t$ and $f(y) > t$. Thus $x \cdot (y \cdot z), y \in U^+(f;t) \neq \varnothing$. By assumption, we have $U^+(f;t)$ is a UP-ideal of A and thus $x \cdot z \in U^+(f;t)$. So $f(x \cdot z) > t = f(x \cdot z)$, a contradiction. Hence $f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\}$, for all $x, y, z \in A$. Therefore f is a fuzzy UP-ideal of A. □

Theorem 2.31. *Let f be a fuzzy set in A. Then f is a fuzzy UP-filter of A if and only if for all $t \in [0,1], U^+(f;t)$ is a UP-filter of A, if $U^+(f;t)$ is nonempty.*

Proof. Assume that f is a fuzzy UP-filter of A. Let $t \in [0,1]$ be such that $U^+(f;t) \neq \varnothing$ and let $a \in U^+(f;t)$. Then $f(a) > t$. Since f is a fuzzy UP-filter of A, we have $f(0) \geq f(a) > t$. Thus $0 \in U^+(f;t)$.

Next, let $x, y \in A$ be such that $x \in U^+(f;t)$ and $x \cdot y \in U^+(f;t)$. Then $f(x) > t$ and $f(x \cdot y) > t$, so t is a lower bound of $\{f(x), f(x \cdot y)\}$. Since f is a fuzzy UP-filter of A, we have $f(y) \geq \min\{f(x), f(x \cdot y)\} > t$. Thus $y \in U^+(f;t)$. So $U^+(f;t)$ is a UP-filter of A.

Conversely, assume that for all $t \in [0,1], U^+(f;t)$ is a UP-filter of A, if $U^+(f;t)$ is nonempty. Assume that there exists $x \in A$ such that $f(0) < f(x)$. Then $f(0) \in [0,1]$. Choose $t = f(0)$. Then $f(x) > t$. Thus $x \in U^+(f;t) \neq \varnothing$. By assumption, we have $U^+(f;t)$ is a UP-filter of A and thus $0 \in U^+(f;t)$. So $f(0) > t = f(0)$, a contradiction. Hence $f(0) \geq f(x)$, for all $x \in A$.

Assume that there exist $x, y \in A$ such that $f(y) < \min\{f(x), f(x \cdot y)\}$. Then $f(y) \in [0,1]$. Choose $t = f(y)$. Then $f(x) > t$ and $f(x \cdot y) > t$. Thus $x, x \cdot y \in U^+(f;t) \neq \varnothing$. By assumption, we have $U^+(f;t)$ is a UP-filter of A and thus $y \in U^+(f;t)$. So $f(y) > t = f(y)$, a contradiction. Hence $f(y) \geq \min\{f(x), f(x \cdot y)\}$, for all $x, y \in A$. Therefore f is a fuzzy UP-filter of A. □

Theorem 2.32. *Let f be a fuzzy set in A. Then f is a prime fuzzy set in A if and only if for all $t \in [0,1], U^+(f;t)$ is a prime subset of A, if $U^+(f;t)$ is nonempty.*

Proof. Assume that f is a prime fuzzy set in A. Let $t \in [0,1]$ be such that $U^+(f;t) \neq \varnothing$. Let $x, y \in A$ be such that $x \cdot y \in U^+(f;t)$. Assume that $x \notin U^+(f;t)$ and $y \notin U^+(f;t)$. Then $f(x) \leq t$ and $f(y) \leq t$. Thus t is an upper bound of $\{f(x), f(y)\}$. Since f is a prime fuzzy set in A, we have $f(x \cdot y) \leq \max\{f(x), f(y)\} \leq t$ and thus $x \cdot y \notin U^+(f;t)$, a contradiction. So $x \in U^+(f;t)$ or $y \in U^+(f;t)$. Hence $U^+(f;t)$ is a prime subset of A.

Conversely, assume that for all $t \in [0,1], U^+(f;t)$ is a prime subset of A if $U^+(f;t)$ is nonempty. Assume that there exist $x, y \in A$ such that $f(x \cdot y) > \max\{f(x), f(y)\}$. Then $\max\{f(x), f(y)\} \in [0,1]$. Choose $t = \max\{f(x), f(y)\}$. Then $f(x \cdot y) > t$. Thus $x \cdot y \in U^+(f;t) \neq \varnothing$. By assumption, we have $U^+(f;t)$ is a prime subset of A and thus $x \in U^+(f;t)$ or $y \in U^+(f;t)$. So $f(x) > t = \max\{f(x), f(y)\}$ or

$f(y) > t = \max\{f(x), f(y)\}$, a contradiction. Hence $f(x \cdot y) \leq \max\{f(x), f(y)\}$, for all $x, y \in A$. Therefore f is a prime fuzzy set in A. $\square$

Theorem 2.33. *Let f be a fuzzy set in A. Then f is a prime fuzzy UP-subalgebra of A if and only if for all $t \in [0,1], U^+(f;t)$ is a prime UP-subalgebra of A, if $U^+(f;t)$ is nonempty.*

Proof. It is straightforward by Theorem 2.29 and 2.32. $\square$

Theorem 2.34. *Let f be a fuzzy set in A. Then f is a prime fuzzy UP-ideal of A if and only if for all $t \in [0,1], U^+(f;t)$ is a prime UP-ideal of A, if $U^+(f;t)$ is nonempty.*

Proof. It is straightforward by Theorem 2.30 and 2.32. $\square$

Theorem 2.35. *Let f be a fuzzy set in A. Then f is a prime fuzzy UP-filter of A if and only if for all $t \in [0,1], U^+(f;t)$ is a prime UP-filter of A, if $U^+(f;t)$ is nonempty.*

Proof. It is straightforward by Theorem 2.31 and 2.32. $\square$

Lemma 2.36. *Let f be a fuzzy set in A. Then the following statements hold for any $x, y \in A$,*

(1) $1 - \max\{f(x), f(y)\} = \min\{1 - f(x), 1 - f(y)\}$,
(2) $1 - \min\{f(x), f(y)\} = \max\{1 - f(x), 1 - f(y)\}$.

Proof. (1) If $\max\{f(x), f(y)\} = f(x)$, then $f(y) \leq f(x)$. Thus $1 - f(y) \geq 1 - f(x)$. So $\min\{1 - f(x), 1 - f(y)\} = 1 - f(x) = 1 - \max\{f(x), f(y)\}$.

Similarly, if $\max\{f(x), f(y)\} = f(y)$, then

$$\min\{1 - f(x), 1 - f(y)\} = 1 - f(y) = 1 - \max\{f(x), f(y)\}.$$

(2) If $\min\{f(x), f(y)\} = f(x)$, then $f(x) \leq f(y)$. Thus $1 - f(x) \geq 1 - f(y)$. So $\max\{1 - f(x), 1 - f(y)\} = 1 - f(x) = 1 - \min\{f(x), f(y)\}$.

Similarly, if $\min\{f(x), f(y)\} = f(y)$, then

$$\max\{1 - f(x), 1 - f(y)\} = 1 - f(y) = 1 - \min\{f(x), f(y)\}.$$

$\square$

Theorem 2.37. *Let f be a fuzzy set in A. Then $\overline{f}$ is a fuzzy UP-subalgebra of A if and only if for all $t \in [0,1], L(f;t)$ is a UP-subalgebra of A, if $L(f;t)$ is nonempty.*

Proof. Assume that $\overline{f}$ is a fuzzy UP-subalgebra of A. Let $t \in [0,1]$ be such that $L(f;t) \neq \varnothing$ and let $x, y \in L(f;t)$. Then $f(x) \leq t$ and $f(y) \leq t$. Thus t is an upper bound of $\{f(x), f(y)\}$. Since $\overline{f}$ is a fuzzy UP-subalgebra of A, we have

$$\overline{f}(x \cdot y) \geq \min\{\overline{f}(x), \overline{f}(y)\}.$$

By Lemma 2.36 (1), we have

$$1 - f(x \cdot y) \geq \min\{1 - f(x), 1 - f(y)\} = 1 - \max\{f(x), f(y)\}.$$

Thus $f(x \cdot y) \leq \max\{f(x), f(y)\} \leq t$. So $x \cdot y \in L(f;t)$. Hence $L(f;t)$ is a UP-subalgebra of A.

Conversely, assume that for all $t \in [0,1], L(f;t)$ is a UP-subalgebra of A, if $L(f;t)$ is nonempty. Let $x, y \in A$. Then $f(x), f(y) \in [0,1]$. Choose $t = \max\{f(x), f(y)\}$. Then $f(x) \leq t$ and $f(y) \leq t$. Thus $x, y \in L(f;t) \neq \varnothing$. By assumption, we have $L(f;t)$ is a UP-subalgebra of A and thus $x \cdot y \in L(f;t)$. So $f(x \cdot y) \leq t = \max\{f(x), f(y)\}$. By Lemma 2.36 (1), we have

$$\begin{aligned}\overline{f}(x \cdot y) &= 1 - f(x \cdot y)\\ &\geq 1 - \max\{f(x), f(y)\}\\ &= \min\{1 - f(x), 1 - f(y)\}\\ &= \min\{\overline{f}(x), \overline{f}(y)\}.\end{aligned}$$

Therefore $\overline{f}$ is a fuzzy UP-subalgebra of A. □

Theorem 2.38. *Let f be a fuzzy set in A. Then $\overline{f}$ is a fuzzy UP-ideal of A if and only if for all $t \in [0,1], L(f;t)$ is a UP-ideal of A, if $L(f;t)$ is nonempty.*

Proof. Assume that $\overline{f}$ is a fuzzy UP-ideal of A. Let $t \in [0,1]$ be such that $L(f;t) \neq \varnothing$ and let $a \in L(f;t)$. Then $f(a) \leq t$. Since $\overline{f}$ is a fuzzy UP-ideal of A, we have $\overline{f}(0) \geq \overline{f}(a)$. Thus $1 - f(0) \geq 1 - f(a)$. so $f(0) \leq f(a) \leq t$. Hence $0 \in L(f;t)$.

Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in L(f;t)$ and $y \in L(f;t)$. Then $f(x \cdot (y \cdot z)) \leq t$ and $f(y) \leq t$. Thus t is an upper bound of $\{f(x \cdot (y \cdot z)), f(y)\}$. Since $\overline{f}$ is a fuzzy UP-ideal of A, we have

$$\overline{f}(x \cdot z) \geq \min\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}.$$

By Lemma 2.36 (1), we have

$$1 - f(x \cdot z) \geq \min\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\} = 1 - \max\{f(x \cdot (y \cdot z)), f(y)\}.$$

So $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} \leq t$ and thus $x \cdot z \in L(f;t)$. Hence $L(f;t)$ is a UP-ideal of A.

Conversely, assume that for all $t \in [0,1], L(f;t)$ is a UP-ideal of A, if $L(f;t)$ is nonempty. Let $x \in A$. Then $f(x) \in [0,1]$. Choose $t = f(x)$. Then $f(x) \leq t$. Thus $x \in L(f;t) \neq \varnothing$. By assumption, we have $L(f;t)$ is a UP-ideal of A and thus $0 \in L(f;t)$. So $f(0) \leq t = f(x)$. Hence $\overline{f}(0) = 1 - f(0) \geq 1 - f(x) = \overline{f}(x)$.

Next, let $x, y, z \in A$. Then $f(x \cdot (y \cdot z)), f(y) \in [0,1]$. Choose $t = \max\{f(x \cdot (y \cdot z)), f(y)\}$. Then $f(x \cdot (y \cdot z)) \leq t$ and $f(y) \leq t$. Thus $x \cdot (y \cdot z), y \in L(f;t) \neq \varnothing$. By assumption, we have $L(f;t)$ is a UP-ideal of A and thus $x \cdot z \in L(f;t)$. So $f(x \cdot z) \leq t = \max\{f(x \cdot (y \cdot z)), f(y)\}$. By Lemma 2.36 (1), we have

$$\begin{aligned}\overline{f}(x \cdot z) &= 1 - f(x \cdot z)\\ &\geq 1 - \max\{f(x \cdot (y \cdot z)), f(y)\}\\ &= \min\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\}\\ &= \min\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}.\end{aligned}$$

Hence $\overline{f}$ is a fuzzy UP-ideal of A. □

Theorem 2.39. *Let f be a fuzzy set in A. Then $\overline{f}$ is a fuzzy UP-filter of A if and only if for all $t \in [0,1], L(f;t)$ is a UP-filter of A, if $L(f;t)$ is nonempty.*

Proof. Assume that $\overline{f}$ is a fuzzy UP-filter of A. Let $t \in [0,1]$ be such that $L(f;t) \neq \varnothing$ and let $a \in L(f;t)$. Then $f(a) \leq t$. Since $\overline{f}$ is a fuzzy UP-filter of A, we have $\overline{f}(0) \geq \overline{f}(a)$. Thus $1 - f(0) \geq 1 - f(a)$, so $f(0) \leq f(a) \leq t$. So $0 \in L(f;t)$.

Next, let $x, y \in A$ be such that $x \in L(f;t)$ and $x \cdot y \in L(f;t)$. Then $f(x) \leq t$ and $f(x \cdot y) \leq t$. Thus t is an upper bound of $\{f(x), f(x \cdot y)\}$. Since $\overline{f}$ is a fuzzy UP-filter of A, we have

$$\overline{f}(y) \geq \min\{\overline{f}(x), \overline{f}(x \cdot y)\}.$$

By Lemma 2.36 (1), we have

$$1 - f(y) \geq \min\{1 - f(x), 1 - f(x \cdot y)\} = 1 - \max\{f(x), f(x \cdot y)\}.$$

So $f(y) \leq \max\{f(x), f(x \cdot y)\} \leq t$ and thus $y \in L(f;t)$. Hence $L(f;t)$ is a UP-filter of A.

Conversely, assume that for all $t \in [0,1], L(f;t)$ is a UP-filter of A if $L(f;t)$ is nonempty. Let $x \in A$. Then $f(x) \in [0,1]$. Choose $t = f(x)$. Then $f(x) \leq t$. Thus $x \in L(f;t) \neq \varnothing$. By assumption, we have $L(f;t)$ is a UP-filter of A and thus $0 \in L(f;t)$. So $f(0) \leq t = f(x)$. Hence $\overline{f}(0) = 1 - f(0) \geq 1 - f(x) = \overline{f}(x)$.

Next, let $x, y \in A$. Then $f(x), f(x \cdot y) \in [0,1]$. Choose $t = \max\{f(x), f(x \cdot y)\}$. Then $f(x) \leq t$ and $f(x \cdot y) \leq t$. Thus $x, x \cdot y \in L(f;t) \neq \varnothing$. By assumption, we have $L(f;t)$ is a UP-filter of A and thus $y \in L(f;t)$. Thus $f(y) \leq t = \max\{f(x), f(x \cdot y)\}$. By Lemma 2.36 (1), we have

$$\begin{aligned}
\overline{f}(y) &= 1 - f(y) \\
&\geq 1 - \max\{f(x), f(x \cdot y)\} \\
&= \min\{1 - f(x), 1 - f(x \cdot y)\} \\
&= \min\{\overline{f}(x), \overline{f}(x \cdot y)\}.
\end{aligned}$$

Hence $\overline{f}$ is a fuzzy UP-filter of A. □

Theorem 2.40. *Let f be a fuzzy set in A. Then $\overline{f}$ is a prime fuzzy set in A if and only if for all $t \in [0,1], L(f;t)$ is a prime subset of A, if $L(f;t)$ is nonempty.*

Proof. Assume that $\overline{f}$ is a prime fuzzy set in A. Let $t \in [0,1]$ be such that $L(f;t) \neq \varnothing$. Let $x, y \in A$ be such that $x \cdot y \in L(f;t)$. Assume that $x \notin L(f;t)$ and $y \notin L(f;t)$. Then $f(x) > t$ and $f(y) > t$. Thus t is a lower bound of $\{f(x), f(y)\}$. Since $\overline{f}$ is a prime fuzzy set in A, we have

$$\overline{f}(x \cdot y) \leq \max\{\overline{f}(x), \overline{f}(y)\}.$$

By Lemma 2.36 (2), we have

$$1 - f(x \cdot y) \leq \max\{1 - f(x), 1 - f(y)\} = 1 - \min\{f(x), f(y)\}.$$

So $f(x \cdot y) \geq \min\{f(x), f(y)\} > t$ and thus $x \cdot y \notin L(f;t)$, a contradiction. Hence $x \in L(f;t)$ or $y \in L(f;t)$. Therefore $L(f;t)$ is a prime subset of A.

Conversely, assume that for all $t \in [0,1], L(f;t)$ is a prime subset of A if $L(f;t)$ is nonempty. Let $x, y \in A$. Then $f(x \cdot y) \in [0,1]$. Choose $t = f(x \cdot y)$. Then $f(x \cdot y) \leq t$. Thus $x \cdot y \in L(f;t) \neq \varnothing$. By assumption, we have $L(f;t)$ is a prime subset of A and thus $x \in L(f;t)$ or $y \in L(f;t)$. So $t \geq f(x)$ or $t \geq f(y)$. Hence $f(x \cdot y) = t \geq \min\{f(x), f(y)\}$. By Lemma 2.36 (2), we have

$$\begin{aligned}\overline{f}(x \cdot y) &= 1 - f(x \cdot y)\\ &\leq 1 - \min\{f(x), f(y)\}\\ &= \max\{1 - f(x), 1 - f(y)\}\\ &= \max\{\overline{f}(x), \overline{f}(y)\}.\end{aligned}$$

Therefore $\overline{f}$ is a prime fuzzy set in A. □

Theorem 2.41. *Let f be a fuzzy set in A. Then $\overline{f}$ is a prime fuzzy UP-subalgebra of A if and only if for all $t \in [0,1], L(f;t)$ is a prime UP-subalgebra of A, if $L(f;t)$ is nonempty.*

Proof. It is straightforward by Theorem 2.37 and 2.40. □

Theorem 2.42. *Let f be a fuzzy set in A. Then $\overline{f}$ is a prime fuzzy UP-ideal of A if and only if for all $t \in [0,1], L(f;t)$ is a prime UP-ideal of A, if $L(f;t)$ is nonempty.*

Proof. It is straightforward by Theorem 2.38 and 2.40. □

Theorem 2.43. *Let f be a fuzzy set in A. Then $\overline{f}$ is a prime fuzzy UP-filter of A if and only if for all $t \in [0,1], L(f;t)$ is a prime UP-filter of A, if $L(f;t)$ is nonempty.*

Proof. It is straightforward by Theorem 2.39 and 2.40. □

Theorem 2.44. *Let f be a fuzzy set in A. Then $\overline{f}$ is a fuzzy UP-subalgebra of A if and only if for all $t \in [0,1], L^{-}(f;t)$ is a UP-subalgebra of A, if $L^{-}(f;t)$ is nonempty.*

Proof. Assume that $\overline{f}$ is a fuzzy UP-subalgebra of A. Let $t \in [0,1]$ be such that $L^{-}(f;t) \neq \varnothing$ and let $x, y \in L^{-}(f;t)$. Then $f(x) < t$ and $f(y) < t$. Thus t is an upper bound of $\{f(x), f(y)\}$. Since $\overline{f}$ is a fuzzy UP-subalgebra of A, we have

$$\overline{f}(x \cdot y) \geq \min\{\overline{f}(x), \overline{f}(y)\}.$$

By Lemma 2.36 (1), we have

$$1 - f(x \cdot y) \geq \min\{1 - f(x), 1 - f(y)\} = 1 - \max\{f(x), f(y)\}.$$

So $f(x \cdot y) \leq \max\{f(x), f(y)\} < t$ and thus $x \cdot y \in L^{-}(f;t)$. Hence $L^{-}(f;t)$ is a UP-subalgebra of A.

Conversely, assume that for all $t \in [0,1], L^{-}(f;t)$ is a UP-subalgebra of A if $L^{-}(f;t)$ is nonempty. Assume that there exist $x, y \in A$ such that $\overline{f}(x \cdot y) < \min\{\overline{f}(x), \overline{f}(y)\}$. By Lemma 2.36 (1), we have

$$1 - f(x \cdot y) < \min\{1 - f(x), 1 - f(y)\} = 1 - \max\{f(x), f(y)\}.$$

Thus $f(x \cdot y) > \max\{f(x), f(y)\}$.

Now $f(x \cdot y) \in [0,1]$, we choose $t = f(x \cdot y)$. Then $f(x) < t$ and $f(y) < t$. Thus $x, y \in L^{-}(f;t) \neq \varnothing$. By assumption, we have $L^{-}(f;t)$ is a UP-subalgebra of A and thus $x \cdot y \in L^{-}(f;t)$. So $f(x \cdot y) < t = f(x \cdot y)$, a contradiction. Hence $\overline{f}(x \cdot y) \geq \min\{\overline{f}(x), \overline{f}(y)\}$, for all $x, y \in A$. Therefore $\overline{f}$ is a fuzzy UP-subalgebra of A. □

Theorem 2.45. *Let f be a fuzzy set in A. Then $\overline{f}$ is a fuzzy UP-ideal of A if and only if for all $t \in [0,1], L^-(f;t)$ is a UP-ideal of A, if $L^-(f;t)$ is nonempty.*

Proof. Assume that $\overline{f}$ is a fuzzy UP-ideal of A. Let $t \in [0,1]$ be such that $L^-(f;t) \neq \varnothing$ and let $a \in L^-(f;t)$. Then $f(a) < t$. Since $\overline{f}$ is a fuzzy UP-ideal of A, we have $\overline{f}(0) \geq \overline{f}(a)$. Thus $1 - f(0) \geq 1 - f(a)$. So $f(0) \leq f(a) < t$. Hence $0 \in L^-(f;t)$.

Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in L^-(f;t)$ and $y \in L^-(f;t)$. Then $f(x \cdot (y \cdot z)) < t$ and $f(y) < t$, so t is an upper bound of $\{f(x \cdot (y \cdot z)), f(y)\}$. Since $\overline{f}$ is a fuzzy UP-ideal of A, we have

$$\overline{f}(x \cdot z) \geq \min\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}.$$

By Lemma 2.36 (1), we have

$$1 - f(x \cdot z) \geq \min\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\} = 1 - \max\{f(x \cdot (y \cdot z)), f(y)\}.$$

Thus $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} < t$. So $x \cdot z \in L^-(f;t)$. Hence $L^-(f;t)$ is a UP-ideal of A.

Conversely, assume that for all $t \in [0,1], L^-(f;t)$ is a UP-ideal of A, if $L^-(f;t)$ is nonempty. Assume that there exists $x \in A$ such that $\overline{f}(0) < \overline{f}(x)$. Then $1 - f(0) < 1 - f(x)$. Thus $f(0) > f(x)$.

Now $f(0) \in [0,1]$, we choose $t = f(0)$. Then $f(x) < t$. Thus $x \in L^-(f;t) \neq \varnothing$. By assumption, we have $L^-(f;t)$ is a UP-ideal of A and thus $0 \in L^-(f;t)$. So $f(0) < t = f(0)$, a contradiction. Hence $\overline{f}(0) \geq \overline{f}(x)$, for all $x \in A$.

Assume that there exist $x, y, z \in A$ such that $\overline{f}(x \cdot z) < \min\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$. By Lemma 2.36 (1), we have

$$1 - f(x \cdot z) < \min\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\} = 1 - \max\{f(x \cdot (y \cdot z)), f(y)\}.$$

Then $f(x \cdot z) > \max\{f(x \cdot (y \cdot z)), f(y)\}$.

Now $f(x \cdot z) \in [0,1]$, we choose $t = f(x \cdot z)$. Then $f(x \cdot (y \cdot z)) < t$ and $f(y) < t$. Thus $x \cdot (y \cdot z), y \in L^-(f;t) \neq \varnothing$. By assumption, we have $L^-(f;t)$ is a UP-ideal of A and thus $x \cdot z \in L^-(f;t)$. So $f(x \cdot z) < t = f(x \cdot z)$, a contradiction. Hence $\overline{f}(x \cdot z) \geq \min\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$, for all $x, y, z \in A$. Therefore $\overline{f}$ is a fuzzy UP-ideal of A. □

Theorem 2.46. *Let f be a fuzzy set in A. Then $\overline{f}$ is a fuzzy UP-filter of A if and only if for all $t \in [0,1], L^-(f;t)$ is a UP-filter of A, if $L^-(f;t)$ is nonempty.*

Proof. Assume that $\overline{f}$ is a fuzzy UP-filter of A. Let $t \in [0,1]$ be such that $L^-(f;t) \neq \varnothing$ and let $a \in L^-(f;t)$. Then $f(a) < t$. Since $\overline{f}$ is a fuzzy UP-filter of A, we have $\overline{f}(0) \geq \overline{f}(a)$. Thus $1 - f(0) \geq 1 - f(a)$. So $f(0) \leq f(a) < t$. Hence $0 \in L^-(f;t)$.

Next, let $x, y \in A$ be such that $x \in L^-(f;t)$ and $x \cdot y \in L^-(f;t)$. Then $f(x) < t$ and $f(x \cdot y) < t$. Thus t is an upper bound of $\{f(x), f(x \cdot y)\}$. Since $\overline{f}$ is a fuzzy UP-filter of A, we have

$$\overline{f}(y) \geq \min\{\overline{f}(x), \overline{f}(x \cdot y)\}.$$

By Lemma 2.36 (1), we have

$$1 - f(y) \geq \min\{1 - f(x), 1 - f(x \cdot y)\} = 1 - \max\{f(x), f(x \cdot y)\}.$$

So $f(y) \leq \max\{f(x), f(x \cdot y)\} < t$ and thus $y \in L^-(f;t)$. Hence $L^-(f;t)$ is a UP-filter of A.

Conversely, assume that for all $t \in [0,1], L^-(f;t)$ is a UP-filter of A, if $L^-(f;t)$ is nonempty. Assume that there exists $x \in A$ such that $\overline{f}(0) < \overline{f}(x)$. Then $1 - f(0) < 1 - f(x)$. Thus $f(0) > f(x)$.

Now $f(0) \in [0,1]$, we choose $t = f(0)$. Then $f(x) < t$. Thus $x \in L^-(f;t) \neq \varnothing$. By assumption, we have $L^-(f;t)$ is a UP-filter of A and thus $0 \in L^-(f;t)$. So $f(0) < t = f(0)$, a contradiction. Hence $\overline{f}(0) \geq \overline{f}(x)$, for all $x \in A$.

Assume that there exist $x, y \in A$ such that $\overline{f}(y) < \min\{\overline{f}(x), \overline{f}(x \cdot y)\}$. By Lemma 2.36 (1), we have

$$1 - f(y) < \min\{1 - f(x), 1 - f(x \cdot y)\} = 1 - \max\{f(x), f(x \cdot y)\}.$$

Then $f(y) > \max\{f(x), f(x \cdot y)\}$.

Now $f(y) \in [0,1]$, we choose $t = f(y)$. Then $f(x) < t$ and $f(x \cdot y) < t$. Thus $x, x \cdot y \in L^-(f;t) \neq \varnothing$. By assumption, we have $L^-(f;t)$ is a UP-filter of A and thus $y \in L^-(f;t)$. So $f(y) < t = f(y)$, a contradiction. Hence $\overline{f}(y) \geq \min\{\overline{f}(x), \overline{f}(x \cdot y)\}$, for all $x, y \in A$. Therefore $\overline{f}$ is a fuzzy UP-filter of A. □

Theorem 2.47. *Let f be a fuzzy set in A. Then $\overline{f}$ is a prime fuzzy set in A if and only if for all $t \in [0,1], L^-(f;t)$ is a prime subset of A, if $L^-(f;t)$ is nonempty.*

Proof. Assume that $\overline{f}$ is a prime fuzzy set in A. Let $t \in [0,1]$ be such that $L^-(f;t) \neq \varnothing$. Let $x, y \in A$ be such that $x \cdot y \in L^-(f;t)$. Assume that $x \notin L^-(f;t)$ and $y \notin L^-(f;t)$. Then $f(x) \geq t$ and $f(y) \geq t$. Thus t is a lower bound of $\{f(x), f(y)\}$. Since $\overline{f}$ is a prime fuzzy set in A, we have

$$\overline{f}(x \cdot y) \leq \max\{\overline{f}(x), \overline{f}(y)\}.$$

By Lemma 2.36 (2), we have

$$1 - f(x \cdot y) \leq \max\{1 - f(x), 1 - f(y)\} = 1 - \min\{f(x), f(y)\}.$$

So $f(x \cdot y) \geq \min\{f(x), f(y)\} \geq t$ and thus $x \cdot y \notin L^-(f;t)$, a contradiction. Hence $x \in L^-(f;t)$ or $y \in L^-(f;t)$. Therefore $L^-(f;t)$ is a prime subset of A.

Conversely, assume that for all $t \in [0,1], L^-(f;t)$ is a prime subset of A, if $L^-(f;t)$ is nonempty. Assume that there exist $x, y \in A$ such that $\overline{f}(x \cdot y) > \max\{\overline{f}(x), \overline{f}(y)\}$. By Lemma 2.36 (2), we have

$$1 - f(x \cdot y) > \max\{1 - f(x), 1 - f(y)\} = 1 - \min\{f(x), f(y)\}.$$

Then $f(x \cdot y) < \min\{f(x), f(y)\}$.

Now $\min\{f(x), f(y)\} \in [0,1]$, we choose $t = \min\{f(x), f(y)\}$. Then $f(x \cdot y) < t$. Thus $x \cdot y \in L^-(f;t) \neq \varnothing$. By assumption, we have $L^-(f;t)$ is a prime subset of A and thus $x \in L^-(f;t)$ or $y \in L^-(f;t)$. So $f(x) < t = \min\{f(x), f(y)\}$ or $f(y) < t = \min\{f(x), f(y)\}$, a contradiction. Hence $\overline{f}(x \cdot y) \leq \max\{\overline{f}(x), \overline{f}(y)\}$, for all $x, y \in A$. Therefore $\overline{f}$ is a prime fuzzy set in A. □

Theorem 2.48. *Let f be a fuzzy set in A. Then $\overline{f}$ is a prime fuzzy UP-subalgebra of A if and only if for all $t \in [0,1], L^-(f;t)$ is a prime UP-subalgebra of A, if $L^-(f;t)$ is nonempty.*

Proof. It is straightforward by Theorem 2.44 and 2.47. □

Theorem 2.49. *Let f be a fuzzy set in A. Then $\overline{f}$ is a prime fuzzy UP-ideal of A if and only if for all $t \in [0,1], L^{-}(f;t)$ is a prime UP-ideal of A, if $L^{-}(f;t)$ is nonempty.*

Proof. It is straightforward by Theorem 2.45 and 2.47. □

Theorem 2.50. *Let f be a fuzzy set in A. Then $\overline{f}$ is a prime fuzzy UP-filter of A if and only if for all $t \in [0,1], L^{-}(f;t)$ is a prime UP-filter of A, if $L^{-}(f;t)$ is nonempty.*

Proof. It is straightforward by Theorem 2.46 and 2.47. □

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Jaruwat Somjanta (jaruwat.st@gmail.com)
Department of Mathematics, School of Science, University of Phayao, Phayao, postal code 56000, Thailand

Natthanicha Thuekaew (aj.natthanicha@gmail.com)
Department of Mathematics, School of Science, University of Phayao, Phayao, postal code 56000, Thailand

Praprisri Kumpeangkeaw (praprisri.k@gmail.com)
Department of Mathematics, School of Science, University of Phayao, Phayao, postal code 56000, Thailand

Aiyared Iampan (aiyared.ia@up.ac.th)
Department of Mathematics, School of Science, University of Phayao, Phayao, postal code 56000, Thailand