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On intuitionistic fuzzy representation of intuitionistic fuzzy G-modules

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Abstract. The concept of intuitionistic fuzzy G-modules and their properties are defined and discussed by the authors in [12]. In this paper, we study intuitionistic fuzzy representation of an intuitionistic fuzzy G-module A_N of a G-module M/N onto an intuitionistic fuzzy G-module B of a general linear space GL(V). This transformation is done on the basis of G-module representation theory. We also prove a fundamental theorem of G-module intuitionistic fuzzy representations.

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1. Introduction

The notion of intuitionistic fuzzy subset was introduced by K.T. Atanassov [1] as a generalization of Zadeh′s fuzzy sets [14]. In [3] R. Biswas introduced the concept of intuitionistic fuzzy subgroups and studies some of its properties. After this, many Mathematicians like K. Hur, H. W. Kang, H. K. Song, B. Davvaz, Y.B. Yun, P. Isaac, P.P. John, S. Rahman and Saikia [5, 6, 7, 8, 9] have studied intuitionistic fuzzy versions of various algebraic structures. G. Frobenius developed the theory of group representations at the end of the 19^{th} century. The theory of G-modules originated in the 20^{th} century. Representation theory was developed on the basis of embedding a group G into a general linear group GL(V) (or Aut(V) is the group of automorphisms of V), where V is a vector space over the field K. As a continuation of authors' work [10, 11, 12, 13], in this paper, we prove a theorem related to the intuitionistic fuzzy homomorphism and a fundamental theorem for G-module intuitionistic fuzzy representations.

2. Preliminaries

In this section, we list some basic concepts and well known results on G-modules, intuitionistic fuzzy subgroups and intuitionistic fuzzy G-modules for the sake of completeness of the topic under study. Throughout the paper, R and C will denote the field of real numbers and field of complex numbers respectively. Unless specified all G-modules are assumed to be taken over the field, K where K is a subfield of field of Complex numbers.

Definition 2.1 ([4]). Let G be a group and let M be a vector space over a field K(a subfield of C). Then M is called a G-module if for every g ∈G and m ∈M, ∃ a product (called the action of G on M), gm ∈M satisfies the following axioms
(i)$1_G.m = m, \forall m$ ∈M (1_G being the identity of G)
(ii)$(g.h).m = g.(h.m), \forall m$ ∈M, g, h ∈G
(iii)$g.(k_1m_1 + k_2m_2) = k_1(g.m_1) + k_2(g.m_2)$, $\forall k_1, k_2$ ∈K; m_1, m_2 ∈M and g ∈G.

Since G acts on M on the left hand side, M may be called a left G-module. In a similar way, we can define right G-module. Here, we shall consider only left G-modules. A parallel study is possible using right G-modules also.

Definition 2.2 ([4]). Let G be a group and let M be a G-module over the field K. Let N be a subspace of the vector space over K. Then N is called a G-submodule of M if $an_1 + bn_2$ ∈N, for all a, b ∈K and n_1, n_2 ∈N.

Definition 2.3 ([4]). Let M and M* be G-modules. A mapping f:M→ M* is a G-module homomorphism if
(i) $f(k_1m_1 + k_2m_2) = k_1f(m_1) + k_2f(m_2)$
(ii)$f(gm) = gf(m), \forall k_1, k_2$ ∈K ; m, m_1, m_2 ∈M and g ∈G.

Definition 2.4 ([4]). Let M be a G-module. A subspace N of M is a G-module if N is also a G-module under the same action of G.

Proposition 2.5 ([4]). *If M is a G-module and let N be a G-submodule of M, then M/N is a G-module.*

Definition 2.6 ([1, 2]). Let X be a non-empty fixed set. An intuitionistic fuzzy set(IFS) A in X is an object having the form A = $\{(x, \mu_A(x), \nu_A(x)) : x \in$ X$\}$ where the function μ_A: X→[0,1] and ν_A: X→[0,1] denote the degree of membership namely $(\mu_A(x))$ and the degree of non-membership namely $(\nu_A(x))$ of each element x ∈X to the set A respectively and 0≤ $\mu_A(x) + \nu_A(x) \leq 1$ for each x ∈X.

Remark 2.7. (i) When $\mu_A(x) + \nu_A(x) = 1$, i.e., $\nu_A(x) = 1 - \mu_A(x) = \mu_{A^c}(x)$, $\forall x \in X$, A is a fuzzy set.

(ii) We denote the IFS A $= \{(x, \mu_A(x), \nu_A(x)) : x \in$ X$\}$ by A $= (\mu_A, \nu_A)$.

(iii) An IFS A is also represented by the mapping $(\mu_A, \nu_A) : X \to I \times I$, where $I = [0, 1]$.

Definition 2.8 ([3])**.** An IFS A $= (\mu_A, \nu_A)$ of a group G is said to be an intuitionistic fuzzy group (IFG) or an intuitionistic fuzzy subgroup (IFSG) of G if

(i) $\mu_A(xy) \geqslant \mu_A(x) \wedge \mu_A(y)$ (ii) $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$

(iii) $\mu_A(x^{-1}) = \mu_A(x)$ (iv) $\nu_A(x^{-1}) = \nu_A(x)$, for all $x, y \in$G.

Definition 2.9 ([10])**.** Let X and Y be two non-empty sets and f:X$\to$Y be a mapping. Let A and B be IFSs of X and Y respectively. Then the image of A under the map f is denoted by f(A) and is defined as

$$\mu_{f(A)}(y) = \begin{cases} \vee\{\mu_A(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

and

$$\nu_{f(A)}(y) = \begin{cases} \wedge\{\nu_A(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \varnothing \\ 1 & \text{otherwise.} \end{cases}$$

Also the pre-image of B under f is denoted by f^{-1}(B) and is defined as
$f^{-1}(B)(x) = B(f(x)); \forall x \in$X.

Remark 2.10. Note that $\mu_A(x) \leq \mu_{f(A)}(f(x))$ and $\nu_A(x) \geqslant \nu_{f(A)}(f(x)); \forall x \in$X, and the equality holds when f is bijective.

Theorem 2.11 ([10])**.** *Let $f : G_1 \to G_2$ be a group homomorphism from a group G_1 onto a group G_2 and A, B be IFSG of groups G_1 and G_2 respectively. Then $f(A)$ and $f^{-1}(B)$ are IFSG of groups G_1 and G_2 respectively.*

Definition 2.12 ([12], [13])**.** Let G and G_1 be groups. Let A be an intuitionistic fuzzy group on G and B be an intuitionistic fuzzy group on G_1. Let f be a group homomorphism of G onto G_1. Then f is called a weak intuitionistic fuzzy homomorphism of A into B if $f(A) \subseteq$ B. The homomorphism f is an intuitionistic fuzzy homomorphism of A onto B if f(A)= B. We say that A is an intuitionistic fuzzy homomorphic to B and we write A $\approx$ B.
Let $f : G \to G_1$ be an isomorphism. Then f is called a weak intuitionistic fuzzy isomorphism $f(A) \subseteq$ B and f is an intuitionistic fuzzy isomorphism if f (A) = B.

Definition 2.13 ([4])**.** Let G be a group and let M be a vector space over a field K. A linear representation of G with representation space M is a homomorphism T : G $\rightarrow$ GL(M), where GL(M) denotes the group of invertible K-linear maps from M to itself is called the general linear group.

Definition 2.14 ([13])**.** Let G be a group and let M be a vector space over the field K and T : G $\rightarrow$ GL(M) be a representation of G in M. Let A be an intuitionistic fuzzy group on G and B be an intuitionistic fuzzy group on the range of T. Then the representation T is an intuitionistic fuzzy representation if T is an intuitionistic fuzzy homomorphism of A onto B i.e., T(A) = B, where T(A) = $(\mu_{T(A)}, \nu_{T(A)})$, defined as $\mu_{T(A)}(T_x) = \vee\{\mu_A(z) : z \in T^{-1}(T_x)\}$ and $\nu_{T(A)}(T_x) = \wedge\{\nu_A(z) : z \in T^{-1}(T_x)\}, \forall T_x \in$ T(G)$\subseteq$ GL(M).

Theorem 2.15 ([13])**.** *Let A be an intuitionistic fuzzy subgroup of G and let N be a normal subgroup of G. Let $A_N = (\mu_{A_N}, \nu_{A_N})$, where $\mu_{A_N}, \nu_{A_N} : G/N \rightarrow$ [0,1] defined by $\mu_{A_N}(xN) = \vee\{\mu_A(xn) : n \in N\}$ and $\nu_{A_N}(xN) = \wedge\{\nu_A(xn) : n \in N\}, \forall x \in G$.*
Then A_N is an intuitionistic fuzzy subgroup of G/N.

Theorem 2.16 ([13])**.** *(A fundamental theorem of intuitionistic fuzzy representation) Let G be a group, M be a vector space over K and T: G $\rightarrow$ GL(M) be a representation of G, then ψ: G/N $\rightarrow$ GL(M) defined by $\psi(xN) = T(x) = T_x, \forall x \in$ G, is an intuitionistic fuzzy representation of G/N, where N is a normal subgroup of G.*

Definition 2.17 ([12])**.** Let G be a group and let M be a G-module over the field K, which is a subfield of C. Then an intuitionistic fuzzy G-module on M is an intuitionistic fuzzy set A = (μ_A, ν_A) of M such that following conditions are satisfied

(i) $\mu_A(ax + by) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(ax + by) \leq \nu_A(x) \vee \nu_A(y), \forall a, b \in$K and $x, y \in$M and

(ii) $\mu_A(gm) \geq \mu_A(m)$ and $\nu_A(gm) \leq \nu_A(m), \forall g \in$ G ; $m \in$M.

Example 2.18 ([12])**.** Let G = $\{1, -1\}$, M = R^n over R. Then M is a G-module. Define the intuitionistic fuzzy set A = (μ_A, ν_A) on M by

$$\mu_A(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.5 & \text{if } x \neq 0 \end{cases} ; \nu_A(x) = \begin{cases} 0 & \text{if } x = 0 \\ 0.25 & \text{if } x \neq 0, \end{cases}$$

where $x = (x_1, x_2, ..., x_n) \in$ R^n. Then A is an intuitionistic fuzzy G-module on M.

Example 2.19 ([12])**.** Consider the G-module M = R(i)= C over the field R and let G = $\{1, -1\}$ be the group. Define the intuitionistic fuzzy set A = (μ_A, ν_A) on M by

$$\mu_A(z) = \begin{cases} 1, & \text{if } z = 0 \\ 0.5, & \text{if } z \in R - \{0\} \\ 0.25, & \text{if } z \in R(i) - R \end{cases} \quad ; \nu_A(z) = \begin{cases} 0, & \text{if } z = 0 \\ 0.25, & \text{if } z \in R - \{0\}. \\ 0.5, & \text{if } z \in R(i) - R. \end{cases}$$

Then A is an intuitionistic fuzzy G-module on M.

3. Intuitionistic fuzzy representation of intuitionistic fuzzy G-modules

Definition 3.1. Let f be a G-module homomorphism of M into M^*. Let A and B be intuitionistic fuzzy G-modules on the G-modules M and M^* respectively. Then f is called a G-module intuitionistic fuzzy homomorphism if f(A) = B, written as A $\approx$ B. If $f(A) \subseteq B$, f is called a weak G-module intuitionistic fuzzy homomorphism. If $f : M \to M^*$ is a G-module isomorphism and $f(A) = B$, then f is a G-module intuitionistic fuzzy isomorphism of A onto B. If $f(A) \subseteq$ B, then f is called a weak G-module intuitionistic fuzzy isomorphism of A onto B.

Example 3.2. Let G = $\{1, -1\}$ and M = C, M^* = R be G-modules over R. Define the IFSs A and B on M and M^* respectively as:

$$\mu_A(x+iy) = \begin{cases} 1, & \text{if } x = y = 0 \\ 0.5, & \text{if } x \neq 0 \text{ and } y = 0 \\ 0.25, & \text{if } y \neq 0 \end{cases} \quad ; \nu_A(x+iy) = \begin{cases} 0, & \text{if } x = y = 0 \\ 0.25, & \text{if } x \neq 0 \text{ and } y = 0 \\ 0.5, & \text{if } y \neq 0 \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.5 & \text{if } x \neq 0 \end{cases} \quad ; \nu_B(x) = \begin{cases} 0 & \text{if } x = 0 \\ 0.25 & \text{if } x \neq 0, \forall x \in R. \end{cases}$$

Then A and B are intuitionistic fuzzy G-modules on M and M* respectively(See Example 2.19 and Example 2.18 for n=1).
Define the mapping $f : M \to M^*$ by $f(x + iy) = x + y$, where $x, y \in$R.

For $a, b \in$R and $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$, where $x_1, x_2, y_1, y_2 \in$R, we have

$$\begin{aligned} f(az_1 + bz_2) &= f\{(ax_1 + bx_2) + i(ay_1 + by_2)\} \\ &= ax_1 + bx_2 + ay_1 + by_2 \\ &= a(x_1 + y_1) + b(x_2 + y_2) \\ &= af(z_1) + bf(z_2). \end{aligned}$$

For $g \in$G and $z = x + iy \in$M, we have
$f(gz) = f\{g(x + iy)\} = f(gx + igy) = gx + gy = g(x + y) = gf(z)$.
Hence f is a G-module homomorphism.

We know that the image of A under f is given by

$$\mu_{f(A)}(r) = \begin{cases} \vee\{\mu_A(x+iy) : x+iy \in f^{-1}(r)\} & \text{if } f^{-1}(r) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

and

$$\nu_{f(A)}(r) = \begin{cases} \wedge\{\nu_A(x+iy) : x+iy \in f^{-1}(r)\} & \text{if } f^{-1}(r) \neq \varnothing \\ 1 & \text{otherwise ,} \end{cases}$$

where $r \in M^*$ and $x+iy \in$ M. Now,

$$\begin{aligned} \mu_{f(A)}(0) &= \vee_{x+iy\in M}\{\mu_A(x+iy) : f(x+iy) = 0\} \\ &= \mu_A(0+i0) \vee_{x\in R-\{0\}} \{\mu_A(x+i(-x))\} \\ &= 1 \vee_{x\in R-\{0\}} \{0.5\} = 1. \end{aligned}$$

Also,

$$\begin{aligned} \mu_{f(A)}(r) &= \vee_{x+iy\in M}\{\mu_A(x+iy) : f(x+iy) = r\} \\ &= \mu_A(r+i0) \vee \mu_A(0+ir) \vee_{p,q\in R s.t. p+q=r} \{\mu_A(p+iq)\} \\ &= (0.5) \vee (0.25) \vee_{p,q\in R s.t. p+q=r} \{0.25\} = 0.5. \end{aligned}$$

Similarly, we can show that $\nu_{f(A)}(r)$=0.25. Therefore, f(A)= B.
Hence f is an G-module intuitionistic fuzzy homomorphism of A onto B.

In group representation, we are embedding a group into a general linear space GL(V), where V is a vector space over the field K. Here, we are extending this notion to the theory of G-modules.

Proposition 3.3. *Let G be a group and let V be a G-module over K. Then GL(V) is a G-module.*

Proof. For g ∈G and f ∈GL(V), define $(gf)(v) = gf(v) = f(gv), v$ ∈V. Then it can be easily shown that GL(V) is a G-module.

□

Definition 3.4. Let G be a group and let M, V be G-modules over K. The representation T: M → GL(V) is called a G-module representation if T is a G-module homomorphism of M into GL(V).

Definition 3.5. Let G be a group and let M , V be G-modules over K. Let T be a G-module homomorphism of M into GL(V). Let A and B be an intuitionistic fuzzy G-modules on M and T(M) respectively. Then T is called a G-module intuitionistic fuzzy representation if T is a G-module intuitionistic fuzzy homomorphism of A onto B.

Example 3.6. Let G = $\{1, -1\}$ and let M = C. Define T : M $\to$ GL(V), where V is a G-module over K, by $T(m) = T_m$, $m \in$ M, where $T(m)(v) = mv, \forall v \in$ V.
We first show that T is a G-module homomorphism.

Let $k_1, k_2 \in$K ; $m_1, m_2 \in$M and $v \in$V be any elements, then
T$(k_1m_1 + k_2m_2) = T_{k_1m_1+k_2m_2}$, where
T$(k_1m_1 + k_2m_2)(v) = (k_1m_1 + k_2m_2)v = k_1m_1v + k_2m_2v = k_1T_{m_1}(v) + k_2T_{m_2}(v) = (k_1T_{m_1} + k_2T_{m_2})(v)$ implies that $T_{k_1m_1+k_2m_2} = k_1T_{m_1} + k_2T_{m_2}$.

Also, for any $g \in$G ; $m \in$M and $v \in$V, we have $T(gm) = T_{gm}$, where $(T_{gm})(v) = (gm)v = gT_m(v)$. Therefore, $T_{gm} = gT_m$ i.e., $T(gm) = gT(m)$.
Hence T is a G-module homomorphism and so T is a G-module representation.

Let A and B be intuitionistic fuzzy sets on M and T(M)respectively defined by

$$\mu_A(x+iy) = \begin{cases} 1, & \text{if } x = y = 0 \\ 0.5, & \text{if } x \neq 0 \text{ and } y = 0 \\ 0.25, & \text{if } y \neq 0 \end{cases} ;\nu_A(x+iy) = \begin{cases} 0, & \text{if } x = y = 0 \\ 0.25, & \text{if } x \neq 0 \text{ and } y = 0 \\ 0.5, & \text{if } y \neq 0 \end{cases}$$

and

$$\mu_B(T_m) = \mu_B(T_{x+iy}) = \begin{cases} 1, & \text{if } x = y = 0 \\ 0.5, & \text{if } x \neq 0 \text{ and } y = 0 \\ 0.25, & \text{if } y \neq 0; \end{cases}$$

$$\nu_B(T_m) = \nu_B(T_{x+iy}) = \begin{cases} 0, & \text{if } x = y = 0 \\ 0.25, & \text{if } x \neq 0 \text{ and } y = 0. \\ 0.5, & \text{if } y \neq 0 \end{cases}$$

Then it is easy to check that A and B are intuitionistic fuzzy G-module on M and T(M) respectively.
Now, we find T(A). As T(A)$(T_{x+iy}) = (\mu_{T(A)}(T_{x+iy}), \nu_{T(A)}(T_{x+iy}))$, where

$\mu_{T(A)}(T_{x+iy}) = \vee\{\mu_A(z) : z \in T^{-1}(T_{x+iy})\}$ and $\nu_{T(A)}(T_{x+iy}) = \wedge\{\nu_A(z) : z \in T^{-1}(T_{x+iy})\}$.
Now,

$$\begin{aligned}
\mu_{T(A)}(T_{x+iy}) &= \vee\{\mu_A(z) : z \in T^{-1}(T_{x+iy})\} \\
&= \vee\{\mu_A(z) : T(z) = T_{x+iy}\} \\
&= \vee\{\mu_A(x+iy) : T(x+iy) = T_{x+iy}\} \\
&= \begin{cases} 1, & \text{if } x = y = 0 \\ 0.5, & \text{if } x \neq 0 \text{ and } y = 0 \\ 0.25, & \text{if } y \neq 0; \end{cases} \\
&= \mu_B(T_{x+iy}).
\end{aligned}$$

Similarly, we can show that $\nu_{T(A)}(T_{x+iy}) = \nu_B(T_{x+iy})$. Therefore, T(A) = B. Hence T is a G-module intuitionistic fuzzy representation of A onto B.

Example 3.7. Let M = C^n, V = C^n be the vector spaces over C. Let G = $\{1, -1, i, -i\}$. Define T : $C^n \to$ GL(C^n) by $T(z) = T_z$, where $z = (z_1, z_2, ..., z_n) \in C^n$ and $(T_z)(v) = zv, \forall v \in$V. Here multiplication zv is defined component wise. Then it is easy to verify that T is a G-module homomorphism.
Let A be an IFS on C^n defined by

$$\mu_A(z) = \begin{cases} 1, & \text{if } z_i = 0 \forall i \\ \frac{1}{2}, & \text{if } z_1 \neq 0, z_2 = ... = z_n = 0 \\ \frac{1}{3}, & \text{if } z_2 \neq 0, z_3 = ... = z_n = 0 \\ ...... & \\ \frac{1}{n+1}, & \text{if } z_n \neq 0 \end{cases}$$

and

$$\nu_A(z) = \begin{cases} 0, & \text{if } z_i = 0 \forall i \\ \frac{1}{n+1}, & \text{if } z_1 \neq 0, z_2 = ... = z_n = 0 \\ \frac{1}{n}, & \text{if } z_2 \neq 0, z_3 = ... = z_n = 0 \\ ...... & \\ \frac{1}{2}, & \text{if } z_n \neq 0. \end{cases}$$

Then it is easy to verify that A is an intuitionistic fuzzy G-module on M.
Define the IFS B on T(M) by

$$\mu_B(T_m) = \begin{cases} 1, & \text{if } z_i = 0 \forall i \\ \frac{1}{2}, & \text{if } z_1 \neq 0, z_2 = ... = z_n = 0 \\ \frac{1}{3}, & \text{if } z_2 \neq 0, z_3 = ... = z_n = 0 \\ ...... & \\ \frac{1}{n+1}, & \text{if } z_n \neq 0 \end{cases}$$

and

$$\nu_B(T_m) = \begin{cases} 0, & \text{if } z_i = 0 \forall i \\ \frac{1}{n+1}, & \text{if } z_1 \neq 0, z_2 = ... = z_n = 0 \\ \frac{1}{n}, & \text{if } z_2 \neq 0, z_3 = ... = z_n = 0 \\ ...... & \\ \frac{1}{2}, & \text{if } z_n \neq 0. \end{cases}$$

It is easy to check that B is an intuitionistic fuzzy G-module on T(M).
Now, we find T(A): Since $T(A)(T_z) = (\mu_{T(A)}(T_z), \nu_{T(A)}(T_z))$, where
$\mu_{T(A)}(T_z) = \vee\{\mu_A(t) : t \in T^{-1}(T_z)\}$ and $\nu_{T(A)}(T_z) = \wedge\{\nu_A(t) : t \in T^{-1}(T_z)\}$.
Now,

$$\begin{aligned} \mu_{T(A)}(T_z) &= \vee\{\mu_A(t) : t \in T^{-1}(T_z)\} \\ &= \vee\{\mu_A(t) : T(t) = T_z\} \\ &= \vee\{\mu_A(z) : T(z) = T_z\} \\ &= \begin{cases} 1, & \text{if } z_i = 0 \forall i \\ \frac{1}{2}, & \text{if } z_1 \neq 0, z_2 = ... = z_n = 0 \\ \frac{1}{3}, & \text{if } z_2 \neq 0, z_3 = ... = z_n = 0 \\ ...... & \\ \frac{1}{n+1}, & \text{if } z_n \neq 0 \end{cases} \\ &= \mu_B(T_z). \end{aligned}$$

Similarly, we can show that $\nu_{T(A)}(T_z) = \nu_B(T_z)$. Therefore, $T(\mathrm{A}) = \mathrm{B}$.
Hence T is a G-module intuitionistic fuzzy representation of A onto B.

Proposition 3.8. *Let M be a G-module over K and let N be a G-submodule of M. Then the IFS A_N on M/N, defined by $\mu_{A_N}(x+N) = \vee\{\mu_A(x+n) : n \in N\}$ and $\nu_{A_N}(x+N) = \wedge\{\mu_A(x+n) : n \in N\}, \forall x \in M$ is an intuitionistic fuzzy G-module on M/N.*

Proof. For $a, b \in$ K and $x, y \in$ M, we have
$\mu_{A_N}\{a(x+N)+b(y+N)\} = \mu_{A_N}\{(ax+by)+N\} = \vee\{\mu_A(\{ax+by\}+n) : n \in N\}$
$= \vee\{\mu_A(\{ax+by\}+an_1+bn_2) : n_1, n_2 \in N\}$, where $n = an_1 + bn_2$, for some $n_1, n_2 \in$N [As N is a G-submodule of M. So, $n = an_1 + bn_2 \in$ N]
$= \vee\{\mu_A(\{a(x+n_1)+b(y+n_2)\}) : n_1, n_2 \in N\}$
$\geq \vee\{\mu_A\{a(x+n_1)\} \wedge \{\mu_A\{b(y+n_2)\} : n_1, n_2 \in N\}$
$\geq \vee\{\mu_A(x+n_1) \wedge \mu_A(y+n_2) : n_1, n_2 \in N\}$
$\geq [\vee\{\mu_A(x+n_1) : n_1 \in N\}] \wedge [\vee\{\mu_A(y+n_2) : n_2 \in N\}]$
$= \mu_{A_N}(x+N) \wedge \mu_{A_N}(y+N)$.

Thus, $\mu_{A_N}\{a(x+N)+b(y+N)\} \geq \mu_{A_N}(x+N) \wedge \mu_{A_N}(y+N)$.
Similarly, we can show that $\nu_{A_N}\{a(x+N)+b(y+N)\} \leq \nu_{A_N}(x+N) \vee \nu_{A_N}(y+N)$.

Also, $\mu_{A_N}\{g(x+N)\} = \mu_{A_N}(gx+N) = \vee\{\mu_A(gx+n) : n \in N\}$
$= \vee\{\mu_A(gx+gn_3) : n_3 \in N\}$, where $n = gn_3$ for some $n_3 \in$ N [As N is a G-submodule of M. So, $n = gn_3 \in$ N]
$=\vee\{\mu_A(g(x+n_3)) : n_3 \in N\} \geq \vee\{\mu_A(x+n_3) : n_3 \in N\} = \mu_{A_N}(x+N)$.

Thus, $\mu_{A_N}\{g(x+N)\} \geq \mu_{A_N}(x+N)$.
Similarly, we can show that $\nu_{A_N}\{g(x+N)\} \leq \mu_{A_N}(x+N)$.
Therefore, A_N is intuitionistic fuzzy G-module on M/N.

□

Remark 3.9. (i) The intuitionistic fuzzy G-module A_N defined on M/N, as above, is called the quotient intuitionistic fuzzy G-module or factor intuitionistic fuzzy G-module of A of M relative the G-submodule N.

(ii) If A = (μ_A, ν_A) and B = (μ_B, ν_B) be intuitionistic fuzzy G-modules on G-module M and M* respectively. Then a function $f : M \to M^*$ is said to be a function from A to B if $\mu_A = \mu_B o f$ and $\nu_A = \nu_B o f$. In other words the following diagram commutes
Further, if f is a G-module homomorphism (or G-epimorphism or G-isomorphism)

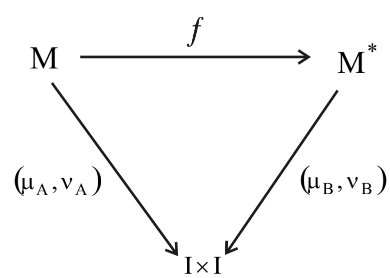

FIGURE 1. Figure-1

from M to M* , then f is said to be intuitionistic fuzzy G-module homomorphism (or G-epimorphism or G-isomorphism)from A to B.

Proposition 3.10. *Let $f : M \to M^*$ be a G-module intuitionistic fuzzy homomorphism of A onto B, where A and B are intuitionistic fuzzy G-modules on M and f(M) respectively. Then the map $\psi : M/N \to M^*$, defined by $\psi(x+N) = f(x), x \in$ M, is a G-module intuitionistic fuzzy homomorphism of A_N onto B, where A_N is an intuitionistic fuzzy G-module on M/N and N is a G-submodule of M.*

Proof. Given that f is a G-module intuitionistic fuzzy homomorphism of A onto B, Therefore,$f(A) = B$. We have to show that $\psi : M/N \to M^*$, defined by $\psi(x+N) = f(x), x \in$ M is a G-module intuitionistic fuzzy homomorphism of A_N onto B.
For this we first show that $\psi : M/N \to M^*$ is a G-module homomorphism.

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "(\mu_A,\nu_A)"'] & M^* \arrow[dl, "(\mu_B,\nu_B)"'] \\
I\times I & M/N \arrow[l, "(\mu_{A_N},\nu_{A_N})"] \arrow[u, "\psi"']
\end{tikzcd}$$

FIGURE 2. Figure-2

Let $a, b \in$K and $x, y \in$ M, we have

$$\begin{aligned}
\psi[a(x+N)+b(y+N)] &= \psi[(ax+by)+N] \\
&= f(ax+by) \\
&= af(x)+bf(y) \\
&= a\psi(x+N)+b\psi(y+N).
\end{aligned}$$

Also, for any $g \in$G and $x+N \in$M/N, We have
$\psi[g(x+N)] = \psi[gx+N] = f(gx) = gf(x) = g\psi(x+N)$.
Hence ψ is a G-module homomorphism.

Next we show that $\psi(A_N) =$ B. Now, $\psi(A_N)(y) = (\mu_{\psi(A_N)}(y), \nu_{\psi(A_N)}(y))$, where
$\mu_{\psi(A_N)}(y) = \vee\{\mu_{A_N}(x+N) : x+N \in \psi^{-1}(y), y \in \psi(M/N)\}$ and
$\nu_{\psi(A_N)}(y) = \wedge\{\nu_{A_N}(x+N) : x+N \in \psi^{-1}(y), y \in \psi(M/N)\}$.

Now,

$$\begin{aligned}
\mu_{\psi(A_N)}(y) &= \vee\{\mu_{A_N}(x+N) : x+N \in \psi^{-1}(y), y \in \psi(M/N)\} \\
&= \vee\{\vee\{\mu_A(z) : z \in x+N\}, \psi(x+N) = y, y \in f(M)\} \\
&= \vee\{\mu_A(z) : z \in x+N, f(x) = y \in f(M)\} \\
&= \mu_{f(A)}(y).
\end{aligned}$$

Similarly, we can show that $\nu_{\psi(A_N)}(y) = \nu_{f(A)}(y)$. Therefore $\psi(A_N) = f(A) = B$.
Hence ψ is a G-module intuitionistic fuzzy homomorphism of A_N onto B.

□

Theorem 3.11. *(A Fundamental Theorem of G-module Intuitionistic Fuzzy Representation)*
Let G be a group and let M,V be G-modules over K. Let T: M $\rightarrow$ GL(V) be a G-module intuitionistic fuzzy representation of A onto B, where A and B are intuitionistic fuzzy G-modules on M and T(M) respectively. Then $\psi : M/N \rightarrow GL(V)$, defined by $\psi(x+N) = T_x, x \in M$, is a G-module intuitionistic fuzzy representation of A_N onto B, where A_N is an intuitionistic fuzzy G-module on M/N, N being a

G-submodule of M.

Proof. Given that T is a G-module intuitionistic fuzzy representation of A onto B, where A and B are intuitionistic fuzzy G-modules on M and T(M) respectively so, T(A) = B. We need to show that ψ : M→ GL(V) is a G-module intuitionistic fuzzy representation of A_N onto B.
Now, by Proposition 3.3, GL(V) is a G-module. Also, by Proposition 3.10, ψ is a

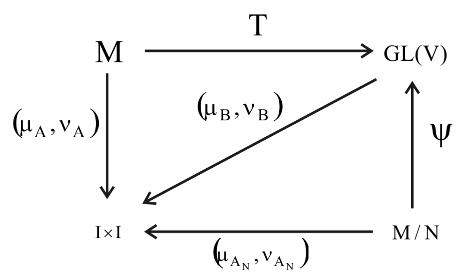

FIGURE 3. Figure-3

G-mdoule intuitionistic fuzzy homomorphism of A_N onto B. So, it remain to show that $\psi(A_N) = $ B.

Now, for $T_x \in T(M) \subseteq$ GL(V). we have $\psi(A_N)(T_x) = (\mu_{\psi(A_N)}(T_x), \nu_{\psi(A_N)}(T_x))$, where $\mu_{\psi(A_N)}(T_x) = \vee\{\mu_{A_N}(x+N) : x+N \in \psi^{-1}(T_x)\}$ and $\nu_{\psi(A_N)}(T_x) = \wedge\{\nu_{A_N}(x+N) : x+N \in \psi^{-1}(T_x)\}$.

Now,

$$\begin{aligned}
\mu_{\psi(A_N)}(T_x) &= \vee\{\mu_{A_N}(x+N) : x+N \in \psi^{-1}(T_x)\} \\
&= \vee\{\vee\{\mu_A(x+n) : n \in N\}, \psi(x+N) = T_x, x \in M\} \\
&= \vee\{\mu_A(z) : z \in x+N, \psi(x+N) = T_x, x \in M\} \\
&= \vee\{\mu_A(z) : T(z) = T_x, x \in M\} \\
&= \vee\{\mu_A(z) : z \in T^{-1}(T_x)\} \\
&= \mu_{T(A)}(T_x).
\end{aligned}$$

Similarly, we can show that $\nu_{\psi(A_N)}(T_x) = \nu_{T(A)}(T_x)$. Thus $\psi(A_N)$= T(A)= B. Hence ψ is a G-module intuitionistic fuzzy representation of A_N onto B.

□

Example 3.12. Let G = $\{1, -1\}$ and M = C. Let N = R. Then N is a G-submodule of M. Consider the G-module homomorphism T : M → GL(V), where V is a G-module over R, defined by $T(m) = T_m, m \in$M. Define the IFSs A and B on M and T(M) respectively as in Example 3.6. Then T is a G-module intuitionistic fuzzy representation of A onto B.

Define ψ : M/N $\rightarrow$ GL(V) by $\psi(u+N) = T(u) = T_u, u \in$ M. Then ψ is G-module homomorphism.

Define an IFS A_N on M/N by

$$\mu_{A_N}(u+N) = \begin{cases} 1, & \text{if } x = y = 0 \\ 0.5, & \text{if } x \neq 0 \text{ and } y = 0 \\ 0.25, & \text{if } y \neq 0 \end{cases} \quad ; \nu_{A_N}(u+N) = \begin{cases} 0, & \text{if } x = y = 0 \\ 0.25, & \text{if } x \neq 0 \text{ and } y = 0 \\ 0.5, & \text{if } y \neq 0 \end{cases}$$

, where $u = x + iy$.

Let $T_u \in T(M) \subseteq$ GL(V). We have $\psi(A_N)(T_u) = (\mu_{\psi(A_N)}(T_u), \nu_{\psi(A_N)}(T_u))$, where $\mu_{\psi(A_N)}(T_u) = \vee\{\mu_{A_N}(u + N) : u + N \in \psi^{-1}(T_u)\}$ and $\nu_{\psi(A_N)}(T_u) = \wedge\{\nu_{A_N}(u + N) : u + N \in \psi^{-1}(T_u)\}$.

Now,

$$\begin{aligned} \mu_{\psi(A_N)}(T_u) &= \vee\{\mu_{A_N}(u + N) : u + N \in \psi^{-1}(T_u)\} \\ &= \vee\{\vee\{\mu_A(z) : z \in u + N\}, \psi(u + N) = T_u, u \in M\} \\ &= \vee\{\mu_A(z) : T(z) = T_u, u \in M\} \\ &= \vee\{\mu_A(z) : z \in T^{-1}(T_u)\} \\ &= \begin{cases} 1, & \text{if } x = y = 0 \\ 0.5, & \text{if } x \neq 0 \text{ and } y = 0 \\ 0.25, & \text{if } y \neq 0 \end{cases} \\ &= \mu_{T(A)}(T_u). \end{aligned}$$

Similarly, we can show that $\nu_{\psi(A_N)}(T_u) = \nu_{T(A)}(T_u)$. Therefore, $\psi(A_N)$= B. Hence ψ is a G-module intuitionistic fuzzy homomorphism of A_N onto B.

Example 3.13. Let M = C^n, V = C^n be the vector spaces over C. Let G = $\{1, -1, i, -i\}$. Let N = $\{w = (z_1, z_2, ..., z_k, 0, 0, ..., 0) : z_i \in$ C and $k < n\}$. Then N is a G-submodule of M. Take T, A and B as in Example (3.7). Then T is a G-module intuitionistic fuzzy representation of A onto B. Now, it remain to show that ψ : M/N $\rightarrow$ GL(V) is a G-module intuitionistic fuzzy representation of A_N onto B, where A_N is an intuitionistic fuzzy G-module on M/N. For this we need to show that $\psi(A_N)$= B.

For $T_z \in \psi$(M/N), we have $\psi(A_N)(T_z) = (\mu_{\psi(A_N)}(T_z), \nu_{\psi(A_N)}(T_z))$, where $\mu_{\psi(A_N)}(T_z) = \vee\{\mu_{A_N}(z + N) : z + N \in \psi^{-1}(T_z)\}$ and $\nu_{\psi(A_N)}(T_z) = \wedge\{\nu_{A_N}(z + N) : z + N \in \psi^{-1}(T_z)\}$.

Now,

$$\begin{aligned}
\mu_{\psi(A_N)}(T_z) &= \vee\{\mu_{A_N}(z+N) : z+N \in \psi^{-1}(T_z)\} \\
&= \vee\{\vee\{\mu_A(v) : v \in z+N, T(v) = T_z\}\} \\
&= \vee\{\mu_A(v) : v \in z+N, v \in T^{-1}(T_z)\} \\
&= \begin{cases} 1, & \text{if } z_i = 0 \forall i \\ \frac{1}{2}, & \text{if } z_1 \neq 0, z_2 = ... = z_n = 0 \\ \frac{1}{3}, & \text{if } z_2 \neq 0, z_3 = ... = z_n = 0 \\ ...... & \\ \frac{1}{n+1}, & \text{if } z_n \neq 0 \end{cases} \\
&= \mu_B(T_z).
\end{aligned}$$

Similarly, we can show that $\nu_{\psi(A_N)}(T_z) = \nu_B(T_z)$. Therefore, $\psi(A_N) =$ B. Hence ψ is a G-module intuitionistic fuzzy representation of A_N onto B.

4. Conclusions

In this article, we have studied the representation of intuitionistic fuzzy G-modules. We have shown that if T: M→GL(V) is a G-module homomorphism from G-module M onto a general linear group GL(V) and if A and B be intuitionistic fuzzy G-modules on M and T(M) respectively, then T is G-module intuitionistic fuzzy repesenation of A onto B if f(A)= B. We have also estabalished a fundamental theorem of G-module intuitioniatic fuzzy representation.

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