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A new fixed point theorem for nonlinear contractions of Alber-Guerre Delabriere type in fuzzy metric spaces

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Abstract. The weak contraction was defined by Alber-Guerre Delabriere is one of the interesting generalizations of Banach contraction. In this paper, we consider some contractions of Alber-Guerre Delabriere type in a fuzzy metric space..

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1. Introduction

Zadeh [26] introduced the concept of a fuzzy set which motivated a lot of mathematical activity on the generalization of the notion of a fuzzy set. Heilpern [8] introduced the concept of fuzzy mapping and proved a fixed point theorem for fuzzy contraction mappings which is a generalization of the fixed point theorem for multivalued mappings of Nadler [13]. In 1984, Kaleva and Seikkala [10] introduced the concept of a fuzzy metric space by setting the distance between two points to be a nonnegative fuzzy real number. The fuzzy metric spaces of both Kaleva and Seikkala type and George- Veeramani type have some relationships whith the Menger probabilistic metric spaces (see [10]). It is well known that the Kaleva and $Seikkala's$ type fuzzy metric space possesses rich structure with suitable choices of binary operations.

Recently, Huang and Wu [6] investigated the completion of the Kaleva and $Seikkala's$ type fuzzy metric space. The weak contraction was defined by Alber and Guerre - Delabriere is one of the interesting generalizations of Banach contraction (see [1]) and Xiao, Zhu and Jin consider some contractions of Alber -Guerre -Delabriere type in a FMS (see [25]) and so establish some fuzzy versions of Kannan-Reich type theorem (see [15, 16, 17]). We refer to [19, 20, 21, 3, 11, 18] for additional results on fuzzy metric spaces.

The aim of this work is to establish the existence and unicity of fixed points for mappings in fuzzy metric spaces. Our result generalizes, improves and extends many known results from related literature [25].

2. Preliminaries

Throughout this paper, let Z^+ be the set of all positive integers, $R = (-\infty, +\infty)$ and $R^+ = [0, +\infty)$. If $\Phi : R^+ \to R^+$ is a function and $r \in R^+$, then $\Phi^n(r)$ denotes the nth iteration of $\Phi(r)$ and $\Phi^{-1}(\{0\}) = \{r \in R^+ : \Phi(r) = 0\}$. For the details of fuzzy real number, we refer the reader to Dubois and Prade [4, 5] , Kaleva and Seikkala [10], Mizumoto and Tanaka [12], Wu and Ma [22], Bag and Samanta [2].

Definition 2.1 (cf. Dubois and Prad [4, 5], Xiao and Zhu [23]). A mapping $\eta : R \to [0, 1]$ is called a fuzzy real number or fuzzy interval, whose $\alpha - level$ set is denoted by $[\eta]_\alpha = \{q \in R : \eta(q) \geq \alpha\}$, if it satisfies two axioms:

(i) there exists $q_0 \in R$ such that $\eta(q_0) = 1$

(i) $[\eta]_\alpha = [\lambda_\alpha, p_\alpha]$ is a closed interval of R for each $\alpha \in (0, 1]$, where $-\infty < \lambda_\alpha \leq p_\alpha < +\infty$.

The set of all such fuzzy real numbers is denoted by $\mathcal{F}$. If $\eta \in \mathcal{F}$ and $\eta(q) = 0$ whenever $q < 0$, then η is called a nonnegative fuzzy real number, and by $\mathcal{F}^+$ we mean the set of all nonnegative fuzzy real numbers. If $\lambda_\alpha = -\infty$ and $p_\alpha = +\infty$ are admissible, then, for the sake of clarity, η is called a generalized fuzzy real number. The sets of all generalized fuzzy real numbers or all generalized nonnegative fuzzy real numbers are denoted by $\mathcal{F}_\infty$ and $\mathcal{F}^+_\infty$, respectively. In that case, if $\lambda_\alpha = -\infty$, for instance, then $[\lambda_\alpha, p_\alpha]$ means the interval $(-\infty, p_\alpha]$.

The notation $\overline{0}$ stands for the fuzzy number satisfying $\overline{0}(0) = 1$ and $\overline{0}(q) = 0$ if $q \neq 0$. Clearly, $\overline{0} \in \mathcal{F}^+$. R can be embedded in $\mathcal{F}$: if $a \in R$, than $\overline{a} \in \mathcal{F}$ satisfies $\overline{a}(q) = \overline{0}(q - a)$.

Lemma 2.1 (Xiao et al [23, 24]). Let $\eta \in \mathcal{F}$, $\alpha \in (0, 1]$, and $[\eta]_\alpha = [\lambda_\alpha, p_\alpha]$. Then

(1) $\lim_{q \to -\infty} \eta(q) = 0 = \lim_{q \to +\infty} \eta(q)$.

(2) $\eta(q)$ is a left continuous and non- increasing function for $q \in (\lambda_1, +\infty)$.

(3) p_α is a left continuous and non- increasing function for $\alpha \in (0, 1]$.

Definition 2.2 (cf. Kaleva and Seikkala [10]). Suppose that X a non- empty set and that d is a mapping from $X \times X$ in to $\mathcal{F}^+$. Let $L, R : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ be two symmetric and nondecreasing functions such that $L(0, 0) = 0$ and $R(1, 1) = 1$. For $\alpha \in (0, 1]$ and $x, y \in X$, define the mapping

$$[d(x, y)]_\alpha = [\lambda_\alpha(x, y), p_\alpha(x, y)].$$

The quadruple (X, d, L, R) is called a fuzzy metric space (briefly, FMS), and d is called a fuzzy metric, if

(D1) $d(x,y) = 0^-$ if and only if $x = y$,
(D2) $d(x,y) = d(y,x)$ for all $x,y \in X$,
(D3) for all $x,y,z \in X$:
(D3L) $d(x,y)(p+q) \geq L(d(x,z)(p), d(z,y)(q))$, whenever $p \leq \lambda_1(x,z), q \leq \lambda_1(z,y)$ and
$p+q \leq \lambda_1(x,y)$,
(D3R) $d(x,y)(p+q) \leq R(d(x,z)(p), d(z,y)(q))$, whenever $p \leq \lambda_1(x,z), q \leq \lambda_1(z,y)$ and
$p+q \leq \lambda_1(x,y)$.
If d is a mapping from X into F^+_∞ and (X,d,L,R) satisfies (D1) -(D3), then (X,d,L,R) is called a generalized fuzzy metric space (briefly, GFMS).

From Lemma 2.1 and Definition 2.2 we obtain the following consequence.

Lemma 2.2. Let (X,d,L,R) be a FMS , $[d(x,y)]_t = [\lambda_t(x,y), p_t(x,y)]$ for $t \in (0,1]$, where $x,y \in X$ are any two fixed elements. Then
(1) $\lim_{q\to-\infty} d(x,y)(q) = 0 = \lim_{q\to+\infty} d(x,y)(q)$,
(2) $d(x,y)(q)$ is a left continuous and non- increasing function for $q \in (\lambda_1(x,y), +\infty)$,
(3) $p(x,y)(q)$ is a left continuous and non- increasing function for $t \in (0,1]$.

Lemma 2.3 . Let (X,d,L,R) be a FMS. Then

(1) (R-1) $\Rightarrow$ for each $t \in (0,1]$ [5],
$p_t(x,y) \leq p_t(x,z) + p_t(z,y)$ for all $x,y,z \in X$.

(2) (R-2) $\Rightarrow$ for each $t \in (0,1]$ there exists $s = s(t) \in (0,t]$ such that [18]
$p_t(x,y) \leq p_s(x,z) + p_t(z,y)$ for all $x,y,z \in X$.

(3) (R-3)$\Rightarrow$ for each $t \in (0,1]$ there exists $s = s(t) \in (0,t]$ such that (cf. [6,17])
$p_t(x,y) \leq p_s(x,z) + p_s(z,y)$ for all $x,y,z \in X$.

Lemma 2.4 ([18]). Let (X,d,L,R) be a FMS with (R-2). Thene for each $t \in (0,1]$, $p_t(x,y)$ is continuous at $(x,y) \in X \times X$.

Using the similar manner given by Jachymski [9], we can obtain the following lemmas whose proofs are omitted.

Lemma 2.5. Let $\Phi : R^+ \to R^+$be a function such that $\Phi^{-1}(\{0\}) = \{0\}$.
(1) If $\Phi(r) < r$ and $\lim_{q\to r} \sup \Phi(q) < r$ for all $r > 0$, then $\lim_{n\to\infty} \Phi^n(r) = 0$ for all $r > 0$,
(2) If Φ is non-decreasing and $\lim_{n\to\infty} \Phi^n(r) = 0$ for all $r > 0$, then $\Phi(r) < r$ for all $r > 0$.

Lemma 2.6. Let $\Psi : R^+ \to R^+$ be a function such that $\Psi^{-1}(\{0\}) = \{0\}$.
(1) If $\Psi(r) > r$ and $\lim_{q\to r} \inf \Psi(q) > r$ for all $r > 0$, then $\lim_{n\to\infty} \Psi^n(r) = +\infty$ for all $r > 0$.

(2) If Ψ is non-decreasing and $\lim_{n\to\infty}\Psi^n(r)=+\infty$ for all $r>0$, then $\Psi(r)>r$ for all $r>0$.

3. Main result

In the section, using the idea of the weak contraction was defined by Alber -Guerre -Delabriere we prove the existence and unicity of fixed points for mappings in fuzzy metric spaces.

Theorem 3.1 [25]**.** Let (X,d,L,R) be a complete FMS with (R-2) . Let $\varphi : R^+ \longrightarrow R^+$ be a nondecreasing function with $\varphi^{-1}(\{0\}) = \{0\}$. Let $T : X \longrightarrow X$ be a mapping such that

$$p_t(Tx,Ty) \leq p_t(x,y) - \varphi(p_t(x,y)) \text{ for all } t \in (0,1] \text{ and } x,y \in X.$$

Then there exists a unique $u \in X$ such that $Tu = u$.

Theorem 3.2 [25]**.** Let(X,d,L,R) be a complete FMS with (R-2). Let $\varphi : R^+ \longrightarrow R^+$ be a lower semi-continuous function with $\varphi^{-1}(\{0\}) = \{0\}$. Let $T : X \longrightarrow X$ be a mapping such that,

$$p_t(Tx,Ty) \leq M_t(x,y) - \varphi(M_t(x,y)) \text{ for all } t \in (0,1] \text{ and } x,y \in X,$$

where $M_t(x,y) = \max\{p_t(x,y), p_t(Tx,x), p_t(Ty,y)\}$. Then there exists a unique $u \in X$ such that $Tu = u$.

Now, we can prove the following Theorem.

Theorem 3.3. Let (X,d,L,R) be a complete FMS with (R-2). Let $\Psi, \Phi : R^+ \to R^+$ be a lower semi-continuous function with $\Psi^{-1}(\{0\}), \Phi^{-1}(\{0\}) = \{0\}$. Let $T : X \longrightarrow X$ be a mapping such that,

$$\Psi(p_t(Tx,Ty)) \leq \Psi M_t(x,y) - \Phi(M_t(x,y),$$

for all $t \in (0,1]$ and $x,y \in X$, where $M_t(x,y) = \max\{p_t(x,y), p_t(Tx,x), p_t(Ty,y)\}$. Then there exists a unique $u \in X$ such that $Tu = u$.

Proof. Taking $x_0 \in X$, we construct a sequence $\{x_n\}_{n=1}^{\infty}$ by $x_n = Tx_{n-1}$. Let $t \in (0,1]$ and $a_n(t) = p_t(x_n, x_{n-1})$. Then, we have

$$\begin{aligned} M_t(x_n,x_{n-1}) &= \max\{p_t(x_n,x_{n-1}), p_t(Tx_n,x_n), p_t(Tx_{n-1},x_{n-1})\} \\ &= \max\{p_t(x_n,x_{n-1}), p_t(x_{n+1},x_n)\} = \max\{a_n(t), a_{n+1}(t)\}. \end{aligned} \tag{3.1}$$

By (1), we have

$$\Psi(a_{n+1}(t)) = \Psi(p_t(Tx_n,Tx_{n-1})\}) \leq \Psi(M_t(x_n,x_{n-1})) - \Phi(M_t(x_n,x_{n-1})). \tag{3.2}$$

Now we show that $a_{n+1}(t) \leq a_n(t)$. Suppose opposite that $a_{n+1}(t) > a_n(t) \geq 0$. Then, from (2) and (3) it follows that $M_t(x_n,x_{n-1}) = a_{n+1}(t)$ and

$$\Psi(a_{n+1}(t)) \leq \Psi(a_{n+1}(t)) - \Phi(a_{n+1}(t)) < \Psi(a_{n+1}(t)),$$

which is a contradiction. Hence , $\{a_n(t)\}$ is a nonnegative non- increasing sequence, and so it possesses a limit $a(t) \geq 0$. By (3), we have $\Psi(a_{n+1}(t)) \leq \Psi(a_n(t)) - \Phi(a_n(t))$. From the lower semi- continuity of Ψ, Φ it follows that

$$\begin{aligned} a(t) &= \lim \Psi(a_{n+1}(t)) \leq \lim_{n\to\infty} \inf[\Psi(a_n(t)) - \Phi(a_n(t))] = \Psi(a(t)) - \lim_{n\to\infty} \sup \Phi(a_n(t)) \\ &\leq \Psi(a_n(t)) - \lim_{q\to a(t)} \inf \Phi(q) \leq \Psi(a(t)) - \Phi(a(t)), \end{aligned}$$

i.e., $\Phi(a(t)) = 0$. Hence, we have

$$\lim_{n\to\infty} \Psi(a_n(t)) = a(t) = 0 \text{ for all } t \in (0,1]. \tag{3.3}$$

In the next step we show that $\{x_n\}$ is a cauchy sequence . Since (X,d,L,R) is with (R-2), by Lemma 2.3(2), there exists $s \in (0,t]$ such that

$$p_t(x,y) \leq p_t(x,z) + p_s(z,y) \quad \text{for all } x,y,z \in X. \tag{3.4}$$

Suppose opposite that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon_0 > 0$ and $t \in (0,1]$ for which we can find two subsequences $\{x_{mi}\}$ and $\{x_{ni}\}$ of $\{x_n\}$, where m_i is the smallest index, such that

$$m_i > n_i \geq i, \qquad p_t(x_{mi}, x_{ni}) \geq \epsilon_0. \tag{3.5}$$

Thus, by (5) and (6), we have

$$\epsilon_0 \leq p_t(x_{mi}, x_{ni}) \leq p_s(x_{mi}, x_{mi-1}) + p_t(x_{mi-1}, x_{ni}) \leq a_{mi}(s) + \epsilon_0.$$

It follows from (4) that $p_t(x_{mi}, x_{ni}) \to \epsilon_0$ as $i \to \infty$. Observe that

$$\begin{aligned} M_t(x_{mi}, x_{ni}) &= \max\{p_t(x_{mi}, x_{ni}), p_t(Tx_{mi}, x_{mi}), p_t(Tx_{ni}, x_{ni})\} \\ &= \max\{p_t(x_{mi}, x_{ni}), a_{mi+1}(t), a_{ni+1}(t)\}. \end{aligned}$$

This implies that $M_t(x_{mi}, x_{ni}) \to \epsilon_0$ as $i \to \infty$. By (5) and(1), we have

(3.6)

$$\begin{aligned} &\Psi(p_t(x_{mi}, x_{ni}) \leq \Psi(p_s(x_{mi}, x_{mi+1}) + p_t(x_{mi+1}, x_{ni+1}) + p_s(x_{ni+1}, x_{ni})) \\ &\qquad \leq \Psi(a_{mi+1}(s)) + \Psi(M_t(x_{mi}, x_{ni})) - \Phi((M_t(x_{mi}, x_{ni})) + \Psi(a_{ni+1}(s)). \end{aligned}$$

Since Ψ, Φ is lower semi-continuous, from (4) and (7) it follows that

$$\begin{aligned} \epsilon_0 &= \lim_{i\to\infty} \Psi(p_t(x_{mi}, x_{ni}) \\ &\leq \lim_{i\to\infty} \inf[\Psi(a_{mi+1}(s)) + \Psi(M_t(x_{mi}, x_{ni})) - \Phi((M_t(x_{mi}, x_{ni})) + \Psi(a_{ni+1}(s))], \\ &= \Psi\epsilon_0 - \lim_{i\to\infty} \sup \Phi(M_t(x_{mi}, x_{ni})) \leq \Psi\epsilon_0 - \lim_{q\to\epsilon_0} \inf \Phi(q) \leq \Psi\epsilon_0 - \Phi(\epsilon_0). \end{aligned}$$

Which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. As (X,d,L,R)is complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Thus, for each $t \in (0,1]$, we have

$$\begin{aligned} M_t(x_{n-1}, u) &= \max\{p_t(x_{n-1}, u), p_t(Tx_{n-1}, x_{n-1}), p_t(Tu, u)\} \\ &= \max\{p_t(x_{n-1}, u)), a_n(t), p_t(Tu, u)\} \to p_t(Tu, u) \ as \ n \to \infty. \end{aligned}$$

Using Lemma 2.4, P_t is continuous on $X \times X$. Hence, from (1) and the lower semi-continuity of Ψ, Φ we have

$$\begin{aligned}\Psi(p_t(u,Tu)) &= \lim_{n\to\infty}\Psi(p_t(x_n,Tu)) \le \lim_{n\to\infty}\inf[\Psi(M_t(x_{n-1},u)) - \Phi((M_t(x_{n-1},u))] \\ &= \Psi(p_t(Tu,u)) - \lim_{n\to\infty}\sup\Phi(M_t(x_{n-1},u)) \\ &\le \Psi(p_t(Tu,u)) - \lim_{q\to\ p_t(Tu,u)}\inf\Phi(q) \le \Psi(p_t(Tu,u)) - \Phi(p_t(Tu,u)).\end{aligned}$$

This shows that $p_t(Tu,u) = 0$ for all $t \in (0,1]$, i.e., $Tu = u$. If $v \in X$ with $Tv = v$, then, from(1) it follows that

$$\Psi(p_t(u,v)) = \Psi(p_t(Tu,Tv) \le \Psi(M_t(u,v)) - \Phi((M_t(u,v)) = \Psi(p_t(u,v)) - \Phi(p_t(u,v))$$

for all $t \in (0,1]$. This shows that $p_t(u,v) = 0$ for all $t \in (0,1]$, i.e., $u = v$. So, the proof of Theorem 3.3 is finished. □

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