Characterizations of near-rings by interval valued $(\alpha, \beta)$-Fuzzy ideals

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Abstract. In this paper, we introduce the concept of interval valued $(\alpha, \beta)$-fuzzy subnear-rings and ideal of near-rings, where $\alpha, \beta$ any two of the $\in, \vee, \lor, \wedge, \ell$ with $\alpha \neq \wedge, \ell$, by using belongs to relation $\in$ and quasi-coincidence with relation $\ell$ between interval valued fuzzy points and interval valued fuzzy sets. We also discussed some characterizations of interval valued $(\alpha, \ell, \vee)$-fuzzy ideals(subnear-rings), mainly discuss interval valued $(\in, \vee)$-fuzzy ideals(subnear-rings) of near-rings.

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1. INTRODUCTION

A near-ring satisfies all axioms of an associative ring, expect commutative of addition and one of the two distributive laws. In 1965, the fundamental concept of a fuzzy set was first initiated by Zadeh[24]. Then the fuzzy sets have been used in the reconsideration of classical mathematics. Ten years later Zadeh[25] introduced the concept of interval valued fuzzy subsets, where the values of the membership functions are the intervals instead of numbers. Rosenfeld[19] introduced the concept of fuzzy subgroup and give some of its properties. The concept of the interval valued fuzzy subgroup was first discussed by Biswas[6] in 1994. Abou-zaid [1] proposed the notion of fuzzy subnear-rings and ideals of near-rings. A new type of fuzzy subgroup, namely, $(\alpha, \beta)$-fuzzy subgroup was introduced by Bhakat and Das[3, 4, 5] using the relation “belongs to” $(\in)$ and “quasi-coincidence” $(\ell)$ of fuzzy points and fuzzy sets initiated by Pu Pao-Ming and Liu-Ming[18]. The $(\in, \vee)$-fuzzy subgroup is an important and useful generalization of Rosenfeld’s fuzzy subgroup. In [12], Dudek et al. introduced the concept of $(\alpha, \beta)$-fuzzy ideals and $(\alpha, \beta)$-fuzzy h-ideals in hemirings. Davvaz[7, 8] used this concept in the theory of near-rings and introduced
that Liu Ming[18] introduced the symbol \( x \) form Definition 2.3. A fuzzy subset of \( R \) is called an (\( \mu \)) fuzzy subnear-ring of near-rings. Young Bae Jun[22, 23], gave some results on (\( \alpha, \beta \))-fuzzy h-ideals in hemirings and discussed some properties of (\( \varepsilon, q \))-fuzzy subalgebras in BCK/BCI- algebras. Narayanana and Manikandan[16] introduced the notion of an (\( \varepsilon, q \))-fuzzy quasi-ideals in near-rings. Deena and Comaressane[11] proposed the notion of (\( \varepsilon, q_k \))-fuzzy subnear-rings and ideals of near-rings which is a generalization of (\( \varepsilon, q \))-fuzzy subnear-rings and ideals. In [13, 14], Zhan et al. have considered the idea of interval valued (\( \alpha, \beta \))-fuzzy hyperideals of hypernear-rings and a new view of fuzzy hypernear-rings. Davvaz[9, 10], discussed few concepts of fuzzy ideals of near-rings and generalized fuzzy \( H_o \)-submodules endowed with interval valued membership functions.

2. Preliminaries

In this section, we present some elementary definitions that we use in the sequel.

Definition 2.1. [8] A near-ring is an algebraic system \((R, +, \cdot)\) consisting of a non empty set \( R \) together with two binary operations called + and \( \cdot \) such that \((R, +)\) is a group not necessarily abelian and \((R, \cdot)\) is a semigroup connected by the following distributive law: \( x \cdot (y + z) = x \cdot y + x \cdot z \) valid for all \( x, y, z \in R \). We will use the word ‘near-ring’ to mean ‘left near-ring’. We denote \( xy \) instead of \( x \cdot y \). An ideal \( I \) of a near-ring \( R \) is the subset of \( R \) such that (i) \((I, +)\) is a normal subgroup of \((R, +)\), (ii) \( IR \subseteq I \), (iii) \((x + a) y - xy \in I \), for any \( a \in I \) and \( x, y \in R \).

Note that \( I \) is a left ideal of \( R \) if \( I \) satisfies (i) and (ii), and right ideal of \( R \) if \( I \) satisfies (i) and (iii).

Definition 2.2. [7] A fuzzy subset \( \mu \) of \( R \) is said to be an \((\varepsilon, q)\)-fuzzy subnear-ring of \( R \) if for all \( x, y \in R \) and \( t, r \in (0, 1): \)

1. \( x_t, y_t \in \mu \) implies \( (x + y)_{\min(t, r)} \in \sqrt{\mu} \),
2. \( x_t \in \mu \) implies \( (-x)_t \in \sqrt{\mu} \),
3. \( x_t, y_t \in \mu \) implies \( (xy)_{\min(t, r)} \in \sqrt{\mu} \).

\( \mu \) is called an \((\varepsilon, q)\)-fuzzy ideal of \( R \) if \( \mu \) is a \((\varepsilon, q)\)-fuzzy subnear-ring of \( R \)

4. \( x_t \in \mu \) implies \( (y + x - y)_t \in \sqrt{\mu} \),
5. \( y_t \in \mu \) and \( x \in R \) implies \( (xy)_t \in \sqrt{\mu} \),
6. \( a_t \in \mu \) and \( x, y \in R \) implies \(((x + a)y - xy)_t \in \sqrt{\mu} \), for any \( x, y, a \in R \).

Definition 2.3. A fuzzy subset of \( R \) is a map \( \mu : R \to [0, 1] \). A fuzzy subset of the form

\[
\mu(y) = \begin{cases} 
0 & \text{if } y = x, \\
1 & \text{if } y \neq x.
\end{cases}
\]

is called a fuzzy point with support \( x \) and value \( t \) and is denoted by \( x_t \).

For a fuzzy point \( x_t \) and a fuzzy subset \( \mu \) of the same set \( R \), Pu Ming and Liu Ming[18] introduced the symbol \( x_t \alpha \mu \), where \( \alpha \in \{\varepsilon, q, \infty, q \in \sqrt{\mu}\} \). A fuzzy point \( x_t \) is said to belong to (resp. quasi-coincident with) a fuzzy subset \( \mu \), written as \( x_t \in \mu_{\square} (\mu \langle x_t \rangle) \) if \( \mu(x) \geq t \) (resp. \( \mu(x) + t > 1 \). The symbol \( x_t \in \sqrt{\mu} \) means that \( x_t \in \mu \) or \( x_t \in \mu \). Similarly, \( x_t \in \infty \mu \) denotes that \( x_t \in \mu \) and \( x_t \in \mu \). \( x_t \in \mu \) means that \( x_t \in \mu \) and \( x_t \in \mu \) do not hold, respectively.

Notation 2.4. [20, 10] By an interval number \( \bar{a} \), we mean an interval \([a^-, a^+]\) such that \( 0 \leq a^- \leq a^+ \leq 1 \) where \( a^- \) and \( a^+ \) are the lower and upper limits of
where \(\mu\) and let \(a\) also identify the interval \([a, a]\) by the number \(a \in [0, 1]\). For any interval numbers \(\tilde{a} = [a_i, a_i], \tilde{b} = [b_i, b_i] \in D[0, 1], i \in I\) we define

\[
\begin{align*}
\max^t\{\tilde{a}, \tilde{b}\} &= \{\max^t\{a_i^+, b_i^\}\}, \max^t\{a_i^+, b_i^\}\}, \\
\min^t\{\tilde{a}, \tilde{b}\} &= \{\min^t\{a_i^+, b_i^\}, \min^t\{a_i^+, b_i^\}\}, \\
\inf\tilde{a} &= \left[\bigcap_{i \in I} a_i^-, \bigcap_{i \in I} a_i^+\right], \sup\tilde{a} = \left[\bigcup_{i \in I} a_i^-, \bigcup_{i \in I} a_i^+\right]
\end{align*}
\]

and let

1. \(\tilde{a} \leq \tilde{b} \iff a^- \leq b^- \text{ and } a^+ \leq b^+\),
2. \(\tilde{a} = \tilde{b} \iff a^- = b^- \text{ and } a^+ = b^+\),
3. \(\tilde{a} < \tilde{b} \iff \tilde{a} \leq \tilde{b} \text{ and } \tilde{a} \neq \tilde{b}\),
4. \(k\tilde{a} = [ka^-, ka^+],\) whenever \(0 \leq k \leq 1\).

**Definition 2.5.** [20] Let \(X\) be a non-empty set. A mapping \(\tilde{\mu} : X \to D[0, 1]\) is called an interval valued fuzzy subset of \(X\). For any \(x \in X\), \(\tilde{\mu}(x) = [\mu^-(x), \mu^+(x)]\), where \(\mu^-\) and \(\mu^+\) are fuzzy subsets of \(X\) such that \(\mu^-(x) \leq \mu^+(x)\). Thus \(\tilde{\mu}(x)\) is an interval subset of \([0, 1]\) and not a number from the interval \([0, 1]\) as in the case of a fuzzy set.

Let \(\tilde{\mu}, \tilde{\nu}\) be interval valued fuzzy subsets of \(X\). The following are defined by

1. \(\tilde{\mu} \leq \tilde{\nu} \iff \tilde{\mu}(x) \leq \tilde{\nu}(x)\).
2. \(\tilde{\mu} = \tilde{\nu} \iff \tilde{\mu}(x) = \tilde{\nu}(x)\).
3. \(\tilde{\mu} \cup \tilde{\nu} = \max\{\tilde{\mu}(x), \tilde{\nu}(x)\}\).
4. \(\tilde{\mu} \cap \tilde{\nu} = \min\{\tilde{\mu}(x), \tilde{\nu}(x)\}\).

**Definition 2.6.** [20] Let \(\tilde{\mu}\) be an interval valued fuzzy subset of \(X\) and \([t_1, t_2] \in D[0, 1]\). Then the set \(\tilde{U}(\tilde{\mu} : [t_1, t_2]) = \{x \in X \mid \tilde{\mu} \geq [t_1, t_2]\}\) is called the upper level set of \(\tilde{\mu}\).

**Definition 2.7.** [20] Let \(I\) be a subset of a near-ring \(R\). Define a function \(\tilde{f}_I : R \to D[0, 1]\) by

\[
\tilde{f}_I(x) = \begin{cases} 
1 & \text{if } x \in I \\
0 & \text{otherwise}
\end{cases}
\]

for all \(x \in R\). Clearly \(\tilde{f}_I\) is an interval valued fuzzy subset of \(R\) and \(\tilde{f}_I\) is called the interval valued characteristic function of \(I\).

3. **Interval Valued \((\alpha, \beta)\)-Fuzzy Ideals**

We now extend the idea of quasi-coincident of fuzzy point with a fuzzy set to the concept of quasi-coincidence of a interval value fuzzy point with an interval valued fuzzy set as follows.

**Definition 3.1.** An interval valued fuzzy set \(\tilde{\mu}\) of a near-ring \(R\) of the form

\[
\tilde{\mu}(y) = \begin{cases} 
t \neq [0, 0], & \text{if } y = x, \\
[0, 0], & \text{if } y \neq x,
\end{cases}
\]

is said to be an interval value fuzzy point with support \(x\) and interval value \(\tilde{x}\) and is denoted by \(x_T\). A interval value fuzzy point \(x_T\) is said to belong to (resp. be quasi-coincident with) an interval valued fuzzy set \(\tilde{\mu}\), written as \(x_T \in \tilde{\mu}\) (resp. \(x_T \not\in \tilde{\mu}\)) if
Throughout this paper $R$ will denote a left near-ring and $\alpha$ and $\beta$ denote any one of $\{\varepsilon, q, \in \bigvee q, \in \bigwedge q\}$ unless otherwise specified. Also $\tilde{\mu}(x) = [\mu^-(x), \mu^+(x)]$ satisfies the following conditions:

1. Any two elements of $D[0,1]$ are comparable.
2. $[\mu^-(x), \mu^+(x)] \geq [0.5,0.5]$ or $[\mu^-(x), \mu^+(x)] < [0.5,0.5]$, for all $x \in R$.

In this section, we present some fundamental concepts and characterizations of interval valued $(\alpha, \beta)$-fuzzy ideals in which the central role is played by $(\varepsilon, \in \bigvee q)$-fuzzy ideals.

We first extend the idea of fuzzy ideals to interval valued $(\alpha, \beta)$-fuzzy ideals of near-rings.

**Definition 3.2.** An interval valued fuzzy set $\tilde{\mu}$ of $R$ is said to be an interval valued $(\alpha, \beta)$-fuzzy subnear-ring of $R$ with $\alpha \neq \in \bigwedge q$ if it satisfies the following conditions:

1. $x_t \alpha \tilde{\mu}$ and $y_t \alpha \tilde{\mu}$ implies $(x + y)_{\min(t, r)} \beta \tilde{\mu}$,
2. $x_t \alpha \tilde{\mu}$ implies $(-x)_{\overline{\gamma}} \beta \tilde{\mu}$,
3. $x_t \alpha \tilde{\mu}$ and $y_t \alpha \tilde{\mu}$ implies $(xy)_{\min(t, r)} \beta \tilde{\mu}$, for all $t, r \in (0, 1]$ and $x, y \in R$.

**Definition 3.3.** An interval valued fuzzy set $\tilde{\mu}$ of $R$ is said to be an interval valued $(\alpha, \beta)$-fuzzy ideals of $R$ with $\alpha \neq \in \bigwedge q$ if the following conditions hold:

4. $\tilde{\mu}$ is an interval valued $(\alpha, \beta)$-fuzzy subnear-ring of $R$,
5. $x_t \alpha \tilde{\mu}$ and $y \in R$ implies $(y + x - y)_{\overline{\gamma}} \beta \tilde{\mu}$,
6. $y_t \alpha \tilde{\mu}$ and $x \in R$ implies $(xy)_{\overline{\gamma}} \beta \tilde{\mu}$,
7. $z_t \alpha \tilde{\mu}$ and $x, y, z \in R$ implies $((x + z)_{\overline{\gamma}} (y - xy)_{\overline{\gamma}} \beta \tilde{\mu}$, for all $t, r \in (0, 1]$ and $x, y, z \in R$.

The conditions (1) and (2) in Definition 3.2 is equivalent to the following condition:

1. $x_t \alpha \tilde{\mu}$ and $y_t \alpha \tilde{\mu}$ implies $(x - y)_{\min(t, r)} \beta \tilde{\mu}$.

Let $\tilde{\mu}$ be an interval valued fuzzy subset of $R$ such that $\tilde{\mu}(x) \leq [0.5,0.5]$ for all $x \in R$. Suppose that $x \in R$ and $t \in (0, 1]$ such that $x_t \in \bigwedge q \tilde{\mu}$. Then $\tilde{\mu}(x) \geq t$ and $\tilde{\mu}(x) + \overline{\gamma} > [1, 1]$. It follows that $[1, 1] < \tilde{\mu}(x) + \overline{\gamma} \leq \tilde{\mu}(x) + \overline{\gamma}(x) = 2\tilde{\mu}(x)$. This means that $\tilde{\mu}(x) > [0.5,0.5]$, and so $\{x_t \in \bigwedge q \tilde{\mu} \} = \emptyset$. Therefore the case $\alpha = \varepsilon \in \bigvee q$ in Definitions 3.2 and 3.3 are omitted.

**Example 3.4.** Let $R = \{a, b, c, d\}$ be a set with two binary operations defined as follows:

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Then $(R, +, \cdot)$ is a near-ring and $I = \{a, b\}$ is its ideal. Let $\tilde{\mu} : R \rightarrow D[0,1]$ be an interval valued fuzzy subset of $R$ defined by $\tilde{\mu}(a) = [0.8,0.9], \tilde{\mu}(b) = [0.6,0.7]$ and $\tilde{\mu}(c) = [0.5,0.5] = \tilde{\mu}(d)$. Then, clearly, $\tilde{\mu}$ is an interval valued $(\varepsilon, \in \bigvee q)$-fuzzy ideal of $R$. But
(1) $\widetilde{\mu}$ is not an interval valued $(\varepsilon, \varepsilon)$-fuzzy ideal of $R$, since $b_{[0.58,0.68]} \in \widetilde{\mu}$ but $((c + b)d - cd)_{[0.58,0.68]} = d_{[0.58,0.68]} = \widetilde{\mu}$.

(2) $\widetilde{\mu}$ is not an interval valued $(q, q)$-fuzzy ideal of $R$, since $a_{[0.2,0.3]}q\widetilde{\mu}$ and $b_{[0.48,0.58]}q\widetilde{\mu}$ but $(a - b)_{[0.2,0.3]} = b_{[0.2,0.3]}q\widetilde{\mu}$.

(3) $\widetilde{\mu}$ is not an interval valued $(q, q)$-fuzzy ideal of $R$, since $a_{[0.2,0.3]}q\widetilde{\mu}$ and $c_{[0.58,0.59]}q\widetilde{\mu}$ but $(a - c)_{[0.2,0.3]} = d_{[0.2,0.3]}q\widetilde{\mu}$.

(4) $\widetilde{\mu}$ is not an interval valued $(\varepsilon, \varepsilon)$-fuzzy ideal of $R$, since $b_{[0.58,0.68]} \in \widetilde{\mu}$ and $c_{[0.48,0.49]} \in \widetilde{\mu}$ but $(b - c)_{[0.48,0.49]} = c_{[0.48,0.49]} \in \widetilde{\mu}$.

(5) $\widetilde{\mu}$ is not an interval valued $(\varepsilon, q)$-fuzzy ideal of $R$, since $b_{[0.58,0.68]} \in \vee q\widetilde{\mu}$ and $c_{[0.48,0.49]} \in \vee q\widetilde{\mu}$ but $(b - c)_{[0.48,0.49]} = c_{[0.48,0.49]} \in \vee q\widetilde{\mu}$.

(6) $\widetilde{\mu}$ is not an interval valued $(\varepsilon, q)$-fuzzy ideal of $R$, since $b_{[0.58,0.68]} \in \widetilde{\mu}$ and $c_{[0.48,0.49]} \in \widetilde{\mu}$ but $(b - c)_{[0.48,0.49]} = c_{[0.48,0.49]} \in \widetilde{\mu}$.

(7) $\widetilde{\mu}$ is not an interval valued $(\varepsilon, q)$-fuzzy ideal of $R$, since $b_{[0.58,0.68]} \in \vee q\widetilde{\mu}$ and $c_{[0.52,0.54]}q\widetilde{\mu}$ but $(b - c)_{[0.52,0.54]} = c_{[0.52,0.54]} \in \vee q\widetilde{\mu}$.

(8) $\widetilde{\mu}$ is not an interval valued $(\varepsilon, q)$-fuzzy ideal of $R$, since $b_{[0.58,0.68]} \in \vee q\widetilde{\mu}$ and $c_{[0.52,0.54]}q\widetilde{\mu}$ but $(b - c)_{[0.52,0.54]} = c_{[0.52,0.54]} \in \vee q\widetilde{\mu}$.

(9) $\widetilde{\mu}$ is not an interval valued $(\varepsilon, q)$-fuzzy ideal of $R$, since $\alpha_{[0.2,0.2]} \in \vee q\widetilde{\mu}$ and $b_{[0.3,0.4]} \in \vee q\widetilde{\mu}$ but $(a - b)_{[0.2,0.2]} = b_{[0.2,0.2]} \in \vee q\widetilde{\mu}$.

In the next theorem, using an interval valued $(\alpha, \beta)$-fuzzy ideal of $R$, we present a method of constructing an ideal of $R$.

**Theorem 3.5.** Let $\widetilde{\mu}$ be an interval valued $(\alpha, \beta)$-fuzzy ideal of $R$. Then the set $S_{\widetilde{\mu}} = \{x \in R \mid \mu(x) > [0, 0]\}$ is an ideal of $R$.

**Proof.** $S_{\widetilde{\mu}} = \{x \in R \mid \mu(x) > [0, 0]\}$. Let $x, y \in S_{\widetilde{\mu}}$ be such that $\mu(x) > [0, 0]$ and $\mu(y) > [0, 0]$. Let $\mu(x - y) = [0, 0]$. If $\alpha \in \{\varepsilon, \varepsilon\} \cap \{\vee q\}$, then $x_{[0, 0]}^\alpha \mu$ and $y_{[0, 0]}^\alpha \mu$ but $\mu(x - y) = [0, 0] < \min\{\mu(x), \mu(y)\}$ and $\mu(x - y) + \min\{\mu(x), \mu(y)\} \leq [0, 0] + [1, 1] = [1, 1]$. So, $\mu(x - y) + \min\{\mu(x), \mu(y)\} \mu(\mu) \forall \mu \in \{\varepsilon, \varepsilon, \vee q\}$, which is a contradiction. Hence $\mu(x - y) > [0, 0]$, that is, $x - y \in S_{\widetilde{\mu}}$. Also, $x_{[1, 1]}^\alpha \mu$ and $y_{[1, 1]}^\alpha \mu$ but $(x - y)_{[1, 1]}^\alpha \mu$ for every $\beta \in \{\varepsilon, \varepsilon, \vee q\}$, a contradiction. Hence $\mu(x - y) > [0, 0]$, that is, $x - y \in S_{\widetilde{\mu}}$. Now, let $x \in S_{\widetilde{\mu}}$, $y \in R$ implies $\mu(x) > [0, 0]$ and we assume that $\mu(x + y) = [0, 0]$. If $\alpha \in \{\varepsilon, \varepsilon\} \cap \{\vee q\}$ then $\mu(x + y) \mu(\mu) \forall \mu \in \{\varepsilon, \varepsilon, \vee q\}$, a contradiction, this means that $(y + x - y) \in S_{\widetilde{\mu}}$. Also, $x_{[1, 1]}^\alpha \mu$ but $(y + x - y)_{[1, 1]}^\alpha \mu$ for every $\beta \in \{\varepsilon, \varepsilon, \vee q\}$, a contradiction. This leads to a contradiction and so $\mu(x + y - x) > [0, 0]$, that is, $y + x - y \in S_{\widetilde{\mu}}$. Again, let $y \in S_{\widetilde{\mu}}$, $x \in R$ implies $\mu(y) > [0, 0]$. Let $\mu(xy) = [0, 0]$. If $\alpha \in \{\varepsilon, \varepsilon\} \cap \{\vee q\}$, then $\mu(xy) \mu(\mu) \forall \mu \in \{\varepsilon, \varepsilon, \vee q\}$, a contradiction, this means that $xy \in S_{\widetilde{\mu}}$. Also, $y_{[1, 1]}^\alpha \mu$ but $(xy)_{[1, 1]}^\alpha \mu$ for every $\beta \in \{\varepsilon, \varepsilon, \vee q\}$, a contradiction. This leads to a contradiction and so $\mu(xy) > [0, 0]$, that is, $xy \in S_{\widetilde{\mu}}$. Let $z \in S_{\widetilde{\mu}}$ and $x, y \in R$. Then $\mu(z) > [0, 0]$. Suppose that $\mu((x + z)y - xy) = [0, 0]$. If $\alpha \in \{\varepsilon, \varepsilon\} \cap \{\vee q\}$, then $\mu((x + z)y - xy) > [0, 0]$, that is, $(x + z)y - xy \in S_{\widetilde{\mu}}$. Also, $z_{[1, 1]}^\alpha \mu$ but $(x + z)y - xy)_{[1, 1]}^\alpha \mu$ for every $\beta \in \{\varepsilon, \varepsilon, \vee q\}$, a contradiction. This leads to a contradiction and so $\mu((x + z)y - xy) > [0, 0]$, that is, $(x + z)y - xy \in S_{\widetilde{\mu}}$. This shows that $S_{\widetilde{\mu}}$ is an ideal of $R$. □
Theorem 3.6. If $I$ is an ideal of $R$, then an interval valued fuzzy subset $\tilde{\mu}$ of $R$ such that

$$\tilde{\mu}(x) = \begin{cases} 
\geq [0.5,0.5] & \text{if } x \in I \\
[0,0] & \text{otherwise}
\end{cases}$$

is an interval valued $(\alpha, \in \vee q)$-fuzzy ideal of $R$.

Proof. (a) Let $x, y \in R$ and $\tilde{t}, \tilde{r} \in D[0,1]$ with $\tilde{t}, \tilde{r} \neq [0,0]$ be such that $x_\tilde{t} \in \tilde{\mu}$ and $y_\tilde{r} \in \tilde{\mu}$. Then $\tilde{\mu}(x) \geq \tilde{t}$ and $\tilde{\mu}(y) \geq \tilde{r}$. Thus $x, y \in I$ and so $x - y \in I$, that is, $\tilde{\mu}(x - y) \geq [0.5,0.5]$. If $\min\{\tilde{t}, \tilde{r}\} \leq [0.5,0.5]$, then $\tilde{\mu}(x - y) \geq [0.5,0.5] \geq \min\{\tilde{t}, \tilde{r}\}$. Hence $(x - y)_{\min\{\tilde{t}, \tilde{r}\}} \in \tilde{\mu}$. If $\min\{\tilde{t}, \tilde{r}\} > [0.5,0.5]$, then $\tilde{\mu}(x - y) + \min\{\tilde{t}, \tilde{r}\} > [0.5,0.5] + [0.5,0.5] = [1,1]$ and so $(x - y)_{\min\{\tilde{t}, \tilde{r}\}} \in \vee q\tilde{\mu}$. Therefore $(x - y)_{\min\{\tilde{t}, \tilde{r}\}} \in \vee q\tilde{\mu}$.

Now, let $x, y \in R$ and $\tilde{t} \in D[0,1]$ with $\tilde{t} \neq [0,0]$ be such that $x_{\tilde{t}} \in \tilde{\mu}$. Then $\tilde{\mu}(x) \geq \tilde{t}$, which implies $x \in I$ and so $y + x - y \in I$. Consequently $\tilde{\mu}(y + x - y) \geq [0.5,0.5]$. If $\tilde{t} \leq [0.5,0.5]$, then $\tilde{\mu}(y + x - y) \geq [0.5,0.5] \geq \tilde{t}$. Hence $(y + x - y)_{\tilde{t}} \in \tilde{\mu}$. If $\tilde{t} > [0.5,0.5]$, then $\tilde{\mu}(y + x - y) + \tilde{t} > [0.5,0.5] + [0.5,0.5] = [1,1]$ and so $(y + x - y)_{\tilde{t}} \in \vee q\tilde{\mu}$. Thus $(y + x - y)_{\tilde{t}} \in \vee q\tilde{\mu}$. Similarly, we can prove that $(y + x - y)_{\tilde{t}} \in \vee q\tilde{\mu}$. Therefore $\tilde{\mu}$ is an interval valued $(\alpha, \in \vee q)$-fuzzy ideal of $R$.

(b) Let $x, y \in R$ and $\tilde{t}, \tilde{r} \in D[0,1]$ with $\tilde{t}, \tilde{r} \neq [0,0]$ be such that $x_{\tilde{t}} q\tilde{\mu}$ and $y_{\tilde{r}} q\tilde{\mu}$. Then $x, y \in I$, $\tilde{\mu}(x) + \tilde{t} > [1,1]$ and $\tilde{\mu}(y) + \tilde{r} > [1,1]$. Since $x - y \in I$, we have $\tilde{\mu}(x - y) \geq [0.5,0.5]$. If $\min\{\tilde{t}, \tilde{r}\} \leq [0.5,0.5]$, then $\tilde{\mu}(x - y) \geq [0.5,0.5] \geq \min\{\tilde{t}, \tilde{r}\}$. Hence $(x - y)_{\min\{\tilde{t}, \tilde{r}\}} \in \tilde{\mu}$. If $\min\{\tilde{t}, \tilde{r}\} > [0.5,0.5]$, then $\tilde{\mu}(x - y) + \min\{\tilde{t}, \tilde{r}\} > [0.5,0.5] + [0.5,0.5] = [1,1]$ and so $(x - y)_{\min\{\tilde{t}, \tilde{r}\}} \in \vee q\tilde{\mu}$. Now let $x, y \in R$ and $\tilde{t} \in D[0,1]$ with $\tilde{t} \neq [0,0]$ be such that $x_{\tilde{t}} q\tilde{\mu}$. This means that $\tilde{\mu}(x) + \tilde{t} > [1,1]$. Thus $x \in I$ and so $y + x - y \in I$. This implies that $\tilde{\mu}(y + x - y) \geq [0.5,0.5]$. If $\tilde{t} \leq [0.5,0.5]$, then $\tilde{\mu}(y + x - y) \geq [0.5,0.5] \geq \tilde{t}$. Hence $(y + x - y)_{\tilde{t}} \in \tilde{\mu}$. If $\tilde{t} > [0.5,0.5]$, then $\tilde{\mu}(y + x - y) + \tilde{t} > [0.5,0.5] + [0.5,0.5] = [1,1]$ and so $(y + x - y)_{\tilde{t}} \in \vee q\tilde{\mu}$. Let $x, y, z \in R$ and $\tilde{t} \in D[0,1]$ with $\tilde{t} \neq [0,0]$ be such that $z_{\tilde{t}} q\tilde{\mu}$. Then $\tilde{\mu}(z) + \tilde{t} > [1,1]$ and it follows that $z \in I$. Then $(x + z) - y \in I$ and so $\tilde{\mu}((x + z) - y) \geq [0.5,0.5]$. If $\tilde{t} \leq [0.5,0.5]$, then $\tilde{\mu}((x + z) - y) \geq [0.5,0.5] \geq \tilde{t}$. Hence $((x + z)y - xy)_{\tilde{t}} \in \tilde{\mu}$. If $\tilde{t} > [0.5,0.5]$, then $\tilde{\mu}((x + z)y - xy) + \tilde{t} > [0.5,0.5] + [0.5,0.5] = [1,1]$ and so $((x + z)y - xy)_{\tilde{t}} \in \vee q\tilde{\mu}$. Thus $((x + z)y - xy)_{\tilde{t}} \in \vee q\tilde{\mu}$. Hence $\tilde{\mu}$ is an interval valued $(\alpha, \in \vee q)$-fuzzy ideal of $R$.

(c) Similar consequence of (a) and (b), we have to prove that $\tilde{\mu}$ is an interval valued $(\alpha, \in \vee q)$-fuzzy ideal of $R$. \qed
Remark 3.7. The following example proves that every interval valued fuzzy set \( \tilde{\mu} \) defined in Theorem 3.6 is an interval valued \((\alpha, \in \lor q)-fuzzy ideal of R \) but \( \tilde{\mu} \) is not an interval valued \((\alpha, \beta)-fuzzy ideal of R \), for every \( \beta \in \{ \epsilon, q, \in q, \in \land q \} \).

Example 3.8. Let \( R = \{ a, b, c, d \} \) be a set with two binary operations defined as follows:

\[
\begin{array}{c|cccc}
+ & a & b & c & d \\
\hline
a & a & b & c & d \\
b & b & a & d & c \\
c & c & d & b & a \\
d & d & c & a & b \\
\end{array}
\quad
\begin{array}{c|cccc}
\cdot & a & b & c & d \\
\hline
a & a & a & a & a \\
b & a & a & a & a \\
c & a & a & a & a \\
d & a & a & a & b \\
\end{array}
\]

Then \((R, +, \cdot)\) is a near-ring and \( I = \{ a, b \} \) is its ideal. Let \( \tilde{\mu} : R \rightarrow D[0, 1] \) be an interval valued fuzzy subset of \( R \) defined by \( \tilde{\mu}(a) = [0.6, 0.7], \tilde{\mu}(b) = [0.5, 0.6] \) and \( \tilde{\mu}(c) = [0, 0] = \tilde{\mu}(d) \). Then, clearly, \( \tilde{\mu} \) is an interval valued \((\epsilon, \in \lor q)-fuzzy ideal of R \). Since, \( a_{[0.26,0.28]} \in \tilde{\mu} \). Then, \( (a-a)_{[0.26,0.28]} = a_{[0.26,0.28]} \in \lor q \tilde{\mu} \) and \( (a-a)_{[0.26,0.28]} = a_{[0.26,0.28]} \in \lor q \tilde{\mu} \), which implies that \( (a-a)_{[0.26,0.28]} = a_{[0.26,0.28]} \in \land q \tilde{\mu} \). Then, \( \tilde{\mu} \) is not an \((\alpha, \in \lor q)-fuzzy ideal of R \).

4. INTERVAL VALUED \((\epsilon, \in \lor q)-FUZZY IDEAL OF NEAR-RINGS

In this section, we introduce the notion of interval valued \((\epsilon, \in \lor q)-fuzzy ideal of near-ring and investigate some of its properties.

Definition 4.1. [20] An i-v fuzzy subset \( \mu \) of a near-ring \( R \) is said to be an i-v \((\epsilon, \in \lor q)-fuzzy subnear-ring of R \) if for all \( x, y \in R \) and \( t, r \in (0, 1) \):

1. \( x_t \in \mu \) and \( y_r \in \mu \) implies \( (x + y)_{\min'(t, r)} \in \lor q \mu \),
2. \( x_{t} \in \mu \) implies \( (-x)_{\overline{t}} \in \lor q \mu \),
3. \( x_{t} \in \mu \) and \( y_{r} \in \mu \) implies \( (xy)_{\min'(t, r)} \in \lor q \mu \).

The conditions (1) and (2) in Definition 4.1 is equivalent to

\[(1') x_{\overline{t}} y_{\overline{r}} \in \mu \] implies \((x - y)_{\min'(t, r)} \in \lor q \mu \).

Definition 4.2. An interval valued fuzzy subset \( \tilde{\mu} \) of \( R \) is said to be an interval valued \((\epsilon, \in \lor q)-fuzzy ideal of R \) if it satisfies the following conditions for all \( t, r \in (0, 1) \) and \( x, y, z \in R \):

1. \( \tilde{\mu} \) is an interval valued \((\epsilon, \in \lor q)-fuzzy subnear-ring of R \),
2. \( x_{\overline{t}} \in \tilde{\mu} \) and \( y \in R \) implies \((y + x - y)_{\overline{t}} \in \lor q \tilde{\mu} \),
3. \( y_{\overline{r}} \in \tilde{\mu} \) and \( x \in R \) implies \((xy)_{\overline{r}} \in \lor q \tilde{\mu} \).
4. \( z_{\overline{t}} \in \tilde{\mu} \) and \( x, y \in R \) implies \((x + z)y - xy)_{\overline{t}} \in \lor q \tilde{\mu} \).

Theorem 4.3. [21] An interval valued fuzzy subset \( \tilde{\mu} \) of \( R \) is an interval valued \((\epsilon, \in \lor q)-fuzzy subnear-ring of R \) if and only if

1. \( \tilde{\mu}(x - y) \geq \min^i\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} \),
2. \( \tilde{\mu}(xy) \geq \min^i\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} \), for all \( x, y \in R \).

Lemma 4.4. Let \( \tilde{\mu} \) be an interval valued fuzzy subset of \( R \) and \( \tilde{\mu} \) is an \((\epsilon, \in \lor q)-fuzzy subnear-ring of R \) and \( \tilde{\mu}(x - y) \geq \min^i\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} \), \( \tilde{\mu}(xy) \geq \min^i\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} \) for all \( x, y \in R \) are equivalent.
(2) (c) $x_\tilde{\varepsilon} \in \tilde{\mu}$ and $y \in R$ implies $(y + x - y)_{\tilde{\varepsilon}} \in \vee q \tilde{\mu}$, and
(d) $\mu(y + x - y) \geq \min^i\{\mu(x), [0.5, 0.5]\}$, for all $x, y \in R$ are equivalent.
(3) (e) $y_{\tilde{\varepsilon}} \in \tilde{\mu}$ and $x \in R$ implies $(xy)_{\tilde{\varepsilon}} \in \vee q \tilde{\mu}$ and
(f) $\mu(xy) \geq \min\{\mu(y), [0.5, 0.5]\}$, for all $x, y \in R$ are equivalent.
(4) (g) $z_{\tilde{\varepsilon}} \in \tilde{\mu}$ and $x, y \in R$ implies $((x + z)y - xy)_{\tilde{\varepsilon}} \in \vee q \tilde{\mu}$ and
(h) $\mu((x + z)y - xy) \geq \min\{\mu(z), [0.5, 0.5]\}$, for all $x, y, z \in R$ are equivalent.

Proof. Let $\tilde{\mu}$ be an interval fuzzy subset of $R$.
(1) (a) $\iff$ (b). Theorem 4.3.
(2) (c) $\implies$ (d): Suppose that (d) is not valid, then there exists $x, y \in R$ such that $\mu(y + x - y) < \min^i\{\mu(x), [0.5, 0.5]\}$. Now, we aries the following two cases:
(i) $\mu(x) \leq [0.5, 0.5]$ (ii) $\mu(x) > [0.5, 0.5]$.
Case (i): We have $\mu(y + x - y) < \mu(x)$. Choose an interval $\tilde{\ell}$ such that $\mu(y + x - y) < \tilde{\ell} < \mu(x)$. This implies $x_{\tilde{\varepsilon}} \in \tilde{\mu}$ and $(y + x - y)_{\tilde{\varepsilon}} \in \vee q \tilde{\mu}$, which contradicts (c). So, $\mu(y + x - y) \geq \mu(x) = \min^i\{\mu(x), [0.5, 0.5]\}$.
Case (ii): We have $\mu(y + x - y) \leq [0.5, 0.5]$. Then $x_{[0.5, 0.5]} \in \tilde{\mu}$ and $(y + x - y)_{[0.5, 0.5]} \in \vee q \tilde{\mu}$, which is a contradiction to (c). Hence $\mu(y + x - y) \geq [0.5, 0.5] = \min [\mu(x), [0.5, 0.5]]$.
(d) $\implies$ (e): Let $x_{\tilde{\varepsilon}} \in \tilde{\mu}$ and $y \in R$. Then $\mu(x) \geq \tilde{\ell}$. Now (d), we have $\mu(y + x - y) \geq \min^i\{\mu(x), [0.5, 0.5]\} \geq \min^i\{\tilde{\ell}, [0.5, 0.5]\}$. If $\tilde{\ell} \leq [0.5, 0.5]$ then $\mu(y + x - y) \geq \tilde{\ell}$ and so $(y + x - y)_{\tilde{\varepsilon}} \in \tilde{\mu}$. If $\tilde{\ell} > [0.5, 0.5]$, then $\mu(y + x - y) + \tilde{\ell} > [1, 1]$ and so $(y + x - y)_{\tilde{\varepsilon}} \in q \tilde{\mu}$. This implies that $(y + x - y)_{\tilde{\varepsilon}} \in \vee q \tilde{\mu}$.
(3) (e) $\implies$ (f): Let us assume that (f) is not valid. Then $x, y \in R$, we can write $\mu(xy) < \min^i\{\mu(y), [0.5, 0.5]\}$. We consider the following two cases:
(i) $\mu(y) \leq [0.5, 0.5]$ (ii) $\mu(y) > [0.5, 0.5]$.
Case (i): We have $\mu(xy) < \mu(y)$. Choose $\tilde{\ell}$ such that $\mu(xy) < \tilde{\ell} < \mu(y)$. Then $y_{\tilde{\varepsilon}} \in \tilde{\mu}$, but $(xy)_{\tilde{\varepsilon}} \notin \vee q \tilde{\mu}$, which contradicts (e).
Case (ii): We have $\mu(xy) < [0.5, 0.5] \leq \mu(y)$. This implies that $y_{[0.5, 0.5]} \in \tilde{\mu}$, but $(xy)_{[0.5, 0.5]} \notin \vee q \tilde{\mu}$, which contradicts (e). Therefore $(xy)_{\tilde{\varepsilon}} \in \vee q \tilde{\mu}$.
(f) $\implies$ (e): Let $y_{\tilde{\varepsilon}} \in \tilde{\mu}$ and $x \in R$ be such that $\tilde{\mu}(y) \geq \tilde{\ell}$. We have $\mu(xy) \geq \min^i\{\mu(y), [0.5, 0.5]\} \geq \min^i\{\tilde{\ell}, [0.5, 0.5]\}$, which implies that $\mu(xy) \geq \tilde{\ell}$ or $\mu(xy) \geq [0.5, 0.5]$ according to $\tilde{\ell} \leq [0.5, 0.5]$ or $\tilde{\ell} > [0.5, 0.5]$. Therefore $(xy)_{\tilde{\varepsilon}} \in \vee q \tilde{\mu}$.
Similarly, we can prove (4)(g) $\implies$ (h) and (h) $\implies$ (g). This completes the proof. \qed

By Definition 4.2 and Lemma 4.4, we obtain the following theorem.

Theorem 4.5. An interval valued fuzzy subset $\tilde{\mu}$ of $R$ is an interval valued $(\varepsilon, \in \vee q)$-fuzzy ideal of $R$ if and only if
(1) $\tilde{\mu}$ is an interval valued $(\varepsilon, \in \vee q)$-fuzzy subnear-ring of $R$,
(2) $\mu(y + x - y) \geq \min^i\{\mu(x), [0.5, 0.5]\}$,
(3) $\mu(xy) \geq \min^i\{\mu(y), [0.5, 0.5]\}$,
(4) $\mu((x + z)y - xy) \geq \min^i\{\mu(z), [0.5, 0.5]\}$, for all $x, y, z \in R$.

In the following theorem, we explain the construction of an interval valued generalized fuzzy ideal form an ideal.
Theorem 4.6. Let $I$ be an ideal of $R$. For every $\bar{t} \in D[0,0.5]$ with $\bar{t} \neq [0,0]$ there exists an interval valued $(\varepsilon, \in \vee q)$-fuzzy ideal $\tilde{\mu}$ of $R$ such that $\tilde{U}(\tilde{\mu} : \bar{t}) = I$.

Proof. Let $\tilde{\mu}$ be an interval valued fuzzy subset in $R$ defined by

$$
\tilde{\mu}(x) = \begin{cases} 
\bar{t} & \text{if } x \in I \\
[0,0] & \text{otherwise}
\end{cases}
$$

for all $x \in R$, where $\bar{t} \in D[0,0.5]$ with $\bar{t} \neq [0,0]$. Obviously, $\tilde{U}(\tilde{\mu} : \bar{t}) = I$. Assume that $\tilde{\mu}(x-y) < \min^i(\tilde{\mu}(x), \tilde{\mu}(y), [0.5,0.5])$, for some $x, y \in R$. Since $|\text{Im}(\tilde{\mu})| = 2$, it follows that $\tilde{\mu}(x-y) = [0,0]$ and $\min^i(\tilde{\mu}(x), \tilde{\mu}(y), [0.5,0.5]) = \bar{t}$. Hence $\tilde{\mu}(x) = \tilde{\mu}(y)$ and so $x, y \in I$. Thus $x-y \in I$, since $I$ is an ideal of $R$ and so $\tilde{\mu}(x-y) = \bar{t}$, which is a contradiction. Therefore $\tilde{\mu}(x-y) \geq \min^i(\tilde{\mu}(x), \tilde{\mu}(y), [0.5,0.5])$. Let us suppose that $\tilde{\mu}(y+x-y) < \min^i(\tilde{\mu}(x), [0.5,0.5])$, for some $x, y \in R$. It follows that $\tilde{\mu}(y+x-y) = [0,0]$ and $\min^i(\tilde{\mu}(x), [0.5,0.5]) = \bar{t}$. Hence $\tilde{\mu}(x) = \tilde{\mu}(y)$ and so $x \in I$. Since $I$ is an ideal of $R$, then $y+x-y \in I$. Thus $\tilde{\mu}(y+x-y) = \bar{t}$, which is a contradiction and hence $\tilde{\mu}(y+x-y) \geq \min^i(\tilde{\mu}(x), [0.5,0.5])$. Assume that $\tilde{\mu}(xy) < \min^i(\tilde{\mu}(x), [0.5,0.5])$, for some $x, y \in R$. Then $\tilde{\mu}(xy) = [0,0]$ and $\min^i(\tilde{\mu}(x), [0.5,0.5]) = \bar{t}$. Hence $\tilde{\mu}(y) = \tilde{\mu}(x)$ and so $y \in I$. Since $I$ is an ideal of $R$, then $xy \in I$. Thus $\tilde{\mu}(xy) = \bar{t}$, which is a contradiction and therefore $\tilde{\mu}(xy) \geq \min^i(\tilde{\mu}(x), [0.5,0.5])$. Similarly, the same procedure we have $\tilde{\mu}((x+z)y-x-y) \geq \min^i(\tilde{\mu}(z), [0.5,0.5])$. 

The next theorem brings out the relationship between interval valued $(\varepsilon, \in \vee q)$-fuzzy ideals of $R$ and the crisp ideals of $R$.

Theorem 4.7. A nonempty subset $I$ of $R$ is an ideal of $R$ if and only if $\tilde{f}_I$ is an interval valued $(\varepsilon, \in \vee q)$-fuzzy ideal of $R$.

Proof. Let $I$ be an ideal of $R$. Then $\tilde{f}_I$ is an interval valued $(\varepsilon, \in \vee q)$-fuzzy ideal of $R$ by Theorem 4.6.

Conversely, assume that $\tilde{f}_I$ is an interval valued $(\varepsilon, \in \vee q)$ fuzzy ideal of $R$. Then clearly, $\tilde{f}_I(x-y) \geq \min^i(\tilde{f}_I(x), \tilde{f}_I(y), [0.5,0.5]) = \min^i([1,1], [0.5,0.5]) = [0.5,0.5]$ $\neq [0,0]$, which implies $\tilde{f}_I(x-y) = [1,1]$ and so $x-y \in I$. Let $x \in I$ and $y \in R$. Then, $\tilde{f}_I(y+x-y) \geq \min^i(\tilde{f}_I(y), [0.5,0.5]) = \min^i([1,1], [0.5,0.5]) = [0.5,0.5] \neq [0,0]$. This implies that $\tilde{f}_I(y+x-y) = [1,1]$ and so $y+x-y \in I$. Let $y \in I$ and $x \in R$ be such that $f_I(y) = [1,1]$. Then, $\tilde{f}_I(xy) \geq \min^i(\tilde{f}_I(x), [0.5,0.5]) = [0.5,0.5] \neq [0,0]$.

This implies that $\tilde{f}_I(xy) = [1,1]$ and so $xy \in I$. Similarly, we proceed like this $(x+z)y-x-y \in I$.

Now, we characterize the interval valued $(\varepsilon, \in \vee q)$-fuzzy ideals using their level ideals.

Theorem 4.8. An interval valued fuzzy subset $\tilde{\mu}$ of $R$ is an interval valued $(\varepsilon, \in \vee q)$-fuzzy ideal of $R$ if and only if the level subset $\tilde{U}(\tilde{\mu} : \bar{t})$ is an ideal of $R$ for all $[0,0] < \bar{t} \leq [0.5,0.5]$.

Proof. Let $\tilde{\mu}$ be an interval valued $(\varepsilon, \in \vee q)$-fuzzy ideal of $R$ and $[0,0] < \bar{t} \leq [0.5,0.5]$. Let $x, y \in \tilde{U}(\tilde{\mu} : \bar{t})$ then $\tilde{\mu}(x) \geq \bar{t}$ and $\tilde{\mu}(y) \geq \bar{t}$. Now by Theorem 4.5, we have $\tilde{\mu}(x-y) \geq \min^i(\tilde{\mu}(x), \tilde{\mu}(y), [0.5,0.5]) \geq \min^i(\bar{t}, \tilde{t}, [0.5,0.5]) = \bar{t}$. So...
\[
x - y \in \tilde{U}(\tilde{\mu} : \tilde{t}). \quad \text{If } x \in \tilde{U}(\tilde{\mu} : \tilde{t}) \text{ and } y \in R. \text{ Then } \tilde{\mu}(x) \geq \tilde{t}. \text{ Consequently by Theorem 4.5, we have } \tilde{\mu}(y + x - y) \geq \min^i \{\tilde{\mu}(x), \min^i \{\tilde{\mu}(y), [0, 0.5]\} = \tilde{t}. \text{ So } y + x - y \in \tilde{U}(\tilde{\mu} : \tilde{t}). \]

Let \( y \in \tilde{U}(\tilde{\mu} : \tilde{t}) \) and \( x \in R. \) Then \( \tilde{\mu}(y) \geq \tilde{t}. \) Since \( \tilde{\mu} \) is an interval valued \( (\in, \in \vee q) \)-fuzzy ideal of \( R, \) we have \( \tilde{\mu}(xy) \geq \min^i \{\tilde{\mu}(y), [0, 0.5]\} \geq \min^i \{\tilde{t}, [0, 0.5]\} = \tilde{t}. \) Thus \( xy \in \tilde{U}(\tilde{\mu} : \tilde{t}). \) and so \( \tilde{U}(\tilde{\mu} : \tilde{t}) \) is a left ideal of \( R. \) Also, for every \( z \in \tilde{U}(\tilde{\mu} : \tilde{t}) \) and \( x, y \in R \) such that \( \tilde{\mu}(z) \geq \tilde{t}. \) Then \( \tilde{\mu}((x + z)y - xy) \geq \min^i \{\tilde{\mu}(z), [0, 0.5]\} \geq \min^i \{\tilde{t}, [0, 0.5]\} = \tilde{t} \) and so \( (x + z)y - xy \in \tilde{U}(\tilde{\mu} : \tilde{t}). \) Therefore \( \tilde{U}(\tilde{\mu} : \tilde{t}) \) is an ideal of \( R. \)

Conversely, assume that \( \tilde{\mu} \) is an interval valued fuzzy subset of \( R \) such that \( \tilde{U}(\tilde{\mu} : \tilde{t})(\neq \emptyset) \) become an ideal of \( R, \) for all \([0, 0] < \tilde{t} \leq [0, 0.5]. \) Let \( x, y \in R. \) Suppose that \( \tilde{\mu}(x-y) \in [0, 0.5]\}. \) Then we can choose \( \tilde{t} \) such that \( \tilde{\mu}(x-y) \in \tilde{t} < \min^i \{\tilde{\mu}(x), \min^i \{\tilde{\mu}(y), [0, 0.5]\}\}. \) This implies that \( x, y \in \tilde{U}(\tilde{\mu} : \tilde{t}). \) Since \( \tilde{U}(\tilde{\mu} : \tilde{t}) \) is an additive subgroup of \( R, \) then \((x - y) \in \tilde{U}(\tilde{\mu} : \tilde{t})\) and \( \tilde{\mu}(x - y) \geq \tilde{t}, \) which is a contradiction. Thus \( \tilde{\mu}(x - y) \geq \min^i \{\tilde{\mu}(x), \tilde{\mu}(y), [0, 0.5]\}. \) Let us assume that \( \tilde{\mu}(y + x - y) < \min^i \{\tilde{\mu}(x), [0, 0.5]\}. \) Choose \( \tilde{t} \) such that \( \tilde{\mu}(y + x - y) \in \tilde{t} < \min^i \{\tilde{\mu}(x), [0, 0.5]\}. \) Then \( x \in \tilde{U}(\tilde{\mu} : \tilde{t}) \) and so \( y + x - y \in \tilde{U}(\tilde{\mu} : \tilde{t}). \) since \( \tilde{U}(\tilde{\mu} : \tilde{t}) \) is a left ideal of \( R. \) This implies that \( \tilde{\mu}(y + x - y) \geq \tilde{t}, \) which contradicts to our hypothesis. Hence \( \tilde{\mu}(y + x - y) \geq \min^i \{\tilde{\mu}(x), [0, 0.5]\}. \) Suppose that \( \mu(xy) \geq \min^i \{\tilde{\mu}(y), [0, 0.5]\}, \) for all \( x, y \in R. \) Then there exist \( t \) such that \( \tilde{\mu}(xy) < \tilde{t} < \min^i \{\tilde{\mu}(y), [0, 0.5]\}. \) Thus \( y \in \tilde{U}(\tilde{\mu} : \tilde{t}) \) and so \( xy \in \tilde{U}(\tilde{\mu} : \tilde{t}). \) since \( \tilde{U}(\tilde{\mu} : \tilde{t}) \) is an ideal of \( R. \) Hence \( \tilde{\mu}(xy) \geq \tilde{t}, \) which contradicts to our hypothesis. Hence \( \tilde{\mu}(xy) \geq \min^i \{\tilde{\mu}(y), [0, 0.5]\}. \) Similarly, we can prove that \( \tilde{\mu}(x + z) - xy \geq \min^i \{\tilde{\mu}(z), [0, 0.5]\}. \) Therefore \( \tilde{\mu} \) is an interval valued \( (\in, \in \vee q) \)-fuzzy ideal of \( R. \)

Next, we discuss the relationship between these generalized interval valued fuzzy ideals.

**Theorem 4.9.** Every interval valued \( (\in, \in \vee q) \)-fuzzy ideal of \( R \) is an interval valued \( (\in, \in \vee q) \)-fuzzy ideal of \( R. \)

**Proof.** Let \( \tilde{\mu} \) be an interval valued \( (\in, \in \vee q) \)-fuzzy ideal of \( R. \) Suppose that \( x, y \in R \) and \( t, r \in D[0, 0.5] \) with \( t, r \neq [0, 0] \) such that \( x \in \tilde{\mu} \) and \( y \in \tilde{\mu}. \) Then \( x \in \in \vee q \tilde{\mu} \) and \( y \in \in \vee q \tilde{\mu}. \) By the hypothesis \((x - y)_{\min^i (\tilde{t}, \tilde{r})} \in \in \vee q \tilde{\mu}. \) Now \( x, y \in R \) and \( t, r \in D[0, 0.5] \) with \( t, r \neq [0, 0] \) such that \( x \in \tilde{\mu} \) and \( y \in \tilde{\mu}. \) Then \( x \in \in \vee q \tilde{\mu}, \) so by hypothesis \((y + x - y)_{\tilde{t}} \in \in \vee q \tilde{\mu}. \) Similarly, we prove \((xy)_{\tilde{t}} \in \in \vee q \tilde{\mu} \) and \((x + z)y - xy)_{\tilde{t}} \in \in \vee q \tilde{\mu}. \) Therefore \( \tilde{\mu} \) is an interval valued \( (\in, \in \vee q) \)-fuzzy ideal of \( R. \)

The following theorem gives the connection between interval valued \( (\in, \in) \)-fuzzy ideal and interval valued fuzzy ideal.

**Theorem 4.10.** An interval valued fuzzy subset \( \tilde{\mu} \) of \( R \) is an interval valued \( (\in, \in) \)-fuzzy ideal of \( R \) if and only if it is an interval valued fuzzy ideal of \( R. \)

**Proof.** Assume that \( \tilde{\mu} \) is an interval valued fuzzy ideal of \( R. \) Let \( x, y \in R \) and \( t, r \in D[0, 1] \) with \( t, r \neq [0, 0] \) be such that \( x, y \in \tilde{\mu}. \) Then \( \tilde{\mu}(x) \geq \tilde{t} \) and \( \tilde{\mu}(y) \geq \tilde{r}. \) Since \( \tilde{\mu} \) is an interval valued fuzzy ideal of \( R, \) we have \( \tilde{\mu}(x - y) \geq \min^i \{\tilde{\mu}(x), \tilde{\mu}(y)\} \geq \min^i \{\tilde{t}, \tilde{r}\}, \) it follows that \( (x - y)_{\min^i (\tilde{t}, \tilde{r})} \in \tilde{\mu}. \) Now let \( x, y \in R \) and \( t \in D[0, 1] \) with
The proof is straightforward. □

The converse part of the above Theorem 4.11 is not true in general as shown in Example 3.4(6).

Theorem 4.12. Every interval valued \((\varepsilon, \varepsilon)\)-fuzzy ideal of \(R\) is an interval valued \((\varepsilon, \varepsilon)\)\(-fuzzy ideal of \(R\).

Proof. The proof is straightforward. □

The converse part of the above Theorem 4.12 is not true in general as shown in Example 3.4(1).

In the following theorem, we give a condition for an interval valued \((\varepsilon, \varepsilon)\)\(-fuzzy ideal of \(R\) to be an interval valued \((\varepsilon, \varepsilon)\)-fuzzy ideal of \(R\).

Theorem 4.13. Let \(\tilde{\mu}\) be an interval valued \((\varepsilon, \varepsilon)\)\(-fuzzy ideal of \(R\) such that \(\tilde{\mu}(x) < [0.5, 0.5] \) for all \(x \in R\). Then \(\tilde{\mu}\) is an interval valued \((\varepsilon, \varepsilon)\)-fuzzy ideal of \(R\).

Proof. Let \(x, y \in R\) and \(\tilde{t}, \tilde{r} \in D[0, 1]\) with \(\tilde{t}, \tilde{r} \neq [0, 0]\) be such that \(x_t, y_t \in \tilde{\mu}\). Then \(\tilde{\mu}(x) \geq \tilde{t}, \tilde{\mu}(y) \geq \tilde{r}\). Since \(\tilde{\mu}\) is an interval valued \((\varepsilon, \varepsilon)\)\(-fuzzy ideal of \(R\), then \(\tilde{\mu}(x-y) \geq \min\{\tilde{r}, \tilde{\mu}(y), [0.5, 0.5]\} \geq \min\{\tilde{t}, \tilde{\mu}(x), [0.5, 0.5]\} = \min\{\tilde{t}, \tilde{r}\}\) and so \((x-y)_{\min\{\tilde{t}, \tilde{r}\}} \in \tilde{\mu}\). Let \(x, y \in R\) and \(\tilde{t} \in D[0, 1]\) with \(\tilde{t} \neq [0, 0]\) be such that \(x_t \in \tilde{\mu}\). Then \(\tilde{\mu}(x) \geq \tilde{t}\). Thus \(\tilde{\mu}(y-x) \geq \min\{\tilde{t}, \tilde{\mu}(x), [0.5, 0.5]\} \geq \tilde{t}\), since \(\tilde{\mu}\) is an interval valued \((\varepsilon, \varepsilon)\)\(-fuzzy ideal of \(R\). Hence \((y-x-y)_{\tilde{t}} \in \tilde{\mu}\). Let \(x, y \in R\) and \(\tilde{t} \in D[0, 1]\) with \(\tilde{t} \neq [0, 0]\). Then \(y_t \in \tilde{\mu}\) implies \(\tilde{\mu}(y) \geq \tilde{t}\). So \(\tilde{\mu}(xy) \geq \min\{\tilde{t}, \tilde{\mu}(y), [0.5, 0.5]\} \geq \tilde{t}\),
since \( \tilde{\mu} \) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy ideal of \( R \). Thus \((xy) \tilde{\mu} \in \tilde{\mu} \). Similarly, we can prove that \((x+z)y-xy \tilde{\mu} \in \tilde{\mu} \). Therefore \( \tilde{\mu} \) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy ideal of \( R \). \( \square \)

**Theorem 4.14.** [21] If \( \{\tilde{\mu}_i \mid i \in \Omega \} \) is a family of interval valued \((\varepsilon, \in \vee q)\)-fuzzy subnear-ring of a near-ring \( R \), then \( \tilde{\mu} = \bigcap_{i \in \Omega} \tilde{\mu}_i \) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy subnear-ring of a near-ring \( R \), where \( \Omega \) is any index set.

**Theorem 4.15.** If \( \{\tilde{\mu}_i \mid i \in \Omega \} \) is a family of interval valued \((\varepsilon, \in \vee q)\)-fuzzy ideal of \( R \), then \( \tilde{\mu} = \bigcap_{i \in \Omega} \tilde{\mu}_i \) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy ideal of \( R \), where \( \Omega \) is any index set.

**Proof.** Let \( x, y, z \in R \). Then, clearly, \( \tilde{\mu} = \bigcap_{i \in \Omega} \tilde{\mu}_i \) is an interval valued \((\varepsilon, \in \vee q)\) fuzzy subnear-ring of \( R \) from Theorem 4.15. Then,

\[
\tilde{\mu}(y + x - y) = \bigcap_{i \in \Omega} \tilde{\mu}_i(y + x - y) = \inf_{i \in \Omega} \{\tilde{\mu}_i(y + x - y) : i \in \Omega\} \\
\geq \inf_{i \in \Omega} \{\min_{i \in \Omega} \{\tilde{\mu}_i(x), [0.5, 0.5]\} : i \in \Omega\} \\
= \min_{i \in \Omega} \{\inf_{i \in \Omega} \{\tilde{\mu}_i(x), [0.5, 0.5]\}\} \\
= \min_{i \in \Omega} \{\tilde{\mu}_i(x), [0.5, 0.5]\}.
\]

\[
\tilde{\mu}(xy) = \bigcap_{i \in \Omega} \tilde{\mu}_i(xy) = \inf_{i \in \Omega} \{\tilde{\mu}_i(xy) : i \in \Omega\} \\
\geq \inf_{i \in \Omega} \{\min_{i \in \Omega} \{\tilde{\mu}_i(y), [0.5, 0.5]\} : i \in \Omega\} \\
= \min_{i \in \Omega} \{\inf_{i \in \Omega} \{\tilde{\mu}_i(y), [0.5, 0.5]\}\} \\
= \min_{i \in \Omega} \{\tilde{\mu}_i(y), [0.5, 0.5]\}.
\]

Similarly, \( \tilde{\mu}(x+z)y-xy \geq \min_{i \in \Omega} \{\tilde{\mu}(z), [0.5, 0.5]\} \). Therefore \( \tilde{\mu} = \bigcap_{i \in \Omega} \tilde{\mu}_i \) is an interval valued \((\varepsilon, \in \vee q)\) fuzzy ideal of \( R \). \( \square \)

**Theorem 4.16.** [21] Let \( \tilde{\mu} \) be an interval valued fuzzy subset of \( R \). \( \tilde{\mu} = [\mu^-, \mu^+] \) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy subnear-ring of \( R \) if and only if \( \mu^-, \mu^+ \) are \((\varepsilon, \in \vee q)\)-fuzzy subnear-ring of \( R \).

The following theorem establishes the connection between interval valued \((\varepsilon, \in \vee q)\)-fuzzy ideal of \( R \) and \((\varepsilon, \in \vee q)\)-fuzzy ideal of \( R \).

**Theorem 4.17.** Let \( \tilde{\mu} \) be an interval valued fuzzy subset of \( R \). \( \tilde{\mu} = [\mu^-, \mu^+] \) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy ideal of \( R \) if and only if \( \mu^-, \mu^+ \) are \((\varepsilon, \in \vee q)\)-fuzzy ideal of \( R \).
Proof. Let \( \tilde{\mu} \) be an interval valued \((\in, \in \lor)\)-fuzzy ideal of \( R \). For any \( x, y, z \in R \),
\[
[\mu^{-}(x - y), \mu^{+}(x - y)] = \tilde{\mu}(x - y) \\
\geq \min^{i}\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} \\
= \min^{i}\{[\mu^{-}(x), \mu^{+}(x)], [\mu^{-}(y), \mu^{+}(y)], [0.5, 0.5]\} \\
= [\min\{\mu^{-}(x), \mu^{-}(y), 0.5\}, \min\{\mu^{+}(x), \mu^{+}(y), 0.5\}].
\]
It follows that \( \mu^{-}(x - y) \geq \min\{\mu^{-}(x), \mu^{-}(y), 0.5\} \) and \( \mu^{+}(x - y) \geq \min\{\mu^{+}(x), \mu^{+}(y), 0.5\} \). And
\[
[\mu^{-}(y + x - y), \mu^{+}(y + x - y)] = \tilde{\mu}(y + x - y) \\
\geq \min^{i}\{\tilde{\mu}(y), [0.5, 0.5]\} \\
= \min^{i}\{[\mu^{-}(y), \mu^{+}(y)], [0.5, 0.5]\} \\
= [\min\{\mu^{-}(y), 0.5\}, \min\{\mu^{+}(y), 0.5\}].
\]
It follows that \( \mu^{-}(y + x - y) \geq \min\{\mu^{-}(y), 0.5\} \) and \( \mu^{+}(y + x - y) \geq \min\{\mu^{+}(y), 0.5\} \). Further,
\[
[\mu^{-}(xy), \mu^{+}(xy)] = \tilde{\mu}(xy) \\
\geq \min^{i}\{\tilde{\mu}(y), [0.5, 0.5]\} \\
= \min^{i}\{[\mu^{-}(y), \mu^{+}(y)], [0.5, 0.5]\} \\
= [\min\{\mu^{-}(y), 0.5\}, \min\{\mu^{+}(y), 0.5\}].
\]
It follows that \( \mu^{-}(xy) \geq \min\{\mu^{-}(y), 0.5\} \) and \( \mu^{+}(xy) \geq \min\{\mu^{+}(y), 0.5\} \). Similarly, \( \mu^{-}((x + z)y - xy) \geq \min\{\mu^{-}(z), 0.5\} \), \( \mu^{+}((x + z)y - xy) \geq \min\{\mu^{+}(z), 0.5\} \). Therefore \( \mu^{+} \) and \( \mu^{-} \) are \((\in, \in \lor)\)-fuzzy ideal of \( R \).

Conversely, assume that \( \mu^{+} \) and \( \mu^{-} \) are \((\in, \in \lor)\)-fuzzy ideal of \( R \). Let \( x, y, z \in R \). Then,
\[
\tilde{\mu}(x - y) = [\mu^{-}(x - y), \mu^{+}(x - y)] \\
\geq [\min\{\mu^{-}(x), \mu^{-}(y), 0.5\}, \min\{\mu^{+}(x), \mu^{+}(y), 0.5\}] \\
= \min^{i}\{[\mu^{-}(x), \mu^{+}(x)], [\mu^{-}(y), \mu^{+}(y)], [0.5, 0.5]\} \\
= \min^{i}\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\}.
\]
Further,
\[
\tilde{\mu}(y + x - y) = [\mu^{-}(y + x - y), \mu^{+}(y + x - y)] \\
\geq [\min\{\mu^{-}(x), 0.5\}, \min\{\mu^{+}(x), 0.5\}] \\
= \min^{i}\{[\mu^{-}(x), \mu^{+}(x)], [0.5, 0.5]\} \\
= \min^{i}\{\tilde{\mu}(x), [0.5, 0.5]\}.
\]
And
\[
\tilde{\mu}(xy) = [\mu^{-}(xy), \mu^{+}(xy)] \\
\geq [\min\{\mu^{-}(y), 0.5\}, \min\{\mu^{+}(y), 0.5\}] \\
= \min^{i}\{[\mu^{-}(y), \mu^{+}(y)], [0.5, 0.5]\} \\
= \min^{i}\{\tilde{\mu}(y), [0.5, 0.5]\}.
\]
Similarly, \( \tilde{\mu}((x + z)y - xy) \geq \min\{\tilde{\mu}(z), [0.5, 0.5]\} \). \( \square \)
Definition 4.18. For any interval valued fuzzy subset $\tilde{\mu}$ of $R$ and $\tilde{t} \in D[0,1]$ with $\tilde{t} \neq [0,0]$ we consider two subsets: $\tilde{Q}(\tilde{\mu};\tilde{t}) = \{ x \in R | x \in \tilde{t}q\tilde{\mu} \}$ and $\tilde{\mu} = \{ x \in R | x \in \tilde{Q}(\tilde{\mu};\tilde{t}) \}$. Obviously, $[\tilde{\mu}]_{\tilde{t}} = \tilde{U}(\tilde{\mu} : \tilde{t}) \cup \tilde{Q}(\tilde{\mu};\tilde{t})$.

We call $[\tilde{\mu}]_{\tilde{t}}$ as an $\in \\forall q$-level ideal and $\tilde{Q}(\tilde{\mu};\tilde{t})$ a $q$-level ideal of $\tilde{\mu}$.

Lemma 4.19. Every interval valued fuzzy subset $\tilde{\mu}$ of $R$ satisfies the following assertion $\tilde{t} \in D[0,0.5]$ with $\tilde{t} \neq [0,0]$ implies $[\tilde{\mu}]_{\tilde{t}} = \tilde{U}(\tilde{\mu} : \tilde{t})$.

Proof. Let $\tilde{t} \in D[0,0.5]$ with $\tilde{t} \neq [0,0]$. Clearly, $\tilde{U}(\tilde{\mu} : \tilde{t}) \subseteq [\tilde{\mu}]_{\tilde{t}}$. Let $x \in [\tilde{\mu}]_{\tilde{t}}$. If $x \notin \tilde{U}(\tilde{\mu} : \tilde{t})$, then $\tilde{\mu}(x) < t$ and so $\tilde{\mu}(x) + t \leq 2t \leq [1,1]$. This implies that $x \notin \tilde{Q}(\tilde{\mu};\tilde{t})$. Thus $x \notin \tilde{U}(\tilde{\mu} : \tilde{t}) \cup \tilde{Q}(\tilde{\mu};\tilde{t})$. This leads to a contradiction and so $x \in \tilde{U}(\tilde{\mu} : \tilde{t})$. Therefore $[\tilde{\mu}]_{\tilde{t}} = \tilde{U}(\tilde{\mu} : \tilde{t})$. □

Using the $(\in \forall q)$-level ideals of near-rings, we characterize the interval valued $(\in, \in \forall q)$-fuzzy ideals of near-rings.

Theorem 4.20. A fuzzy subset $\tilde{\mu}$ of $R$ is an $(\in, \in \forall q)$-fuzzy ideal of $R$ if and only if $[\tilde{\mu}]_{\tilde{t}}$ is an ideal of $R$.

Proof. Assume that $\tilde{\mu}$ is an interval valued $(\in, \in \forall q)$-fuzzy ideal of $R$ and let $\tilde{t} \in D[0,0.5]$ with $\tilde{t} \neq [0,0]$ be such that $[\tilde{\mu}]_{\tilde{t}}$ is an ideal of $R$.

We can consider four cases:

(i) $\tilde{\mu}(x) \geq t$ and $\tilde{\mu}(y) \geq t$,
(ii) $\tilde{\mu}(x) \geq t$ and $\tilde{\mu}(y) + t > [1,1]$,
(iii) $\tilde{\mu}(x) + t > [1,1]$, and $\tilde{\mu}(y) \geq t$,
(iv) $\tilde{\mu}(x) + t > [1,1]$ and $\tilde{\mu}(y) + t > [1,1]$.

Consider Case (i): $\tilde{\mu}(x) \geq t$ and $\tilde{\mu}(y) \geq t$. This implies that $\tilde{\mu}(x) \geq t$ and $\tilde{\mu}(y) \geq t$.

If $\tilde{t} \geq [0.5,0.5]$, then $\tilde{\mu}(x-y) \geq 0.5,0.5$ and so $\tilde{\mu}(x-y) + t \geq [0.5,0.5] + [0.5,0.5] = \tilde{t}$, that is, $(x-y)q\tilde{\mu}$. If $\tilde{t} \leq [0.5,0.5]$, then $\tilde{\mu}(x-y) \geq \tilde{t}$ and thus $(x-y) \in \tilde{\mu}$.

Therefore, $(x-y) \in \forall q\tilde{\mu}$, that is, $(x-y) \in [\tilde{\mu}]_{\tilde{t}}$. Case(ii): $\tilde{\mu}(x) \geq t$ and $\tilde{\mu}(y) + t > [1,1]$. If $\tilde{t} > [0.5,0.5]$, then $\tilde{\mu}(x-y) \geq \min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5,0.5] \} \geq \min\{\tilde{t}, [1,1] - \tilde{t}, [0.5,0.5] \} = [1,1] - \tilde{t}$, that is, $\tilde{\mu}(x-y) + t > [1,1]$ and thus $(x-y) \in \forall q\tilde{\mu}$.

If $\tilde{t} \leq [0.5,0.5]$, then $\tilde{\mu}(x-y) \geq \min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5,0.5] \} \geq \min\{\tilde{t}, [1,1] - \tilde{t}, [0.5,0.5] \} = \tilde{t}$, that is, $(x-y) \in \tilde{\mu}$ and thus $(x-y) \in \forall q\tilde{\mu}$. This means that $x-y \in [\tilde{\mu}]_{\tilde{t}}$. Similarly, we can prove the result for the case(iii). Next we consider the case(iv): $\tilde{\mu}(x) + t > [1,1]$ and $\tilde{\mu}(y) + t > [1,1]$. If $\tilde{t} \geq [0.5,0.5]$, then $\tilde{\mu}(x-y) \geq \min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5,0.5] \} \geq \min\{\tilde{t}, [1,1] - \tilde{t}, [0.5,0.5] \} = [1,1] - \tilde{t}$, and $\tilde{\mu}(y) + t \geq [1,1]$, that is, $(x-y)q\tilde{\mu}$. If $\tilde{t} \leq [0.5,0.5]$, then $\tilde{\mu}(x-y) \geq \min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5,0.5] \} \geq \min\{\tilde{t}, [1,1] - \tilde{t}, [0.5,0.5] \} = [0.5,0.5] \geq t$, that is, $(x-y) \in \tilde{\mu}$ and hence $(x-y) \in \forall q\tilde{\mu}$.

This means that $x-y \in [\tilde{\mu}]_{\tilde{t}}$. Consequently, $[\tilde{\mu}]_{\tilde{t}}$ is a subnear-ring of $(R,+)$. Let $x \in [\tilde{\mu}]_{\tilde{t}}$ and $y \in R$ such that $\tilde{\mu}(x) \geq t$ and $\tilde{\mu}(x) + t > [1,1]$ and we consider two cases.
Case(i): $\bar{\mu}(x) \geq t$. Since $\bar{\mu}$ is an $(\in, \in \forall q)$-fuzzy ideal of $R$, we have $\bar{\mu}(y + x - y) \geq \min^i\{\bar{\mu}(x), [0.5, 0.5]\} \geq \min^i\{t, [0.5, 0.5]\}$. If $t > [0.5, 0.5]$, then $\bar{\mu}(y + x - y) \geq [0.5, 0.5]$ and so $\bar{\mu}(y + x - y) + t > [0.5, 0.5] + [0.5, 0.5] = [1, 1]$, that is, $\bar{\mu}(y + x - y) + t > [1, 1]$. Thus $(y + x - y) \notin q\bar{\mu}$. If $t \leq [0.5, 0.5]$, then $\bar{\mu}(y + x - y) \geq t$. Hence $(y + x - y) \in \bar{\mu}$.

Case(ii): $\bar{\mu}(x) + t > [1, 1]$. Since $\bar{\mu}$ is an interval valued $(\in, \in \forall q)$-fuzzy ideal of $R$, we have $\bar{\mu}(y + x - y) \geq \min^i\{\bar{\mu}(x), [0.5, 0.5]\} > \min^i\{[1, 1] - t, [0.5, 0.5]\}$. If $t > [0.5, 0.5]$, then $\bar{\mu}(y + x - y) > [1, 1] - t$. Thus $(y + x - y) \notin q\bar{\mu}$. If $t \leq [0.5, 0.5]$, then $\bar{\mu}(y + x - y) \geq t$. Hence $(y + x - y) \in \bar{\mu}$. This means that $(y + x - y) \in \forall q\bar{\mu}$, that is, $y + x - y \in [\bar{\mu}]^i$ and therefore $[\bar{\mu}]^i$ is an ideal of $R$. Let $y \in [\bar{\mu}]^i$ and $x \in R$. Then $\bar{\mu}(y) \geq t$ or $\bar{\mu}(y) + t > [1, 1]$.

Assume that $\bar{\mu}(y) \geq t$. Since $\bar{\mu}$ is an $(\in, \in \forall q)$-fuzzy ideal of $R$, we have $\bar{\mu}(xy) \geq \min^i\{\bar{\mu}(y), [0.5, 0.5]\} \geq \min^i\{t, [0.5, 0.5]\}$. If $t > [0.5, 0.5]$, then $\bar{\mu}(xy) \geq [0.5, 0.5]$ implies $\bar{\mu}(xy) + t > [0.5, 0.5] + [0.5, 0.5] = [1, 1]$. So, $\bar{\mu}(xy) + t > [1, 1]$ and thus $(xy) \notin q\bar{\mu}$. If $t \leq [0.5, 0.5]$, then $\bar{\mu}(xy) \geq [0.5, 0.5]$ implies $\bar{\mu}(xy) + t > [0.5, 0.5]$. Thus $(xy) \notin q\bar{\mu}$. This means that $(xy) \in \forall q\bar{\mu}$, that is, $xy \in [\bar{\mu}]^i$ and $[\bar{\mu}]^i$ is an ideal of $R$. Now, let $x, y \in R$ and $z \in [\bar{\mu}]^i$ for $[0, 0] < t \leq [1, 1]$. Then $z \in \forall q\bar{\mu}$, that is, $z \in \forall q\bar{\mu}$ and $z \in [\bar{\mu}]^i$. Since $\bar{\mu}$ is an interval valued $(\in, \in \forall q)$-fuzzy ideal of $R$, then we have $\bar{\mu}((x + z)y - xy) \geq \min^i\{z\bar{\mu}(y), [0.5, 0.5]\}$. Similarly, we can prove that $(x + z)y - xy \in [\bar{\mu}]^i$ and $[\bar{\mu}]^i$ the ideal of $R$. Therefore, $[\bar{\mu}]^i$ is an ideal of $R$.

Conversely, assume that $\bar{\mu}$ be an interval valued fuzzy subset in $R$ and let $[0, 0] < t \leq [1, 1]$ be such that $[\bar{\mu}]^i$ is an ideal of $R$.

Suppose that $\bar{\mu}(x - y) < \min^i\{\bar{\mu}(x), \bar{\mu}(y), [0.5, 0.5]\}$. Choose $\bar{\mu}$ such that $\bar{\mu}(x - y) < t < \min^i\{\bar{\mu}(x), \bar{\mu}(y), [0.5, 0.5]\}$. Then $[0, 0] < t \leq [0.5, 0.5]$ and $x, y \in \bar{\mu}(x - y) = [\bar{\mu}]^i$. Since $[\bar{\mu}]^i$ is an ideal of $R$, then $x - y \in [\bar{\mu}]^i$ and we have $\bar{\mu}(x - y) \geq t$ or $\bar{\mu}(x - y) + t > [1, 1]$, which is a contradiction. Thus $\bar{\mu}(x - y) \geq \min^i\{\bar{\mu}(x), \bar{\mu}(y), [0.5, 0.5]\}$, for all $x, y \in R$.

Now, let $x, y \in R$ be such that $\bar{\mu}(y + x - y) < t < \min^i\{\bar{\mu}(x), [0.5, 0.5]\}$. Then $[0, 0] < t \leq [0.5, 0.5]$ and $x \in \bar{\mu}(x - y) \subseteq [\bar{\mu}]^i$. Since $[\bar{\mu}]^i$ is an ideal of $R$, then $y + x - y \in [\bar{\mu}]^i$ and so $\bar{\mu}(y + x - y) \geq t$ or $\bar{\mu}(y + x - y) + t > [1, 1]$. This is a contradiction to our assumption. Hence $\bar{\mu}(y + x - y) \geq \min^i\{\bar{\mu}(x), [0.5, 0.5]\}$, for all $x, y \in R$. For, let $x, y \in R$ be such that $\mu(xy) + t < \min^i\{\bar{\mu}(y), [0.5, 0.5]\}$. Then $[0, 0] < t \leq [0.5, 0.5]$ and $y \in \bar{\mu}(x - y) \subseteq [\bar{\mu}]^i$. Since $[\bar{\mu}]^i$ is an ideal of $R$, then $xy \in [\bar{\mu}]^i$ and so $\bar{\mu}(xy) \geq t$ or $\bar{\mu}(xy) + t > [1, 1]$ which is a contradiction to our assumption. Hence $\bar{\mu}(xy) \geq \min^i\{\bar{\mu}(y), [0.5, 0.5]\}$, for all $x, y \in R$. Similarly we have to prove $\bar{\mu}(x + z)y - xy \geq \min^i\{\bar{\mu}(z), [0.5, 0.5]\}$ and therefore $\bar{\mu}$ is an interval valued $(\in, \in \forall q)$-fuzzy ideal of $R$.

\textbf{Theorem 4.21.} If $\bar{\mu}$ is an interval valued $(\in, \in \forall q)$-fuzzy ideal of $R$, then the set $Q(\bar{\mu}; t)(\neq \emptyset)$ is an ideal of $R$ for all $[0.5, 0.5] < t \leq [1, 1]$.

\textbf{Proof.} Assume that $\bar{\mu}$ is an interval valued $(\in, \in \forall q)$-fuzzy ideal of $R$ and let $[0.5, 0.5] < t \leq [1, 1]$ be such that $Q(\bar{\mu}; t) \neq \emptyset$. Let $x, y \in Q(\bar{\mu}; t)$ be such that $\bar{\mu}(x) - \bar{\mu}(y)$...
\[\tilde{t} > [1, 1] \text{ and } \bar{\mu}(y) + \tilde{t} > [1, 1] \text{ and we have } \bar{\mu}(x - y) \geq \min\{\bar{\mu}(x), \bar{\mu}(y), [0.5, 0.5]\}. \]

If \(\min\{\bar{\mu}(x), \bar{\mu}(y)\} \geq [0.5, 0.5]\), then \(\bar{\mu}(x - y) \geq [0.5, 0.5] > [1, 1] - \tilde{t}\). If \(\min\{\bar{\mu}(x), \bar{\mu}(y)\} < [0.5, 0.5]\), then \(\bar{\mu}(x - y) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\} > [1, 1] - \tilde{t}\). This implies that \(x - y \in \bar{Q}(\bar{\mu}; \bar{t})\).

Now, let \(x \in \bar{Q}(\bar{\mu}; \bar{t})\) and \(y \in R\) be such that \(\bar{\mu}(x + \tilde{t}) > [1, 1]\). Since \(\bar{\mu}\) is an interval valued \((\in, \in \vee)\)-fuzzy ideal of \(R\), then we have \(\bar{\mu}(y + x - y) \geq \min\{\bar{\mu}(x), [0.5, 0.5]\}\). If \(\bar{\mu}(x) \geq [0.5, 0.5]\), then \(\bar{\mu}(y + x - y) \geq [0.5, 0.5] > [1, 1] - \tilde{t}\). If \(\bar{\mu}(x) < [0.5, 0.5]\), then \(\bar{\mu}(y + x - y) \geq \bar{\mu}(x) > [1, 1] - \tilde{t}\). Thus \(y + x - y \in \bar{Q}(\bar{\mu}; \bar{t})\). Similarly, let \(y \in \bar{Q}(\bar{\mu}; \bar{t})\) and \(x \in R\), then \(xy \in \bar{Q}(\bar{\mu}; \bar{t})\). Again let \(x, y \in R\) and \(z \in \bar{Q}(\bar{\mu}; \bar{t})\) be such that \(\bar{\mu}(z + \tilde{t}) > [1, 1]\). Since \(\bar{\mu}\) is an interval valued \((\in, \in \vee)\)-fuzzy ideal of \(R\), then we have \(\bar{\mu}((x + z)y - xy) \geq \min\{\bar{\mu}(z), [0.5, 0.5]\}\). If \(\bar{\mu}(z) \geq [0.5, 0.5]\), then \(\bar{\mu}((x + z)y - xy) \geq [0.5, 0.5] > [1, 1] - \tilde{t}\) and if \(\bar{\mu}(z) < [0.5, 0.5]\), then \(\bar{\mu}((x + z)y - xy) \geq \bar{\mu}(z) > [1, 1] - \tilde{t}\) and thus \(x(y + z) - xy \in \bar{Q}(\bar{\mu}; \bar{t})\). Therefore \(\bar{Q}(\bar{\mu}; \bar{t})\) is an ideal of \(R\).


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