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L-fuzzy prime spectrum of l-groups

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Abstract. The set of all L-fuzzy Prime convex sub l-groups of an l-group (not necessarily abelian) with a strong order unit is topologized and the certain of its subspaces as well its functorial nature are discussed.

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1. Introduction

The notion of fuzzy subset of a set was introduced by Zadeh[22]. Rosenfeld [18] applied this concept to the theory of groups and groupoids. Since then, so many have been applied these ideas to various algebraic structures. Swamy and Swamy studied Fuzzy Prime ideals of rings[21]. For further study see [3, 12, 14, 17]. Goguen [8]initiated a more abstract study of fuzzy sets by replacing the values set $[0, 1]$, by a complete lattice in an attempt to make a generalized study of fuzzy set theory by studying L-fuzzy sets. Most of the authors considered fuzzy subsets by taking values in a complete lattice. Fuzzy algebra is now a well developed part of algebra. Partially ordered algebraic systems play an important role in algebra. Especially l-groups, l-rings, Vector lattices, and f-rings are important concepts in algebra [2, 4, 5, 6, 7, 9, 16]. we introduced L-fuzzy sub l-groups and L-fuzzy l-ideals in [19] and we introduced the concepts of L-fuzzy convex sub l-groups, L-fuzzy prime convex sub l-groups and L-fuzzy maximal convex sub l-groups in[20].

A study of prime spectrum or the topological space obtained by introducing Zariski topology, on the set of prime l-ideals of a commutative l-group with strong order unit, plays an important role in the field of commutative algebra, algebraic geometry and lattice theory. In the last few years a considerable amount of work has been done on fuzzy ideals in general and prime fuzzy ideals. Also a considerable amount of work has been done on L-fuzzy prime spectrum on prime ideals of a commutative ring with unity [10, 11, 13, 15, 23]. Now, it is natural to attempt to introduce a topology on the set of L-fuzzy prime convex sub l-groups of an l-group. In this paper, G is a lattice ordered group (not necessarily abelian) and $LSpec(G) = X$ be the set of all L-fuzzy prime convex sub l-groups of G. The space $LSpec(G) = X$ has many interesting properties. The topological space $Spec(G)$ is the spectrum of (non fuzzy) prime convex l-subgroups of G. we prove that $X(x_a)$, $x \in G,\ a \in L$ is a basis of X. When L is regular, we show that (i)$X(x_a) = \varnothing$, then $x = 0$ and (ii) For any $a \in L - \{0\}$, $X(0_a) = \varnothing$. Also, we characterize X in terms of $X(x_a)$ as, $X(x_a) = X$ where $x \in G^+$ if and only if x is a strong order unit in G and a is not contained in any prime element in L. If a is contained in a prime element in L, the above result is not true as in example. Throughout this paper $X_a = \{\lambda \in X \mid Im\lambda = \{1, a\}\}$, where $a \in L - \{1\}$ is prime. We show that X_a is compact, when G has a strong order unit e and X_a is homeomorphic to $Spec(G)$. We have a characterization of X_a as ,X_a is T_1 if and only if every element of X_a is a L-fuzzy maximal convex sub l-group of G. Also, we prove that as a subset the space $Spec(G)$ is a dense subspace of $LSpec(G) = X$. We prove that whenever l-groups G and G' are homomorphic, $LSpec(G)$ and $LSpec(G')$ are homeomorphic. If $f : G \to G'$ is a homomorphism of l-groups, then $LSpec(G')$ is homeomorphic to the subspace of $LSpec(G)$ consisting of L-fuzzy prime convex l-groups, which are constant on $Kerf$. Also, we prove that $LSpec(G)$ is T_0 and X is compact if and only if G has a strong order unit. But, the compactness of $LSpec(G)$ is depends on the space $I(L)$ set of all irreducible elements of L.

Throughout this paper, let $G \neq 0$ be an l-group and L stands for a nontrivial complete lattice in which the infinite meet distributive law, $a \wedge (\vee_{s \in S} s) = \vee_{s \in S}(a \wedge s)$ for any $S \subseteq L$ and $a \in L$ holds. Throughout the paper we consider meet irreducible elements of L only.

2. Preliminaries

Definition 2.1 ([5]). A lattice ordered group is a system $G = (G, +, \leq)$, where
(i) $(G, +)$ is a group,
(ii) $(G, \leq)$ is a lattice and
(iii) the inclusion is invariant under all translations $x \mapsto a + x + b$ i.e, $x \leq y \Rightarrow a + x + b \leq a + y + b$, for all $a, b \in G$.

Definition 2.2 ([5]). If a is an element of l-group G, then $a \vee (-a)$ is called the absolute value of a and is denoted by $|a|$. Any element a of an l-group G can be written as $a = a \vee 0 + a \wedge 0$, i.e., $a = a^+ + a^-$, where a^+ is called positive part of a and a^- is called negative part of a.

Theorem 2.3 ([5]). *In any l-group G, for all $a \in G$, (i) $|a| \geq 0$, moreover $|a| > 0$ unless $a = 0$, (ii) $a^+ \wedge (-a^+) = 0$, (iii) $|a| = a^+ - a^-$.*

Theorem 2.4 ([2]). *If G is an l-group and $M \in \mathcal{C}(G)$, then the following are equivalent :*
(1)If $A, B \in \mathcal{C}(G)$ and $M \supseteq A \cap B$, then $M \supseteq A$ or $M \supseteq B$.
(2)If $A, B \in \mathcal{C}(G)$, $A \supset M$ and $B \supset M$, then $A \cap B \supset M$.
(3)If $a, b \in G^+ - M$, then $a \wedge b \in G^+ - M$.
(4)If $a, b \in G^+ - M$, then $a \wedge b > 0$.

(5)The lattice of right cosets of M is totally ordered.
(6)$\{C \in \mathcal{C}(G) \mid C \supseteq M\}$ is chain.
(7)M is the intersection of a chain of regular convex l-subgroups.

Definition 2.5 ([2])**.** A convex l-subgroup that satisfies any one of the conditions of above theorem will be called prime.

Lemma 2.6 ([7])**.** *If G is a l-group and $g \in G$, then $\langle g \rangle = \{f \in G \mid |f| \leq n|g|, n \in \mathbb{Z}^+\}$.*

Corollary 2.7 ([7])**.** *If G is a l-group and $f, g \in G^+$, then $\langle f \wedge g \rangle = \langle f \rangle \wedge \langle g \rangle$ and $\langle f \vee g \rangle = \langle f \rangle \vee \langle g \rangle$.*

Corollary 2.8 ([7])**.** *Let $g \in G$. Then, $\langle |g| \rangle = \langle g \rangle$.*

Definition 2.9 ([7])**.** Let $e \in G$ be called a strong unit if for any $a \in G$, $|a| < n|e|$, for some $n \in \mathbb{Z}^+$. Clearly $\langle e \rangle = G$.

Definition 2.10 ([17])**.** An L-Fuzzy subset λ of X is a mapping from X into L, where L is a complete lattice satisfying the infinite meet distributive law. If L is the unit interval [0,1] of real numbers, there are the usual fuzzy subsets of X. A L-fuzzy subset $\lambda : G \to L$ is said to be a nonempty, if it is not the constant map which assumes the values 0 of L.

Definition 2.11 ([17])**.** Let $\lambda : X \to L$ be a L-fuzzy sub set of X. Then the set $\{\lambda(x) \mid x \in X\}$ is called the image of λ and is denoted by $\lambda(x)$ or $Im(\lambda)$. The set $\{x \mid x \in X, \lambda(x) > 0\}$ is called the support of λ and is denoted by $Supp(\lambda)$. The set $X_\lambda = \{x \in X \mid \lambda(x) = \lambda(0)\}$. For $t \in L$, $\lambda_t = \{x \in X \mid \lambda(x) \geq t\}$ is called a t-cut or t-level set of λ.

Definition 2.12 ([17])**.** Let λ, μ be two L-fuzzy subsets of X. If $\lambda(x) \leq \mu(x)$ for all $x \in X$, then we say that λ is contained in μ and we write $\lambda \subseteq \mu$. Define $\lambda \cup \mu$ and $\lambda \cap \mu$ are L-fuzzy subsets of X by for all $x \in X$, $(\lambda \cup \mu)(x) = \lambda(x) \vee \mu(x)$, $(\lambda \cap \mu)(x) = \lambda(x) \wedge \mu(x)$. Then $\lambda \cup \mu$ and $\lambda \cap \mu$ are called the union and intersection of λ and μ, respectively.

Definition 2.13 ([17])**.** Let f be a mapping from X into Y, and let λ and μ be L-fuzzy subsets of X and Y respectively. The L-fuzzy subsets $f(\lambda)$ of Y and $f^{-1}(\mu)$ of X, defined by

$$f(\lambda)(y) = \begin{cases} \vee\{\lambda(x) \mid x \in X, f(x) = y\} & \text{if } f^{-1}(y) \neq \varnothing; \\ 0 & \text{otherwise.} \end{cases}$$

Where $y \in Y$, and $f^{-1}(\mu)(x) = \mu(f(x))$, for all $x \in X$, are called the image of λ under f and the pre-image of μ under f, respectively.

Definition 2.14 ([17])**.** A L-fuzzy subset λ of X is said to have *sup* property if, for any subset A of X, there exists $a_0 \in A$ such that $\lambda(a_0) = \vee_{a \in A} \lambda(a)$.

Definition 2.15 ([17])**.** Let f be any function from a set X to a set Y, and let λ be any L-fuzzy subset of X. Then λ is called f-invariant if $f(x) = f(y)$ implies $\lambda(x) = \lambda(y)$, where $x, y \in X$.

Definition 2.16 ([17]). Let X be nonempty set. Let $Y \subseteq X$ and $a \in Y$. We define, a L-fuzzy set a_Y is defined as follows :

$$a_Y(x) = \begin{cases} a & \text{if } x \in Y \\ 0 & \text{if } x \in X - Y. \end{cases}$$

In particular, if Y is a singleton, say, $\{y\}$, then a_y is called as L-fuzzy point.

Definition 2.17 ([19]). A L-fuzzy subset λ of an l-group $(G, +, \vee, \wedge)$ is said to be a L-fuzzy sub l-group of G, if

(1) λ is a L-fuzzy subgroup of $(G, +)$, i.e,
  i) $\lambda(x + y) \geq \lambda(x) \wedge \lambda(y)$, for all $x, y \in G$ and ii) $\lambda(-x) = \lambda(x)$, for all $x \in G$,
(2) λ is a L-fuzzy sublattice of $(G, \vee, \wedge)$, i.e,
  i) $\lambda(x \vee y) \geq \lambda(x) \wedge \lambda(y)$, and ii) $\lambda(x \wedge y) \geq \lambda(x) \wedge \lambda(y)$, for all $x, y \in G$.

Theorem 2.18 ([19]). *If λ is a L-fuzzy sub l-group of G, then*
i) $\lambda(0) \geq \lambda(x)$, for all $x \in G$,
ii) $\lambda(x^+) \geq \lambda(x), \lambda(x^-) \geq \lambda(x)$ and $\lambda(|x|) \geq \lambda(x)$, for all $x \in G$.

Definition 2.19 ([20]). A L-fuzzy sub l-group λ of G is said to be a L-fuzzy convex sub l-group of G if $x, a \in G$, $0 \leq x \leq a \Rightarrow \lambda(x) \geq \lambda(a)$ (*Convexity condition*)

Definition 2.20 ([20]). Let λ be a L-fuzzy convex sub l-group of G. Then, λ is called a L-fuzzy maximal convex sub l-group of G, if λ is a maximal element in the set of all non constant L-fuzzy convex sub l-groups of G under point wise partial ordering.

Theorem 2.21 ([20]). *Let λ be a L-fuzzy subset of an l-group G. Then λ is a L-fuzzy maximal convex sub l-group of G if and only if there exist, a maximal convex l-subgroup M of G and maximal element α in L such that*

$$\lambda(x) = \begin{cases} 1, & \textit{if } x \in M \\ \alpha, & \textit{otherwise.} \end{cases}$$

Definition 2.22 ([20]). A non constant L-fuzzy convex sub l-group of an l-group G is called L-fuzzy prime convex sub l-group if and only if for any L-fuzzy convex sub l-groups μ and ν, $\mu \cap \nu \subseteq \lambda \Rightarrow$ either $\mu \subseteq \lambda$ or $\nu \subseteq \lambda$.

Lemma 2.23 ([20]). *If λ is a L-fuzzy prime convex sub l-group of G, then $\lambda(0) = 1$.*

Theorem 2.24 ([20]). *Let λ be a L-fuzzy subset of G. Then, λ is a L-fuzzy prime convex sub l-group of G if and only if there exists a pair (P, α), where P is a prime convex l-subgroup and α is an irreducible element of L, such that*

$$\lambda(x) = \begin{cases} 1, & \textit{if } x \in P \\ \alpha, & \textit{otherwise,} \end{cases}$$

Corollary 2.25 ([20]). *Each L-fuzzy maximal convex sub l-group is L-fuzzy prime convex sub l-group.*

Definition 2.26 ([19]). A L-fuzzy subgroup λ of a l-group G is said to be a L-fuzzy l-ideal of G if i) $\lambda(x + y) = \lambda(y + x)$ for all $x, y \in G$ and ii) $x, a \in G, |x| \leq |a| \Rightarrow \lambda(x) \geq \lambda(a)$.

3. The spectrum of G

Prime spectrum of an l-ring is studied in [9] by Keimel. The Prime spectrum concerning the prime l-ideals of an abelian lattice ordered group is studied in [1, 7]. In this section, following the notion of a prime convex l-subgroup of an l-group presented by Conrad in [2], and we study, the spectrum of a lattice ordered group G (not necessarily abelian) as the set of all prime convex l-subgroups of G which is a larger space than the $Spec(G)$ considered earlier by Bigard, Keimel and Wolfenstein [1] and the spectrum of an abelian l-group presented in [7]. Now, we prove the following results in l-groups, which are useful in the further study of this paper.

Lemma 3.1. *Let G be an l-group with strong order unit e. Then, the following hold.*

(i) If H is a convex l-subgroup of G containing e, then $H = G$.

(ii) Every proper convex l-subgroup is contained in a maximal convex l-subgroup.

(iii) If $A \subseteq G$ such that A is not contained in any prime convex l-subgroup of G, then $\langle A\rangle = G$, where $\langle A\rangle$ is a convex l-subgroup generated by A.

Proof. (i) Let H be a convex l-subgroup of G containing e. Let $g \in G$. Since e is a strong order unit, there exists $n \in \mathbb{Z}^+$ such that $ne > g^+ \geq 0$. Since, H is convex l-subgroup of G, $g^+ \in H$. Similarly, we have $g^- \in H$. Thus $g = g^+ - g^- \in H$. So $G \subseteq H$. Hence $G = H$.

(ii) Let I be a proper convex l-subgroup of G. By (i), $e \notin I$, so $I \neq G$. Write $\mathcal{F} = \{J \mid J$ is a proper convex l-subgroup of $G, e \notin J, I \subseteq J\}$. Clearly, $I \in \mathcal{F}$. Then $\mathcal{F} \neq \varnothing$. Clearly, $\mathcal{F}$ is a poset under set inclusion. Let $\{J_\alpha \mid \alpha \in \Delta\}$ be a chain in $\mathcal{F}$. Put $J = \cup_{\alpha\in\Delta} J_\alpha$. By(i), $e \notin J_\alpha, \alpha \in \Delta$. Thus $e \notin J$. Clearly, J is a convex l-subgroup of G. So $J \in \mathcal{F}$, i.e, $J_\alpha \subseteq J$, for all $\alpha \in \Delta$. Hence J is an upper bound of $\{J_\alpha \mid \alpha \in \Delta\}$ in $\mathcal{F}$. Thus every chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$. By Zorn's lemma, $\mathcal{F}$ contains a maximal element-M say. Clearly, M is a maximal convex l-subgroup of G. Therefore every proper convex l-subgroup is contained in a maximal convex l-subgroup.

(iii) Let $A \subseteq G$ such that A is not contained in any prime convex l-subgroup of G. Suppose, $\langle A\rangle \neq G$. By(ii), $\langle A\rangle$ is contained in a maximal convex l-subgroup M of G. We know that every maximal convex l-subgroup is prime. Thus $A \subseteq \langle A\rangle \subseteq M$, which is contradiction to our assumption. So $\langle A\rangle = G$. □

Theorem 3.2. *Let G be an l-group and $x \in G^+$. Then the following are equivalent :*

(i) x is a strong order unit.

(ii) x is not in any proper convex l-subgroup of G.

(iii) x is not in any prime convex l-subgroup of G.

Proof. (i)⇒(ii) and (ii)⇒(iii) are clear.

(iii)⇒(i) Assume that x is not in any prime convex l-subgroup of G. Suppose x is not a strong order unit. Then there exists $y \in G$ such that $nx \ngeq y$, for all $n \in \mathbb{Z}^+$. Thus $nx \ngeq |y|$, for all $n \in \mathbb{Z}^+$, since $|y| \geq y$. Let $H = \langle x\rangle = \{z \mid |z| \leq n|x| = nx$, for some $n \in \mathbb{Z}^+\}$ be the convex l-subgroup generated by x. Clearly, $y \notin H$. So, $H \neq G$. Write $\mathcal{F} = \{K \mid K$ is a convex l-subgroup of G, $H \subseteq K, y \notin K\}$. Clearly, $H \in F$. So, $\mathcal{F} \neq \varnothing$. Clearly, $\mathcal{F}$ is a poset under set inclusion. Let $\{K_\alpha\}_{\alpha\in\Delta}$ be a chain in $\mathcal{F}$.

Put $K = \cup_{\alpha\in\Delta} K_\alpha$. Clearly, K is a convex l-subgroup of G and $y \notin K$. Therefore, $K \in \mathcal{F}$. Clearly, K is an upper bound of $\{K_\alpha\}_{\alpha\in\Delta}$ in $\mathcal{F}$. Thus every chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$. By Zorn's lemma, $\mathcal{F}$ contains a maximal element, say M. Now, we prove that M is a value of y. Let N be a convex l-subgroup of G such that $M \subseteq N$ and $y \notin N$. Then, $H \subseteq M \subseteq N$, so that $N \in \mathcal{F}$ and thus, $M = N$, since M is a maximal element in $\mathcal{F}$. So M is a value of y, i.e, M is regular and hence, M is prime. Clearly, $x \in H \subseteq M$, a contradiction to x is not in any prime convex l-subgroup of G. Hence x is a strong unit. □

Let G be a lattice ordered group (not necessarily abelian). The spectrum of G, denoted by $Spec(G)$ is the set of all prime convex l-subgroups of G. If $A \subseteq G$, let $S(A) = \{P \in Spec(G) \mid A \nsubseteq P\}$ and $H(A) = Spec(G) - S(A) = \{P \in Spec(G) \mid A \subseteq P\}$. We write for $g \in G$, $S(g) = S(\{g\}) = \{P \in Spec(G) \mid g \notin P\}$ and $H(g) = H(\{g\}) = \{P \in Spec(G) \mid g \in P\}$ respectively.

Theorem 3.3. *Let $A, B \subseteq G$.*
(i) If $A \subseteq B$, then $H(B) \subseteq H(A)$ and $S(A) \subseteq S(B)$.
(ii) $H(A) \cup H(B) \subseteq H(A \cap B)$.

Theorem 3.4. *If A and B are convex l-subgroups of G, then $H(A) \cup H(B) = H(A \cap B)$.*

Theorem 3.5. *Let $A \subseteq G$ and $\langle A\rangle$ be a convex l-subgroup generated by A. Then, $H(A) = H(\langle A\rangle)$ and $S(A) = S(\langle A\rangle)$.*

Theorem 3.6. *Let $\mathcal{T} = \{S(A) \mid A \subseteq G\}$. Then the pair $(Spec(G), \mathcal{T})$ is a topological space.*

Theorem 3.7. *$\{S(a) \mid a \in G\}$ is a base for $\mathcal{T}$.*

Proof. Let $S(A) \in \tau$ and let $P \in S(A)$. Then $A \nsubseteq P$. Thus there exists $a \in A$ such that $a \notin P$. So $P \in S(a)$. Now, we prove that $S(a) \subseteq S(A)$. Since $a \in A$, i.e, $\{a\} \subseteq A$, we have $S(a) \subseteq S(A)$. Thus $S(a) \subseteq S(A)$. So $P \in S(a) \subseteq S(A)$. Hence $\{S(a) \mid a \in G\}$ is a base for τ. □

Theorem 3.8. *If $a, b \in G^+$, then,*
(i) $S(a \vee b) = S(a) \cup S(b)$,
(ii) $S(a \wedge b) = S(a) \cap S(b)$.

Theorem 3.9. *To each $a \in G$, $S(a)$ is compact.*

Proof. Let $\{S(a_i)\}_{i\in\Delta}$ be an open cover for $S(a)$. Then $S(a) \subseteq \cup_{i\in\Delta} S(a_i) = S(\cup_{i\in\Delta}\{a_i\})$. Thus $H(a) \supseteq H(\cup_{i\in\Delta}\{a_i\})$, i.e, $\cap_{P\in H(a)} P \subseteq \cap_{Q\in H(\cup_{i\in\Delta}\{a_i\})} Q$, i.e, $\langle a\rangle \subseteq \langle \cup_{i\in\Delta}\{a_i\}\rangle$. So there exists $i_1, i_2, \cdots, i_n \in \Delta$ such that $|a| \le |a_{i_1}| + |a_{i_2}| + \cdots + |a_{i_n}|$. Hence $\langle a\rangle \subseteq \langle\{a_{i_1}, a_{i_2}, \cdots, a_{i_n}\}\rangle$ and thus $H(a) = H(\langle a\rangle) \supseteq H(\langle\{a_{i_1}, a_{i_2}, \cdots, a_{i_n}\}\rangle) = H(\{a_{i_1}, a_{i_2}, \cdots, a_{i_n}\}) = \cap_{j=1}^n H(a_{i_j})$. Therefore, $S(a) \subseteq \cup_{j=1}^n S(a_{i_j})$. Therefore $S(a)$ is compact. □

Theorem 3.10. *Let e be a strong order unit in G. Then, $S(e) = Spec(G)$.*

Theorem 3.11. *$Spec(G)$ is compact if and only if G has a strong order unit.*

Proof. Assume that $Spec(G)$ is compact. Then $Spec(G) \subseteq \cup_{a\in G} S(a)$. Let $a \in G$. We know that $a = a^+ - (-a)^+$. Now, $S(a) \subseteq S(a^+) \cup S((-a)^+)$ (since, $P \notin S(a^+) \cup S((-a)^+) \Rightarrow a^+ \in P, (-a)^+ \in P \Rightarrow a = a^+ - (-a)^+ \in P \Rightarrow P \notin S(a)$). So, $Spec(G) = \cup_{a\in G^+} S(a)$. Since $Spec(G)$ is compact, there exists $a_1, a_2, \cdots, a_n \in G^+$ such that $Spec(G) = \cup_{i=1}^{n} S(a_i) = S(a_1 \vee a_2 \vee \cdots \vee a_n) = S(b)$, where $b = a_1 \vee a_2 \vee \cdots \vee a_n \in G^+$. Suppose b is not a strong order unit. So, there exists an element $a \in G$ such that $nb \ngeq a$ for all positive integers n. Then $a \notin \langle b\rangle$. So, there exists a value P of a such that $\langle b\rangle \subseteq P$. Thus P is prime. So, $P \in Spec(G)$ and $P \notin S(b)$ (since, $b \in \langle b\rangle \subseteq P$), a contradiction. Hence b is a strong order unit. Converse is clear. □

4. Topological space $LSpec(G)$

In this section, we introduce a topology on the set of all L-fuzzy prime convex sub l-groups of a lattice ordered group G (not necessarily abelian) i.e, L-fuzzy prime spectrum of a l-group G (not necessarily abelian) as set of L-fuzzy prime convex sub l-groups of G. Let G be an l-group and $\theta : G \to L$ be any L-fuzzy subset of G. Let $X = \{\lambda \mid \lambda$ is a L-fuzzy prime convex sub l-group of $G\}$;
$V(\theta) = \{\lambda \in X \mid \theta \subseteq \lambda\}$; $X(\theta) = X - V(\theta)$, the complement of $V(\theta)$ in X.

Theorem 4.1. *Let $\lambda : G \to L$ and $\mu : G \to L$ be two L-fuzzy subsets.*
(i) If $\lambda \subseteq \mu$, then $V(\mu) \subseteq V(\lambda)$ and $X(\lambda) \subseteq X(\mu)$.
(ii) $V(\mu) \cup V(\lambda) \subseteq V(\mu \cap \lambda)$.

Theorem 4.2. *If μ and λ are L-fuzzy convex sub l-groups of G, then $V(\mu) \cup V(\lambda) = V(\mu \cap \lambda)$.*

Corollary 4.3. *$V(\chi_A) \cup V(\chi_B) = V(\chi_{A\cap B})$, where A and B are convex l-subgroups of G.*

Theorem 4.4. *Let $\lambda : G \to L$ be a L-fuzzy subset and $\langle\lambda\rangle$ be a smallest L-fuzzy sub l-group generated by λ. Then, $V(\lambda) = V(\langle\lambda\rangle)$ and $X(\lambda) = X(\langle\lambda\rangle)$.*

Corollary 4.5. *$V(x_a) = V(\langle x_a\rangle)$ and $X(x_a) = X(\langle x_a\rangle)$, for any L-fuzzy point x_a of G.*

Theorem 4.6. *If $\{\lambda_i \mid i \in I\}$ is a family of L-fuzzy subsets of G, then $V(\cup\{\lambda_i \mid i \in I\}) = \cap V(\{\lambda_i \mid i \in I\})$.*

Theorem 4.7. *Let $\tau = \{X(\theta) \mid \theta$ is any L-fuzzy subset of $G\}$. Then the pair (X, τ) is a topological space.*

Definition 4.8. The topological space (X, τ), is called L-fuzzy prime spectrum of l-group G and is denoted by $LSpec(G)$ or X.

5. Base of $LSpec(G)$

Theorem 5.1. *Let $a, b \in L - \{0\}$ and $x, y \in G^+$. Then $X(x_a) \cap X(y_b) = X((x \wedge y)_c)$, where $c = a \wedge b$.*

Proof. $X(x_a) \cap X(y_b) = X(\langle x_a\rangle \cap \langle y_b\rangle) = X(\langle (x \wedge y)_c\rangle)$, where $c = a \wedge b$. □

Theorem 5.2. *$\{X(x_a) \mid x \in G, a \in L - \{0\}\}$ is a basis for τ.*

Proof. Let $X(\mu) \in \tau$ and $\lambda \in X(\mu)$. Then, $\mu \not\subseteq \lambda$. Then $\mu(x) \not\leq \lambda(x)$, for some $x \in G$. Let $\mu(x) = a$. Clearly, $a \neq 0$ and $a \not\leq \lambda(x)$, i.e, $x_a(x) \not\leq \lambda(x)$, i.e, $x_a \not\leq \lambda$. Thus $\lambda \in X(x_a)$. Now, we prove that $X(x_a) \subseteq X(\mu)$. Then $\nu \in V(\mu) \Rightarrow \mu \subseteq \nu \Rightarrow \mu(x) \leq \nu(x) \Rightarrow a \leq \nu(x) \Rightarrow x_a(x) \leq \nu(x) \Rightarrow x_a \leq \nu \Rightarrow \nu \in V(x_a)$. Thus $V(\mu) \subseteq V(x_a)$. S0o $X(x_a) \subseteq X(\mu)$. Hence $\lambda \in X(x_a) \subseteq X(\mu)$. Therefore $\{X(x_a) \mid x \in G, a \in L - \{0\}\}$ is a basis for $\Im$. □

Lemma 5.3. *$X(x_a) \subseteq X(x_a^+) \cup X(x_a^-)$, where $x^- = (-x) \vee 0 = (-x)^+$.*

Proof. Suppose $\lambda \notin X(x_a^+) \cup X(x_a^-)$ Then $\lambda \notin X(x_a^+)$ and $\lambda \notin X(x_a^-)$. Thus $x_a^+ \subseteq \lambda$ and $x_a^- \subseteq \lambda$. So $a \leq \lambda(x^+)$ and $a \leq \lambda(x^-)$. Hence $a \leq \lambda(x^+ - (x^-)) = \lambda(x)$ and thus $x_a \leq \lambda$, i.e., $\lambda \notin X(x_a)$. Therefore $X(x_a) \subseteq X((x^+)_a) \cup X((x^-)_a)$. □

So, we have $X = \cup_{x \in G} X(x_a) \subseteq \cup_{x \in G}(X((x^+)_a) \cup X((x^-)_a)) = \cup_{x \in G^+} X(x_a)$. We can easily observe that $\{X(x_a) \mid x \in G^+, a \in L - \{0\}\}$ is a basic open cover of X.

Theorem 5.4. *If $a \in L - \{0\}$ such that $X(x_a) = X$, then*
(i) no prime convex l-subgroup of G contains x and
(ii) a is not prime.

Proof. (i) Let $a \in L - \{0\}$. Suppose $X(x_a) = X$. Since $X \neq \varnothing$, X contains a L-fuzzy prime convex sub l-group, say ν. $G_\nu = \{x \in G \mid \nu(x) = \nu(0) = 1\}$ is a prime convex l-subgroup of G and there exist a prime element $b \in L$ such that

$$\nu(y) = \begin{cases} 1, & \text{if } y \in G_\nu \\ b, & \text{otherwise,} \end{cases}$$

Let P be a prime convex l-subgroup of G. Define $\lambda : G \to L$ by,

$$\lambda(y) = \begin{cases} 1, & \text{if } y \in P \\ b, & \text{otherwise,} \end{cases}$$

Then $\lambda \in X = X(x_a)$. Thus $x_a \not\leq \lambda$, i.e, $a \not\leq \lambda(x) = b$. So $x \notin P$ (Otherwise, $x \in P$, and hence $\lambda(x) = 1$, so that $a \leq 1 = \lambda(x)$, which is a contradiction to $a \not\leq \lambda(x)$). Hence x is not a member of any prime convex l-subgroup of G.

(ii) Suppose that a is a prime. Let I be any prime convex l-subgroup of G. Define $\mu : G \to L$ by,

$$\mu(y) = \begin{cases} 1, & \text{if } y \in I \\ a, & \text{otherwise,} \end{cases}$$

μ is a L-fuzzy prime convex sub l-group of G, i.e, $\mu \in X$. Clearly, $x_a \leq \mu$ i.e, $\mu \notin X(x_a)$. Then $X(x_a) \neq X$, a contradiction. Thus a is not a prime. □

Theorem 5.5. *Suppose that L is regular.*
(i) If for some $a \in L - \{0\}$ and $x \in G$, $X(x_a) = \varnothing$, then $x = 0$.
(ii)For any $a \in L - \{0\}$, $X(0_a) = \varnothing$.

Proof. (i) Let $a \in L - \{0\}$ be such that $X(x_a) = \varnothing$. Suppose $x \neq 0$. x has a value M (say) in G. Then M is a prime convex l-subgroup of G. Since L is regular, 0 is a prime element in L. Thus $\chi_M \in X$. Since $X(x_a) = \varnothing$, $\chi_M \notin X(x_a)$, i.e, $x_a \leq \chi_M$. So $a \leq \chi_M(x)$. Since $a \neq 0$, $\chi_M(x)$ must be equal to 1, i.e, $x \in M$, a contradiction. Hence $x = 0$.

(ii) Let $a \in L - \{0\}$. Let $\lambda \in X$. Clearly, $0_a \leq \lambda$, i.e, $\lambda \in V(0_a)$, i.e, $\lambda \notin X(0_a)$. Thus $X(0_a) = \varnothing$. □

Theorem 5.6. *Let $a \in L - \{0\}$ and $x \in G^+$. Then $X(x_a) = X$ if and only if x is a strong order unit in G and a is not contained in any prime element in L.*

Proof. Assume that $X(x_a) = X$. Then $x_a \nleq \mu$, for all $\mu \in X$, i.e, $a \nleq \mu(x)$, for all $\mu \in X$, i.e, $a \nleq b$, for all prime element b in L. Thus x is not a member of every prime convex l-subgroup of G. So, $x \notin \cup\{P \mid P$ is a prime convex l-subgroup of $G\}$. Hence x is a strong order unit.

Conversely, assume that x is a strong order unit and a is not contained in any prime element in L. Clearly, $X(x_a) \subseteq X$. Let $\mu \in X$. Then there exists a prime element b in L such that G_μ is prime convex l-subgroup of G and

$$\mu(y) = \begin{cases} 1, & \text{if } y \in G_\mu \\ b, & \text{otherwise,} \end{cases}$$

Now, we have to prove that $\mu \in X(x_a)$, i.e, $x_a \nsubseteq \mu$, i.e, $a \nleq \mu(x)$, i.e, $a \nleq b$, which is true. Thus $\mu \in X(x_a)$. So $X \subseteq X(x_a)$. Hence $X = X(x_a)$. □

If a is contained in a prime element in L, the above theorem does not hold. We have, the following example.

Example 5.7. Let $L = [0, 1]$. Let $G = C(X)$ be the set of all bounded continuous real valued functions on X, where X is a Hausdroff space. If all the operations defined pointwise i.e, for $f, g \in C(X)$ and for each $x \in X$, $(f + g)(x) = f(x) + g(x), (f \vee g)(x) = f(x) \vee g(x), (f \wedge g)(x) = f(x) \wedge g(x)$, then $C(X)$ becomes an l-group. Clearly, the function $f(x) = 1$ $(x \in X)$ is a strong unit in G. Fix $x \in X$. Clearly, $M_x = \{f \in G \mid f(x) = 0\}$ is a prime convex l-subgroup of G. Let $a = \frac{1}{2} \in (0, 1)$. Clearly, every element b with $a \leq b < 1$ is prime in $L = [0, 1]$. Define, $\mu \in LSpec(G)$ as follows :

$$\mu(g) = \begin{cases} 1, & \text{if } g \in M_x \\ \frac{2}{3}, & \text{otherwise,} \end{cases}$$

and

$$(f_a)(g) = \begin{cases} \frac{1}{2} = a, & \text{if } g = f \\ 0, & \text{otherwise,} \end{cases}$$

Clearly, $f_a \subseteq \mu$. Then $\mu \notin X(f_a)$. Thus $X(f_a) \neq X$.

6. Subspace X_a of X

Throughout this paper, $X_a = \{\lambda \in X \mid Im\lambda = \{1, a\}\}$, where $a \in L - \{1\}$ is prime. If a is not prime, then X_a becomes empty.

Lemma 6.1. *Let $a \in L$ be prime. If $b \leq a$, then $X(x_b) \cap X_a = \varnothing$.*

Proof. Let $b \in L$ such that $b \leq a$. Suppose $X(x_b) \cap X_a \neq \varnothing$. Let $\lambda \in X(x_b) \cap X_a$, i.e, $\lambda \in X(x_b)$ and $\lambda \in X_a$, i.e, $x_b \nsubseteq \lambda$ and $\lambda \in X_a$, i.e, $b \nleq \lambda(x)$ and $\lambda \in X_a$ i.e, $b \nleq a$ (otherwise $\lambda(x) = 1$, and thus $b \leq 1 = \lambda(x)$ a contradiction), a contradiction. Then $X(x_b) \cap X_a = \varnothing$. □

Lemma 6.2. *Let $a, b \in L - \{0\}$ and $a \leq b$. Then,*
(i) $X(x_a) \subseteq X(x_b)$.
(ii) $V(x_b) \subseteq V(x_a)$.

Proof. (i)Let $\lambda \in X(x_a)$. Suppose $\lambda \notin X(x_b)$. Then $x_b \subseteq \lambda$, i.e., $b \leq \lambda(x)$. Thus $a \leq b \leq \lambda(x)$, i.e., $a \leq \lambda(x)$. So $x_a \leq \lambda$, i.e., $\lambda \notin X(x_a)$. Hence $X(x_a) \subseteq X(x_b)$.
(ii) $V(x_b) = X - X(x_b) \subseteq X - X(x_a) = V(x_a)$ □

Lemma 6.3. *Suppose a is prime and $b \nleq a$. If $b \leq c$, then $X(x_b) \cap X_a = X(x_c) \cap X_a$.*

Proof. Let $b \leq c$. Then $X(x_b) \subseteq X(x_c)$, i.e, $X(x_b) \cap X_a \subseteq X(x_c) \cap X_a$. Let $\lambda \in X(x_c) \cap X_a$. Then $\lambda \in X_a$ and $c \nleq \lambda(x)$. Thus $\lambda(x) = a$ (otherwise $\lambda(x) = 1 > c$, a contradiction). If $\lambda \notin X(x_b)$, then $b \leq \lambda(x) = a$, a contradiction to $b \nleq a$. So $\lambda \in X(x_b) \cap X_a$. Hence $X(x_c) \cap X_a \subseteq X(x_b) \cap X_a$. Therefore $X(x_b) \cap X_a = X(x_c) \cap X_a$. □

Theorem 6.4. *Let G be an l-group with a strong order unit e. Then, X_a is compact.*

Proof. (i) $\{X(x_b) \cap X_a \mid x \in G, b \in L - \{0\}, b \nleq a\}$ is a basis for the subspace X_a. Let $\{X((x_i)_{b_i}) \cap X_a \mid x_i \in G, b_i \nleq a, i \in \Delta\}$ be a basic open cover of X_a. Let $K = \{b_i \mid i \in \Delta\}$ and $c = \vee\{b_i \mid b_i \in K\}$. For any $i \in \Delta$, $b_i \leq c$. Then $X((x_i)_{b_i}) \subseteq X((x_i)_c)$. Thus $X_a = \cup_{i \in \Delta}(X((x_i)_{b_i}) \cap X_a) \subseteq \cup_{i \in \Delta} X((x_i)_{b_i}) \subseteq \cup_{i \in \Delta}(X(x_i)_c)$.

(ii) $\{x_i \mid i \in \Delta\}$ is not contained in any prime convex l-subgroup of G. Let P be any prime convex l-subgroup of G. Define $\mu : G \to L$ by

$$\mu(y) = \begin{cases} 1, & \text{if } y \in P \\ a, & \text{otherwise.} \end{cases}$$

Then $\mu \in X_a$. Thus $\mu \in \cup_{i \in \Delta} X((x_i)_c)$. So there exists $j \in \Delta$ such that $\mu \in X((x_i)_c)$, i.e, $(x_j)_c \nsubseteq \mu$, i.e, $c \nleq \mu(x_j)$. Hence $\mu(x_j) \neq 1$ (otherwise $c \leq 1$, which is a contradiction) and thus $x_j \notin P$. Therefore $\{x_i \mid i \in \Delta\}$ is not contained in any prime convex l-subgroup of G. Hence $\langle\{x_i \mid i \in \Delta\}\rangle = G$. Since e is a strong order unit in G,

$$|e| \leq \sum_{k=1}^{n} |x_{i_k}| \text{ for some } i_1, i_2, \cdots, i_n. \tag{6.4.1}$$

(iii) $X_a \subseteq \cup_{k=1}^{n} X((x_{i_k})_c)$.
Assume that this is not true. Then there exists $\lambda \in X_a$ such that $\lambda \notin \cup_{k=1}^{n} X((x_{i_k})_c)$, i.e, $(x_{i_k})_c \subseteq \lambda$, for all $k = 1, 2, \cdots, n$, i.e, $c \leq \lambda(x_{i_k})$, for all $k = 1, 2, \cdots, n$. Thus $\lambda(e) \geq \wedge_{k=1}^{n} \lambda(x_{i_k}) \geq c$. Now, we have to show that $\lambda(x_{i_k}) = 1$, for all $k = 1, 2, \cdots, n$. Suppose $\lambda(x_{i_1}) \neq 1$. Since e is a strong order unit, there exists $n \in \mathbb{Z}^+$ such that $ne \geq |x_{i_1}|$. So $\lambda(x_{i_1}) \geq \lambda(e)$ (since λ is a convex) $\geq c$. Since $\lambda \in x_a$, $\lambda(x_{i_1}) = a$. Which is a contradiction to $b_i \nleq a$, for all $i \in \Delta$. Hence $\lambda(x_{i_1}) = 1$. Similarly, we can show that $\lambda(x_{i_k}) = 1$, for all $k = 1, 2, \cdots, n$. Then $x_{i_k} \in G_\lambda$ $(k = 1, 2, \cdots, n)$. Thus $\sum_{k=1}^{n} |x_{i_k}| \in G_\lambda$. From (6.4.1), $e \in G_\lambda$. So $G_\lambda = G$. Hence λ is constant. Which is a contradiction to $\lambda \in X_a$. Therefore

$$X_a \subseteq \cup_{k=1}^{n} X((x_{i_k})_c). \tag{6.4.2}$$

From (6.4.1) and (6.4.2), $X_a = \cup_{k=1}^{n}(x_a) \cap X((x_{i_k})_{b_k})$. So $\{X((x_{i_k})_{b_k}) \mid k = 1, 2, \cdots, n\}$ is a finite subcover of X_a. Hence X_a is compact. □

Theorem 6.5. *For all $\mu \in X$, $V(\mu)$ = The closure of μ in $X = \overline{\{\mu\}}$.*

Proof. Since $V(\mu)$ is a closed set in X and $\mu \in V(\mu)$, $\overline{\{\mu\}} \subseteq V(\mu)$. Let $\sigma \notin \overline{\{\mu\}}$. Then $\sigma \in \overline{\{\mu\}}^c$. Since $\overline{\{\mu\}}^c$ is open, there exists $X(x_a)$ such that $\sigma \in X(x_a) \subseteq \overline{\{\mu\}}^c$. Thus $\mu \notin X(x_a)$, i.e, $\mu \in V(x_a)$, i.e, $x_a \leq \mu$. If $\sigma \in V(\mu)$, i.e, $\mu \leq \sigma$, then $x_a \subseteq \mu \subseteq \sigma$, and thus $\sigma \notin X(x_a)$, a contradiction to $\sigma \in X(x_a)$. So $\sigma \notin V(\mu)$. Hence $V(\mu) \subseteq \overline{\{\mu\}}$. Therefore $V(\mu) = \overline{\{\mu\}}$. □

Theorem 6.6. *Let F be a subset of X and let $\overline{F}$ denote the closure of F. Then $F \subseteq V(\chi_M)$, where $M = \cap_{\lambda \in F} G_\lambda$ and hence $\overline{F} \subseteq V(\chi_M)$.*

Proof. Clearly, $\chi_M(x) = 1$ if and only if $\lambda(x) = 1$, for all $\lambda \in F$. Now, if $\mu \in F$, then $\chi_M \subseteq \mu$, i.e, $\mu \in V(\chi_M)$. Thus $F \subseteq V(\chi_M)$. So $V(\chi_M)$ is a closed set containing F. Hence $\overline{F} \subseteq V(\chi_M)$. □

Now, the following example shows that $V(\chi_M) \subseteq \overline{F}$ need not hold.

Example 6.7. Let $G = \mathbb{Z} \times \mathbb{Z}$ be an l-group with ordering $(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow x_1 \leq x_2, y_1 \leq y_2$. We can easily observe that $I_1 = \mathbb{Z} \times \{0\}$, $I_2 = \{0\} \times \mathbb{Z}$ are the only prime convex l-subgroups of G. Let $L = \{0, a, b, 1\}$, where $1 > a > 0, 1 > b > 0$ and $a \parallel b$ be the lattice. Define $\eta^a_{I_1}, \eta^b_{I_1}, \eta^a_{I_2}, \eta^b_{I_2}$ from G to L as follows :

$$\eta^a_{I_1}(x, y) = \begin{cases} 1, & \text{if } (x, y) \in I_1 \\ a, & \text{otherwise,} \end{cases}$$

$$\eta^b_{I_1}(x, y) = \begin{cases} 1, & \text{if } (x, y) \in I_1 \\ b, & \text{otherwise,} \end{cases}$$

$$\eta^a_{I_2}(x, y) = \begin{cases} 1, & \text{if } (x, y) \in I_2 \\ a, & \text{otherwise.} \end{cases}$$

$$\eta^b_{I_2}(x, y) = \begin{cases} 1, & \text{if } (x, y) \in I_2 \\ b, & \text{otherwise,} \end{cases}$$

Clearly, above L-fuzzy subsets are L-fuzzy prime convex l-subgroups of G. Then $X = \{\eta^a_{I_1}, \eta^b_{I_1}, \eta^a_{I_2}, \eta^b_{I_2}\}$. Take $F = \{\eta^a_{I_1}, \eta^a_{I_2}\}$. Now $G_{\eta^a_{I_1}} \cap G_{\eta^a_{I_2}} = I_1 \cap I_2 = \{(0, 0)\} = M$, say. Now $V(\chi_M) = X$. Take $(2, 3) \in G$. Then $X((2, 3)_a) = X - V((2, 3)_a) = \{\eta^b_{I_1}, \eta^b_{I_2}\}$. Thus $X((2, 3)_a)$ is a basic open set and $\eta^b_{I_2} \in X((2, 3)_a)$. On the other hand $X((2, 3)_a) \cap (F - \{\eta^b_{I_2}\}) = \varnothing$. Thus $X((2, 3)_a)$ is a neighborhood of $\eta^b_{I_2}$ not containing any point of F other than $\eta^b_{I_2}$. So $\eta^b_{I_2}$ is not a limit point of F. Hence $\overline{F} \subset V(\chi_M)$.

Theorem 6.8. *If $\mu \in X_a$, then $\{\mu\}$ is closed in X_a if and only if μ is L-fuzzy maximal convex sub l-group of G in X_a. In other words, X_a is T_1 if and only if every element of X_a is a L-fuzzy maximal convex sub l-group of G.*

Proof. Let $\mu \in X_a$. Assume that $\{\mu\}$ is closed in X_a. Then $V(\mu) \cap X_a = \overline{\{\mu\}} \cap X_a = \{\mu\} \cap X_a = \{\mu\}$. Now, we prove that μ is a L-fuzzy maximal convex sub l-group in X_a. Let $\lambda \in X_a$ such that $\mu \leq \lambda$. Then clearly, $\lambda \in V(\mu) \cap X_a = \{\mu\}$. Thus $\lambda = \mu$. So μ is a maximal element of X_a.

Conversely, assume that μ is a L-fuzzy maximal convex sub l-group of G in X_a. Then the convex l-subgroup $G_\mu = \{x \in G \mid \mu(x) = 1\}$ is maximal. Now, we have to show that $V(\mu) \cap X_a = \{\mu\}$. Clearly, $\{\mu\} \subseteq V(\mu) \cap X_a$. Let $\sigma \in V(\mu) \cap X_a$. Then $\mu \subseteq \sigma$ and $G_\mu \subseteq G_\sigma$. Thus $G_\mu = G_\sigma$, since G_μ is a maximal convex l-subgroup of G. So $\mu = \sigma$, since $Im\mu = Im\sigma = \{1, a\}$. Hence $V(\mu) \cap X_a = \{\mu\}$, i.e, $\{\mu\}$ is a closed subset of X_a. □

Theorem 6.9. *If $a \in L - \{0\}$ is prime, then corresponding to every prime convex l-subgroup P of G, there exists a $\lambda \in X_a$ such that $P = G_\lambda$, vise verse. (i.e, if $\lambda \in X_a$, then $G_\lambda \in Spec(G)$).*

Proof. Let $a \in L - \{0\}$ be prime. Let P be a prime convex l-subgroup of G. Then there exists a L-fuzzy prime convex sub l-group λ with $Im\lambda = \{a, 1\}$. Thus $\lambda \in X_a$ with $G_\lambda = P$. Converse is clear. □

Theorem 6.10. *$X_a = \{\lambda \in X \mid Im\lambda = \{1, a\}\}$ is homeomorphic to $Spec(G)$.*

Proof. Define $\phi : X_a \to Spec(G)$ as $\phi(\lambda) = G_\lambda$.

(i) ϕ is one-one.

Let $\lambda, \mu \in X_a$ be such that $\phi(\lambda) = \phi(\mu)$, i.e, $G_\lambda = G_\mu$, i.e, $\lambda(x) = \mu(x)$, for all $x \in G$ (since $Im\lambda = Im\mu$). Then $\lambda = \mu$. Thus, ϕ is one-one.

(ii) ϕ is onto.

Let P be any prime convex l-subgroup of G, i.e, $P \in Spec(G)$. Consider the L-fuzzy convex sub l-group of G defined by

$$\mu(x) = \begin{cases} 1, & \text{if } x \in P \\ a, & \text{otherwise.} \end{cases}$$

Then $\mu \in X_a$. Thus $\phi(\mu) = P$. So ϕ is onto.

(iii) ϕ continuous.

Now consider an open set $S(A) = \{P \in Spec(G) \mid A \nsubseteq P\}$ in $Spec(G)$, where $A \subseteq G$. We have $\phi^{-1}(S(A)) = \{\lambda \in X_a \mid G_\lambda \in S(A) i.e, A \nsubseteq G_\lambda\} = \cup_{x\in A}\{\lambda \in X_a \mid x \notin G_\lambda\} = \cup_{x\in A} X(x_1)$ (since $x \notin G_\lambda \Leftrightarrow 1 \nleq \lambda(x) \Leftrightarrow x_1 \nsubseteq \lambda \Leftrightarrow \lambda \in X(x_1)$) is open. Then ϕ is continuous.

(iv) ϕ is open.

Let $X_a \cap X(x_b)$ be basic open set in X_a, where $b \nleq a$. Now $\phi(X_a \cap X(x_b)) = \{\phi(\lambda) \mid \lambda \in X_a \cap X(x_b)\} = \{G_\lambda \mid x_b \nsubseteq \lambda, \lambda \in X_a\} = \{G_\lambda \mid x \notin G_\lambda, \lambda \in X_a\} = S\{x\}$. Then ϕ is open.

Therefore X_a is homeomorphic to $Spec(G)$. □

Theorem 6.11. *Suppose that L is regular. Then, $Spec(G)$ is a dense subspace of X.*

Proof. Let $Y = \{\chi_I \mid I \in Spec(G)\}$. Clearly, $I \mapsto \chi_I : Spec(G) \to Y$ is a bijection. To prove that Y is dense in X, we have to prove every non empty open sub set of X intersects Y. To prove this it is enough to prove that every non empty basic open subset of X intersects Y. Let $X(x_b)$(where $b \in L - \{0\}$) be a non empty basic open sub set of X. Now, we have to prove that $X(x_b) \cap Y \neq \varnothing$. Let $\lambda \in X(x_b)$ i.e, $x_b \nsubseteq \lambda$, i.e, $b \nleq \lambda(x)$, i.e, $x \notin G_\lambda$. Define $\mu : G \to L$ by,

$$\mu(y) = \begin{cases} 1, & \text{if } y \in G_\lambda \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\mu = \chi_{G_\lambda} \in Y$. Now, we have to show that $\mu \in X(x_b) \cap Y$. Since $x \notin G_\lambda$, $\mu(x) = 0$. Thus $b \nleq 0 = \mu(x)$, i.e, $x_b \nsubseteq \mu$, i.e, $\mu \in X(x_b)$. So $\mu \in X(x_b) \cap Y$. Hence $Spec(G)$ is a dense subspace of X. □

7. Compactness of $LSpec(G)$

Lemma 7.1. *Let f be a homomorphism of G onto G', then $f(x_\beta) = (f(x))_\beta$, for all $x \in G$, and $\beta \in L - \{0\}$.*

Proof. We have $(f(x_\beta)(y) = \vee_{t\in f^{-1}(y)} x_\beta(t) = \begin{cases} \beta, & \text{if } t = x \\ 0, & \text{otherwise} \end{cases} = (f(x))_\beta(y)$. Thus $f(x_\beta) = (f(x))_\beta$. □

Theorem 7.2. *Let f be a homomorphism of G onto G', $X = LSpec(G), X' = LSpec(G')$, $X^* = \{\mu \in X \mid \mu$ is f-invariant$\}$. Define $g : X' \to X^*$, by $g(\mu') = f^{-1}(\mu')$, $\mu' \in X'$. Then*
(i) g is continuous.
(ii) g is open.
(iii) g is injective.
(iv) g is homoeomorphic to the closed subset $V(\chi_{Kerf})$. If f is an isomorphism of G onto G', then $g : X' \to X$ defined by $g(\mu') = f^{-1}(\mu')$, for all $\mu' \in X'$, is a homeomorphic.

Proof. (i) Let $X(x_\beta) \cap X^*$ be a basic open subset of X^* (where $\beta \in L - \{0\}, x \in G$). Then $\mu' \in g^{-1}(X(x_\beta) \cap X^*) \Leftrightarrow g(\mu') \in X(x_\beta) \cap X^* \Leftrightarrow f^{-1}(\mu') \in X(x_\beta) \cap X^* \Leftrightarrow f^{-1}(\mu') \in X(x_\beta)$ and $f^{-1}(\mu) \in X^* \Leftrightarrow x_\beta \nsubseteq f^{-1}(\mu') \Leftrightarrow f(x_\beta) \nsubseteq f(f^{-1}(\mu')) \Leftrightarrow f(x_\beta) \nsubseteq \mu' \Leftrightarrow f(x)_\beta \nsubseteq \mu' \Leftrightarrow \mu' \in X(f(x)_\beta)$. Thus inverse image of every basic open set of X^* is open in X'. So g is continuous.

(ii) Let $X'(f(x_\beta))$ be any basic open set in X'. For $x \in G, \beta \in L - \{0\}$, let $\mu \in g(X'(f(x_\beta))$. Then $\mu = g(\mu')$ for some $\mu' \in X'(f(x_\beta))$
$\Rightarrow \mu = f^{-1}(\mu')$ and $f(x)_\beta \nsubseteq \mu'$
$\Rightarrow f(\mu) = f(f^{-1}(\mu'))$ and $f(x_\beta) \nsubseteq \mu'$
$\Rightarrow f(\mu) = \mu'$ and $f(x_\beta) \nsubseteq \mu'$ (since, f is onto)
$\Rightarrow f(x_\beta) \nsubseteq f(\mu)$
$\Rightarrow x_\beta \nsubseteq f^{-1}(f(\mu)) = \mu$ (since μ is $f-$ invariant)
$\Rightarrow x_\beta \nsubseteq \mu$
$\Rightarrow \mu \in X(x_\beta) \cap X^*$.
Thus $g(X'(f(x_\beta))) \subseteq X(x_\beta) \subseteq X^*$.
Now let $\mu \in X(x_\beta) \cap X^*$. Then $\mu \in X(x_\beta)$ and $\mu \in X^*$
$\Rightarrow x_\beta \nsubseteq \mu$ and $\mu \in X^*$
$\Rightarrow x_\beta \nsubseteq \mu = f^{-1}(f(\mu))$ (since μ is $f-$invariant)
$\Rightarrow f(x_\beta) \nsubseteq f(\mu) = \mu'$ (say)
$\Rightarrow \mu' \in X'(f(x)_\beta)$.
On the other hand,
$x_\beta \nleq \mu = f^{-1}(f(\mu)) = f^{-1}(\mu') = g(\mu'))$, where $\mu' = f(\mu)$.
$\Rightarrow f(x_\beta) \nleq f(f^{-1}(\mu')) = \mu'$ (since f is onto, and $\mu = g(\mu')$)
$\Rightarrow f(x_\beta) \nleq \mu'$ and $\mu = g(\mu')$
$\Rightarrow \mu' \in X'(f(x)_\beta)$ and $\mu = g(\mu')$

$\Rightarrow \mu \in g(X'(f(x)_\beta))$.
So $X(x_\beta) \cap X^* \subseteq g(X'(f(x)_\beta))$. Hence $X(x_\beta) \cap X^* = g(X'(f(x)_\beta))$. Therefore g is open.

(iii) Let $\mu', \lambda' \in X'$ such that $g(\mu') = g(\lambda')$. Then $g(\mu') = g(\lambda') \Rightarrow f^{-1}(\mu') = f^{-1}(\lambda') \Rightarrow f(f^{-1}(\mu')) = f(f^{-1}(\lambda')) \Rightarrow \mu' = \lambda'$. Thus g is injective.

(iv) Let $\mu \in V(\chi_{kerf})$. Then $\chi_{kerf} \subseteq \mu$ and $\mu \in X$. Since $\chi_{kerf} \subseteq \mu$, μ is f-invariant. Thus $\mu \in X^*$. Now, we prove that $g(X') = V(\chi_{kerf})$. For this, we have to show that $\mu = g(\lambda)$, for some $\lambda \in X'$. Put $\lambda = f(\mu)$. Clearly $\lambda \in X'$. Now, $g(\lambda) = g(f(\mu)) = f^{-1}(f(\mu)) = \mu$ (since f is onto). Then $\mu \in g(X')$. Thus $V(\chi_{kerf}) \subseteq g(X')$. Now, we prove the reverse inequality. Let $\mu' \in X'$. Now, we show that $g(\mu') \in V(\chi_{kerf})$, i.e, $\chi_{kerf} \subseteq f^{-1}(\mu')$. If $x \notin kerf$, then it is clear. If $x \in kerf$, then $\chi_{kerf}(x) = 1 = \mu'(f(0)) = \mu'(f(x)) = f^{-1}(\mu'(x)) = (f^{-1}(\mu'))(x)$. Thus $\chi_{kerf} \subseteq f^{-1}(\mu') = g(\mu')$. So $g(\mu') \subseteq V(\chi_{kerf})$. Hence the reverse inequality holds and thus $g(X') = V(\chi_{kerf})$. Therefore g is homeomorphic to the closed subset $V(\chi_{kerf})$. □

Theorem 7.3. *Let f be an homomorphism of G onto G'. If each L-fuzzy prime convex sub l-group of G is constant on $kerf$ i.e, $X^* = X$, then X' is homeomorphic to X.*

Theorem 7.4. *Let L be such that for all $a \in L - \{0\}$, there exists $b \in L, a \leq b < 1$, such that b is prime. Let $\Lambda \subseteq G \times (L - \{1\})$ be such that $X = \cup_{(x_i,t_i)\in\Lambda} X((x_i)_{t_i})$. Then, $\vee t_i = 1$.*

Proof. Let $\vee t_i = a$. Suppose $a < 1$. By hypothesis, there exists a prime element $b \in L$ such that $a \leq b < 1$. Let P be a prime convex l-subgroup of G. Define $\mu : G \to L$, by

$$\mu(x) = \begin{cases} 1, & \text{if } x \in P \\ b, & \text{otherwise},. \end{cases}$$

Then $\mu \in X$. Thus $\mu \in X((x_i)_{t_i})$ (for some $(x_i, t_i) \in \Lambda$), i.e, $((x_i)_{t_i}) \not\subseteq \mu$, i.e, $t_i \not\leq \mu(x_i)$. So $\mu(x_i) \neq 1$. Hence $\mu(x_i) = b \ngeq t_i$. But $\mu(x_i) = b \geq a > t_i$. Which is a contradiction. Therefore $a = 1$. □

Corollary 7.5. *Let L be a chain.Let $\Lambda = G \times (L - \{1\})$ be such that $X = \cup_{(x_i,t_i)\in\Lambda} X((x_i)_{t_i})$. Then $\vee t_i = 1$.*

Theorem 7.6. *The space X is T_0.*

Proof. Let $\lambda, \mu \in X, \lambda \neq \mu$. Then $\mu \not\subseteq \lambda$ or $\lambda \not\subseteq \mu$. Suppose $\mu \not\subseteq \lambda$. Then $\lambda \in X(\mu)$ and $\mu \notin X(\mu)$ and $X(\mu)$ is open. Thus X is T_0. □

Let $I(L)$ be the set of all irreducible elements of L. If $\alpha \in L$, then $\tau = \{W(\alpha) \mid \alpha \in L\}$ is the hull kernel topology on $I(L)$, Where $W(\alpha) = \{\beta \in I(L) \mid \alpha \nleq \beta\}$.

Theorem 7.7. *Let G be an l-group with a strong order unit. Then, $LSpec(G)$ is homeomorphic with the product space $Spec(G) \times I(L)$*

Theorem 7.8. *Let G be an l-group with a strong order unit. Then, the space $LSpec(G)$ is compact $\Leftrightarrow$ $Spec(G) \times I(L)$ is compact in the product topology $\Leftrightarrow$ both $Spec(G)$ and $I(L)$ are compact.*

Now, we note that, if we choose a lattice L such that $I(L)$ is not compact, then $LSpec(G)$ is not compact. For example if we take $L = [0,1]$, then $I(L) = [0,1)$ is not compact. For, $\cup_{n=2}^{\infty}[0, 1 - \frac{1}{n}) = [0,1)$ and $\{[0, 1 - \frac{1}{n})\}_{n=2}^{\infty}$ is an open cover of $[0,1)$. Suppose $[0,1)$ is compact in this topology there exists $n_1, n_2, \cdots, n_k$ such that $[0,1) = \cup_{i=1}^{k}[0, 1 - \frac{1}{n_i})$. Without loss of generality, we can assume that $n_1 < n_2 < \cdots < n_k$. So, $\frac{1}{n_1} > \frac{1}{n_2} > \cdots > \frac{1}{n_k}$ i.e, $1 - \frac{1}{n_1} < 1 - \frac{1}{n_2} < \cdots < 1 - \frac{1}{n_k}$. So, $[0,1) = [0, 1 - \frac{1}{n_k})$. Choose x such that $1 - \frac{1}{n_k} < x < 1$. Clearly, $x \in [0,1)$ and $x \in [0, 1 - \frac{1}{n_k})$, a contradiction. So, $[0,1)$ is not compact. If we choose an l-group G with a strong order unit e (hence $Spec(G)$ is compact) and if we choose L such that $I(L)$ is not compact, then $LSpec(G)$ is not compact. So, the compactness of $LSpec(G)$ depends on the space $I(L)$ also.

Theorem 7.9. *$LSpec(G)$ is a T_1-space if and only if every prime convex l-subgroup of G is maximal and every irreducible element of L is dual atom.*

Theorem 7.10. *Let $Y \subseteq X$. Define $\mathcal{F}(Y) = \cap_{\mu \in Y}\mu$. Then $\mathcal{F}(Y)$ is a L-fuzzy convex sub l-group of G.*

Theorem 7.11. *For all $Y \subseteq X$, $V(\mathcal{F}(Y)) = \overline{Y}$, the closure of Y in X.*

Proof. Clearly, $V(\mathcal{F}(Y))$ is a closed set such that $V(\mathcal{F}(Y)) = V(\cap_{\mu \in Y}\mu) \supseteq Y$. Let $V(\mu)$ be a closed subset of X contains Y. Then $\mu \subseteq \eta$, for all $\eta \in V(\mu)$ and thus $\mu \subseteq \eta$ for all $\eta \in Y$. Thus $\mu \subseteq \cap_{\eta \in Y}\eta = \mathcal{F}(Y)$. So $V(\mu) \supseteq V(\mathcal{F}(Y))$. So $V(\mathcal{F}(Y))$ is the smallest closed set contains Y. Hence $V(\mathcal{F}(Y)) = \overline{Y}$. □

Theorem 7.12. *Let Y be a closed subset of X such that $\mathcal{F}(Y)$ is nonconstant. Then Y is irreducible if and only if $\mathcal{F}(Y)$ is an L-fuzzy prime convex sub l-group of G.*

Proof. Clearly, $V(\mathcal{F}(Y)) = \overline{Y} = Y$ (since, Y is closed). Assume that Y is irreducible. Let λ and ν be a L-fuzzy convex sub l-groups of G such that $\lambda \cap \nu \subseteq \mathcal{F}(Y)$, i.e, $\lambda \cap \nu \subseteq \mu$ for all $\mu \in Y$, i.e, to each $\mu \in Y$, either $\lambda \subseteq \mu$ or $\nu \subseteq \mu$, i.e, to each $\mu \in Y$, either $\mu \in V(\lambda)$ or $\mu \in V(\nu)$. We have $Y \cap V(\lambda) = \{\eta \in Y \mid \lambda \subseteq \eta\}, Y \cap V(\nu) = \{\eta \in Y \mid \nu \subseteq \eta\}$. Then $Y = (Y \cap V(\lambda)) \cup (Y \cap V(\nu))$, by our assumption $Y = Y \cap V(\lambda)$ or $Y = Y \cap V(\nu)$. Thus $Y \subseteq V(\lambda)$ or $Y \subseteq V(\nu)$, i.e, $\lambda \subseteq \mathcal{F}(Y)$ or $\nu \subseteq \mathcal{F}(Y)$. So $\mathcal{F}(Y)$ is a L-fuzzy prime convex sub l-group of G.

Conversely, assume that $\mathcal{F}(Y)$ is an L-fuzzy prime convex sub l-group of G. Let A and B be closed subsets of Y such that $Y = A \cup B$. Since Y is closed in X, A and B are closed in X. We have $V(\mathcal{F}(A)) = \overline{A} = A$, $V(\mathcal{F}(B)) = \overline{B} = B$ and $V(\mathcal{F}(Y)) = \overline{Y} = Y$. Then $\mathcal{F}(Y) = \cap_{\eta \in Y}\eta = \cap_{\eta \in A \cup B}\eta = (\cap_{\eta \in A}\eta) \cap (\cap_{\eta \in B}\eta) = \mathcal{F}(A) \cap \mathcal{F}(B)$. Thus $\mathcal{F}(A) \subseteq \mathcal{F}(Y)$ or $\mathcal{F}(B) \subseteq \mathcal{F}(Y)$, i.e, $\mathcal{F}(A) = \mathcal{F}(Y)$ or $\mathcal{F}(B) = \mathcal{F}(Y)$, i.e, $A = Y$ or $B = Y$. So Y is irreducible. □

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