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Fixed point theorems on modified intuitionistic fuzzy quasi-metric spaces with application to the domain of words

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Abstract. In this paper, we introduce the concept of modified intuitionistic fuzzy-quasi metric space and prove modified intuitionistic fuzzy quasi-metric version of the Banach contraction principle which extend the famous Grabiec fixed point theorem. By using this result we show the existence of fixed point for contraction mappings on the domain of words and apply this approach to deduce the existence of solution for some recurrence equations associated to the analysis of Quicksort algorithms and divide and Conquer algorithms, respectively.

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Key words: Modified intuitionistic fuzzy quasi-metric space, common fixed point, G-complete modified intuitionistic fuzzy quasi-metric space, B-contraction, domain of words.

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1 Introduction

It turned out to be milestone in the development of mathematics, when the concept of fuzzy set was introduced by Zadeh [47] in 1965. After that many authors developed the theory of fuzzy sets and their application. On the other hand the concept of fuzzy sets was generalized as intuitionistic fuzzy set by Atanassov [4] in 1984, which found applications in various fields.

Many authors introduced the concept of fuzzy metric space in different ways. George and Veeramani [16] modified the concept of fuzzy metric space given by Kramosil and Michalek [21] with the help of continuous t-norm and defined Hausdorff topology of metric spaces which is later proved to be metrizable, they proved that every metric induces a fuzzy metric. In [17], Grabiec proved fuzzy versions of celebrated Banach fixed point theorem and Edelstein fixed point theorem. Many authors proved fixed point theorems in fuzzy metric spaces including [5, 9, 23, 25, 29, 36, 37, 38, 39, 46, 48].

The concept of fuzzy metric space given by Kramosil and Michalek [21] generalized by Gregori and Romaguera [18] and introduced the notion of fuzzy quasi-metric space. Romaguera, Sapena and Tirado [31] proved the Banach fixed point theorem in fuzzy quasi-metric spaces and applied the result to the domain of words.

Park [27] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space by generalizing the notion of fuzzy metric space given by George and Veeramani [16] with the help of continuous t-norm and continuous t-conorm, while Alaca et al. [3] defined the notion of intuitionistic fuzzy metric space as a generalization of fuzzy metric space given by Kramosil and Michalek [21]. Alaca, Turkoglu and Yildiz [3] proved intuitionistic fuzzy versions of the celebrated Banach fixed point theorem and Edelstein fixed point theorem by using the notion of intuitionistic fuzzy metric space. In [40], Sintunavarat and Kumam introduced a new intuitionistic fuzzy contraction mapping which is more general than intuitionistic fuzzy contraction mapping given by a Rafi and Noorani [28] and establish the new fixed point and common fixed point theorems in intuitionistic fuzzy metric spaces. Many authors proved fixed point theorems in intuitionistic fuzzy metric spaces including [1, 2, 13, 26, 30, 35, 45].

The concept of intuitionistic fuzzy quasi-metric space was introduced by Tirado [44] by generalizing the notion of intuitionistic fuzzy metric space given by Alaca, Turkoglu and Yildiz [3] to the quasi-metric setting and gave intuitionistic fuzzy quasi-metric version of the Banach contraction principle.

On the other hand, Saadati et al. [32] modified the notion of intuitionistic fuzzy metric space and defined the notion of modified intuitionistic fuzzy metric spaces with the help of continuous t-representable. Many authors proved coincidence and common fixed point theorems in modified intuitionistic fuzzy metric spaces including [6, 7, 8, 19, 20, 24, 33, 41, 43].

In this paper, we introduce the concept of modified intuitionistic fuzzy quasi-metric space by generalizing the notion of modified intuitionistic fuzzy metric space given by Saadati, Sedghi and Shobe [32] and prove Banach fixed point theorem in modified intuitionistic fuzzy quasi-metric space. Our results are the genuine generalization of the results of Deshpande, Sharma and Handa [14]. The existence of a solution for a recurrence equation which appears in the average case analysis of Quicksort algorithms is obtained as an application.

2 Preliminaries

Lemma 2. 1. (**Deschrijver and Kerre** [**11**]). Consider the set L^* and operation $\leq_{L^*}$ defined by

$$L^* = \left\{(x_1,\ x_2) : (x_1,\ x_2) \in [0,\ 1]^2 \text{ and } x_1 + x_2 \leq 1\right\},$$

$(x_1,\, x_2) \leq_{L^*} (y_1,\, y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$ for every $(x_1,\, x_2),\, (y_1,\, y_2) \in L^*$. Then $(L^*,\, \leq_{L^*})$ is a complete lattice.

Definition 2. 1. (**Atanassov** [**4**]). An intuitionistic fuzzy set $\mathcal{A}_{\zeta,\,\eta}$ in a universe U is an object $\mathcal{A}_{\zeta,\,\eta} = \{\zeta_{\mathcal{A}}(u),\, \eta_{\mathcal{A}}(u)\}$, where, for all $u \in U$, $\zeta_{\mathcal{A}}(u) \in [0,\, 1]$ and $\eta_{\mathcal{A}}(u) \in [0,\, 1]$ are called the membership degree and non-membership degree respectively of u in $\mathcal{A}_{\zeta,\,\eta}$ and further they satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

For every $z_i = (x_i,\, y_i) \in L^*$ if $c_i \in [0,\, 1]$ such that $\sum_{j=1}^{n} c_j = 1$ then it is easy to see that

$$c_1(x_1,\ y_1) + ... + c_n(x_n,\ y_n) = \sum_{j=1}^{n} c_j(x_j,\ y_j) = \left(\sum_{j=1}^{n} c_j x_j,\ \ \sum_{j=1}^{n} c_j y_j \right) \in L^*. \tag{2.1}$$

We denote its units by $0_{L^*} = (0,\, 1)$ and $1_{L^*} = (1,\, 0)$. Classically, a triangular norm $* = T$ on $[0,\, 1]$ is defined as an increasing, commutative, associative mapping $T : [0,\, 1]^2 \to [0,\, 1]$ satisfying $T(1,\, x) = 1 * x = x$, for all $x \in [0,\, 1]$. A triangular conorm $S = \Diamond$ is defined as an increasing, commutative, associative mapping $S : [0,\, 1]^2 \to [0,\, 1]$ satisfying $S(0,\, x) = 0 \Diamond x = x$, for all $x \in [0,\, 1]$. Using the lattice $(L^*,\, \leq_{L^*})$ these definitions can be straightforwardly extended.

Definition 2. 2. (**Deschrijver**, **Cornelis and Kerre** [**12**]). A triangular norm (t-norm) on L^* is a mapping $\mathcal{T} : (L^*)^2 \to L^*$ satisfying the following conditions:

$(\forall x \in L^*)$ $(\mathcal{T}(x,\, 1_{L^*}) = x)$ (boundary condition),
$(\forall (x,\, y) \in (L^*)^2)$ $(\mathcal{T}(x,\, y) = \mathcal{T}(y,\, x))$ (commutativity),
$(\forall (x,\, y,\, z) \in (L^*)^3)$ $(\mathcal{T}(x,\, \mathcal{T}(y,\, z)) = \mathcal{T}(\mathcal{T}(x,\, y),\, z))$ (associativity),
$(\forall (x,\, x',\, y,\, y') \in (L^*)^4)$ $(x \leq_{L^*} x'$ and $y \leq_{L^*} y' \Rightarrow \mathcal{T}(x,\, y) \leq_{L^*} \mathcal{T}(x',\, y'))$ (monotonicity).

Definition 2. 3. (**Deschrijver and Kerre** [**11**], **Deschrijver**, **Cornelis and Kerre** [**12**]). A continuous t-norm $\mathcal{T}$ on L^* is called continuous t-representable if and only if there exist a continuous t-norm $*$ and a continuous t-conorm $\Diamond$ on $[0,\, 1]$ such that, for all $x = (x_1,\, x_2)$, $y = (y_1,\, y_2) \in L^*$,

$$\mathcal{T}(x,\ y) = (x_1 * y_1,\ x_2 \Diamond y_2).$$

Now define a sequence $\mathcal{T}^n$ recursively by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x^{(1)},\ ...,\ x^{(n+1)}) = \mathcal{T}(\mathcal{T}^{n-1}(x^{(1)},\ ...,\ x^{(n)}),\ x^{(n+1)}),$$

for $n \geq 2$ and $x^{(i)} \in L^*$.

Definition 2. 4. (**Deschrijver and Kerre** [**11**], **Deschrijver**, **Cornelis and Kerre** [**12**]). A negator on L^* is any decreasing mapping $\mathcal{N} : L^* \to L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L^*$, then $\mathcal{N}$ is called an involutive negator. A negator on [0, 1] is a decreasing mapping $N : [0,\ 1] \to [0,\ 1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the standard negator on [0, 1] defined as $N_s(x) = 1 - x$ for all $x \in [0,\ 1]$.

Definition 2. 5. Let M, N are fuzzy sets from $X^2 \times (0, +\infty)$ to [0, 1] such that $M(x,\ y,\ t) + N(x,\ y,\ t) \leq 1$ for all $x,\ y \in X$ and $t > 0$. The 3-tuple (X, $\mathcal{M}_{M,\ N}$, $\mathcal{T}$) is said to be a modified intuitionistic fuzzy quasi-metric space if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t-representable and $\mathcal{M}_{M,\ N}$ is a mapping $X^2 \times (0,\ +\infty) \to L^*$ satisfying the following conditions for every $x,\ y,\ z \in X$ and $t,\ s > 0$:

(*a*) $\mathcal{M}_{M,\ N}(x,\ y,\ t) >_{L^*} 0_{L^*}$,
(*b*) $\mathcal{M}_{M,\ N}(x,\ y,\ t) = \mathcal{M}_{M,\ N}(y,\ x,\ t) = 1_{L^*}$ if and only if $x = y$,
(*c*) $\mathcal{M}_{M,\ N}(x,\ y,\ t+s) \geq_{L^*} \mathcal{T}(\mathcal{M}_{M,\ N}(x,\ z,\ t),\ \mathcal{M}_{M,\ N}(z,\ y,\ s))$,
(*d*) $\mathcal{M}_{M,\ N}(x,\ y,\ \cdot) : (0,\ \infty) \to L^*$ is continuous.

In this case $\mathcal{M}_{M,\ N}$ is called a modified intuitionistic fuzzy quasi-metric (a modified ifqm). Here,

$$\mathcal{M}_{M,\ N}(x,\ y,\ t) = (M(x,\ y,\ t),\ N(x,\ y,\ t)).$$

If in addition $\mathcal{M}_{M,\ N}$ satisfy $\mathcal{M}_{M,\ N}(x,\ y,\ t) = \mathcal{M}_{M,\ N}(y,\ x,\ t)$ for all $x,\ y \in X$ and $t > 0$, then $\mathcal{M}_{M,\ N}$ is called modified intuitionistic fuzzy metric on X and (X, $\mathcal{M}_{M,\ N}$, $\mathcal{T}$) is called a modified intuitionistic fuzzy metric space.

Definition 2. 6. A modified intuitionistic fuzzy quasi-metric space (X, $\mathcal{M}_{M,\ N}$, $\mathcal{T}$) is called a non-Archimedean modified intuitionistic fuzzy quasi-metric space if ($\mathcal{M}_{M,\ N}$, $\mathcal{T}$) is a non-Archimedean modified intuitionistic fuzzy quasi-metric on X, that is, $\mathcal{M}_{M,\ N}(x,\ y,\ t) \geq \min\{\mathcal{M}_{M,\ N}(x,\ z,\ t),\ \mathcal{M}_{M,\ N}(z,\ y,\ t)\}$, for all $x,\ y,\ z \in X$ and $t > 0$.

Definition 2. 7. Let (X, $\mathcal{M}_{M,\ N}$, $\mathcal{T}$) be a modified intuitionistic fuzzy quasi-metric space. For $t > 0$, define the open ball $B(x,\ r,\ t)$ with center $x \in X$ and radius $0 < r < 1$, as

$$B(x,\ r,\ t) = \{y \in X : \mathcal{M}_{M,\ N}(x,\ y,\ t) >_{L^*} (N_s(r),\ r)\}$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x,\ r,\ t) \subset A$. Let $\tau_{\mathcal{M}_{M,\ N}}$ denote the family of all open subsets of X. $\tau_{\mathcal{M}_{M,\ N}}$ is called the topology induced by modified intuitionistic fuzzy quasi-metric.

Remark 2. 1. If ($\mathcal{M}_{M,\ N}$, $\mathcal{T}$) is a modified ifqm on X, then ($\mathcal{M}^{-1}_{M,\ N}$, $\mathcal{T}$) is also a modified ifqm on X, where $\mathcal{M}^{-1}_{M,\ N}$ is the fuzzy sets in $X \times X \times [0,\ +\infty)$ defined by $\mathcal{M}^{-1}_{M,\ N}(x,\ y,\ t) = \mathcal{M}_{M,\ N}(y,\ x,\ t)$. Moreover, if we denote $\mathcal{M}^{i}_{M,\ N}$ the fuzzy sets on $X^2 \times [0,\ +\infty)$ given by $\mathcal{M}^{i}_{M,\ N}(x,\ y,\ t) = \min\{\mathcal{M}_{M,\ N}(x,\ y,\ t),\ \mathcal{M}^{-1}_{M,\ N}(x,\ y,\ t)\}$. Then ($\mathcal{M}^{i}_{M,\ N}$, $\mathcal{T}$) is a modified intuitionistic fuzzy metric on X.

Example 2. 1. Let (X, d) be a quasi-metric space. Denote $\mathcal{T}(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$. Let M and N be fuzzy sets on $X^2 \times (0, +\infty)$ defined as follows:

$$\begin{aligned}\mathcal{M}_{M,\,N}(x,\ y,\ t) &= (M(x,\ y,\ t),\ N(x,\ y,\ t)) \\ &= \left(\frac{ht^n}{ht^n + md(x,\ y)},\ \frac{md(x,\ y)}{ht^n + md(x,\ y)}\right),\end{aligned}$$

for all $t,\ h,\ m,\ n \in R^+$. Then $(X, \mathcal{M}_{M,\,N}, \mathcal{T})$ is a modified intuitionistic fuzzy quasi-metric space.

Definition 2. 8. (**Saadati**, **Sedghi and Shobe** [**32**]). A sequence $(x_n)_n$ in a modified intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,\,N}, \mathcal{T})$ is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in N$ such that

$$\mathcal{M}_{M,\,N}(x_n,\ x_m,\ t) >_{L^*} (N_s(\varepsilon),\ \varepsilon),$$

and for each $n,\ m \geq n_0$, here N_s is the standard negator. The sequence $(x_n)_n$ is said to be convergent to $x \in X$ in the modified intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,\,N}, \mathcal{T})$ and denoted by $x_n \xrightarrow{\mathcal{M}_{M,\,N}} x$, if $\mathcal{M}_{M,\,N}(x_n, x, t) \to 1_{L^*}$ whenever $n \to \infty$, for every $t > 0$. A modified intuitionistic fuzzy metric space is said to be complete if and only if every Cauchy sequence is convergent.

Lemma 2. 2. (**Saadati and Park** [**33**]). Let $\mathcal{M}_{M,\,N}$ be a modified intuitionistic fuzzy metric. Then for any $t > 0$, $\mathcal{M}_{M,\,N}(x, y, t)$ is non-decreasing with respect to t in $(L^*, \leq_{L^*})$, for all x, y in X.

3 The Banach fixed point theorem in a modified intuitionistic fuzzy quasi-metric space

Definition 3. 1. A sequence $(x_n)_n$ in a modified intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,\,N}, \mathcal{T})$ is said to be G-Cauchy if $\lim_{n\to\infty} \mathcal{M}_{M,\,N}(x_n, x_{n+p}, t) = 1_{L^*}$ for each $t > 0$ and $p \in \mathbb{N}$. We say that $(X, \mathcal{M}_{M,\,N}, \mathcal{T})$ is G-complete if every G-Cauchy sequence is convergent.

Definition 3. 2. A B-contraction on a modified intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,\,N}, \mathcal{T})$ is a self-mapping f on X such that there is a constant $k \in (0, 1)$ satisfying

$$\mathcal{M}_{M,\,N}(f(x),\ f(y),\ kt) \geq_{L^*} \mathcal{M}_{M,\,N}(x,\ y,\ t),\ \text{for all } x,\ y \in X \text{ and } t > 0.$$

Generalizing in a natural way the notions of B-contraction, completeness and G-completeness to modified intuitionistic fuzzy quasi-metric spaces are defined as follows:

Definition 3. 3. A sequence $(x_n)_n$ in a modified intuitionistic fuzzy quasi-metric space $(X, \mathcal{M}_{M,\,N}, \mathcal{T})$ is said to be Cauchy if it is a Cauchy sequence in the modified intuitionistic fuzzy metric space $(X, \mathcal{M}^i_{M,\,N}, \mathcal{T})$. A modified intuitionistic fuzzy quasi-metric space $(X, \mathcal{M}_{M,\,N}, \mathcal{T})$ is called bicomplete if the modified intuitionistic fuzzy metric space $(X, \mathcal{M}^i_{M,\,N}, \mathcal{T})$ is complete.

Definition 3. 4. A sequence $(x_n)_n$ in a modified intuitionistic fuzzy quasi-metric space $(X, \mathcal{M}_{M,\,N}, \mathcal{T})$ is said to be G-Cauchy if it is a G-Cauchy sequence in the modified intuitionistic fuzzy metric space $(X, \mathcal{M}^i_{M,\,N}, \mathcal{T})$. A modified intuitionistic fuzzy quasi-metric space $(X, \mathcal{M}_{M,\,N}, \mathcal{T})$ is called G-bicomplete if the modified intuitionistic fuzzy metric space $(X, \mathcal{M}^i_{M,\,N}, \mathcal{T})$ is G-complete.

Definition 3. 5. A B-contraction on a modified intuitionistic fuzzy quasi-metric space $(X, \mathcal{M}_{M,\,N}, \mathcal{T})$ is a self-mapping f on X such that there is a constant $k \in (0, 1)$ satisfying

$$\mathcal{M}_{M,\,N}(f(x),\ f(y),\ kt) \geq_{L^*} \mathcal{M}_{M,\,N}(x,\ y,\ t),\ \text{for all}\ x,\ y \in X\ \text{and}\ t > 0.$$

The number k is called a contraction constant of f.

Theorem 3. 1. Let $(X, \mathcal{M}_{M,\,N}, \mathcal{T})$ be a G-bicomplete modified intuitionistic fuzzy quasi-metric space such that $\lim_{t\to\infty} \mathcal{M}_{M,\,N}(x, y, t) = 1_{L^*}$ for all x, $y \in X$. Then every B-contraction on X has a unique fixed point.

Proof. Let $f : X \to X$ be a B-contraction on X with contraction constant $k \in (0, 1)$. Then

$$\mathcal{M}_{M,\,N}(f(x),\ f(y),\ kt) \geq_{L^*} \mathcal{M}_{M,\,N}(x,\ y,\ t),\ \forall x,\ y \in X\ \text{and}\ t > 0. \tag{3.1}$$

It immediately follows that

$$\mathcal{M}^i_{M,\,N}(f(x),\ f(y),\ kt) \geq_{L^*} \mathcal{M}^i_{M,\,N}(x,\ y,\ t),\ \forall x,\ y \in X\ \text{and}\ t > 0. \tag{3.2}$$

Hence f is a B-contraction on the G-complete modified intuitionistic fuzzy metric space $(X, \mathcal{M}^i_{M,\,N}, \mathcal{T})$. Let $x_0 \in X$ and $x_n = f^n x_0$ $(n \in \mathbb{N})$. Now, we get

$$\mathcal{M}^i_{M,\,N}(x_n,\ x_{n+1},\ t) \geq_{L^*} \mathcal{M}^i_{M,\,N}(x_0,\ x_1,\ \frac{t}{k^n}), \tag{3.3}$$

for all $n \in \mathbb{N}$ and $t > 0$. Thus for any positive integer p, we have by (3.3)

$$\begin{aligned}
\mathcal{M}^i_{M,\,N}(x_n,\ x_{n+p},\ t) &\geq {}_{L^*}\mathcal{T}^{p-1}\left(\begin{array}{c} \mathcal{M}^i_{M,\,N}\left(x_n,\ x_{n+1},\ \frac{t}{p}\right), \\ ...,\ \mathcal{M}^i_{M,\,N}\left(x_{n+p-1},\ x_{n+p},\ \frac{t}{p}\right) \end{array}\right) \\
&\geq {}_{L^*}\mathcal{T}^{p-1}\left(\begin{array}{c} \mathcal{M}^i_{M,\,N}\left(x_0,\ x_1,\ \frac{t}{pk^n}\right), \\ ...,\ \mathcal{M}^i_{M,\,N}\left(x_0,\ x_1,\ \frac{t}{pk^{n+p-1}}\right) \end{array}\right).
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} \mathcal{M}^i_{M,\,N}(x_n,\ x_{n+p},\ t) \geq_{L^*} \mathcal{T}^{p-1}\left(1_{L^*},\ ...,\ 1_{L^*}\right) = 1_{L^*}.$$

Thus $(x_n)_n$ is a G-Cauchy sequence in the G-complete modified intuitionistic fuzzy metric space $(X,\ \mathcal{M}^i_{M,\ N},\ \mathcal{T})$. Thus there exists a point $y \in X$ such that $x_n \to y$ as $n \to \infty$. Thus we have

$$\begin{aligned}
\mathcal{M}^i_{M,\ N}(fy,\ y,\ t) &\geq_{L^*} \mathcal{T}\left(\mathcal{M}^i_{M,\ N}\left(fy,\ fx_n,\ \frac{t}{2}\right),\ \mathcal{M}^i_{M,\ N}\left(fx_n,\ y,\ \frac{t}{2}\right)\right)\\
&\geq_{L^*} \mathcal{T}\left(\mathcal{M}^i_{M,\ N}\left(y,\ x_n,\ \frac{t}{2k}\right),\ \mathcal{M}^i_{M,\ N}\left(x_{n+1},\ y,\ \frac{t}{2}\right)\right).
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get

$$\mathcal{M}^i_{M,\ N}(fy,\ y,\ t) \geq_{L^*} \mathcal{T}\left(1_{L^*},\ 1_{L^*}\right) = 1_{L^*}.$$

Thus we get $fy = y$, that is, y is a fixed point of f. To show uniqueness, assume $fz = z$ for some $z \in X$. Then

$$\begin{aligned}
\mathcal{M}^i_{M,\ N}(z,\ y,\ t) &= \mathcal{M}^i_{M,\ N}(fz,\ fy,\ t)\\
&\geq_{L^*} \mathcal{M}^i_{M,\ N}\left(z,\ y,\ \frac{t}{k}\right)\\
&\geq_{L^*} \mathcal{M}^i_{M,\ N}\left(fz,\ fy,\ \frac{t}{k}\right)\\
&\geq_{L^*} \mathcal{M}^i_{M,\ N}\left(z,\ y,\ \frac{t}{k^2}\right)\\
&\geq_{L^*} \mathcal{M}^i_{M,\ N}\left(fz,\ fy,\ \frac{t}{k^2}\right)\\
&\cdots\\
&\geq_{L^*} \mathcal{M}^i_{M,\ N}\left(z,\ y,\ \frac{t}{k^n}\right) \to 1_{L^*} \text{ as } n \to \infty.
\end{aligned}$$

Thus $y = z$, that is, f has a unique fixed point.

4 G-bicompleteness in non-Archimedean modified intuitionistic fuzzy quasi-metric space

Lemma 4. 1. Each G-Cauchy sequence in a non-Archimedean modified intuitionistic fuzzy quasi-metric space is a Cauchy sequence.

Proof. Let $(x_n)_n$ be a G-Cauchy sequence in the non-Archimedean modified intuitionistic fuzzy quasi-metric space $(X,\ \mathcal{M}_{M,\ N},\ \mathcal{T})$, then it is a G-Cauchy sequence in the non-Archimedean modified intuitionistic fuzzy metric space $(X,\ \mathcal{M}^i_{M,\ N},\ \mathcal{T})$. Thus, for each $t > 0$, we have

$$\lim_{n\to\infty} \mathcal{M}^i_{M,\ N}(x_n,\ x_{n+1},\ t) = 1_{L^*},$$

which implies that, for each $\varepsilon \in (0,\ 1)$, there is $n_0 \in N$ such that

$$\mathcal{M}^i_{M,\ N}(x_n,\ x_{n+1},\ t) >_{L^*} (N_s(\varepsilon),\ \varepsilon) \text{ for each } n \geq n_0.$$

Now, let $m > n \geq n_0$. Then $m = n + j$, for some $j \in N$. So

$$\begin{aligned} & \mathcal{M}^i_{M,\ N}(x_n,\ x_m,\ t) \\ \geq & \ _{L^*}\min\left\{ \begin{array}{c} \mathcal{M}^i_{M,\ N}(x_n,\ x_{n+1},\ t),\ \mathcal{M}^i_{M,\ N}(x_{n+1},\ x_{n+2},\ t), \\ ...,\ \mathcal{M}^i_{M,\ N}(x_{n+j-1},\ x_{n+j},\ t) \end{array} \right\} \\ > & \ _{L^*}(N_s(\varepsilon),\ \varepsilon). \end{aligned}$$

We conclude that $(x_n)_n$ is a Cauchy sequence in $(X,\ \mathcal{M}_{M,\ N},\ \mathcal{T})$.

Theorem 4. 1. Each bicomplete non-Archimedean modified intuitionistic fuzzy quasi-metric space is G-bicomplete.

Proof. Let $(x_n)_n$ be a G-Cauchy sequence in the bicomplete non-Archimedean modified intuitionistic fuzzy quasi-metric space $(X,\ \mathcal{M}_{M,\ N},\ \mathcal{T})$. By Lemma 4.1, $(x_n)_n$ is a Cauchy sequence in $(X,\ \mathcal{M}_{M,\ N},\ \mathcal{T})$. Hence there is $x \in X$ such that $\lim_{n\to\infty} \mathcal{M}^i_{M,\ N}(x,\ x_n,\ t) = 1_{L^*}$, for all $t > 0$. We conclude that $(X,\ \mathcal{M}^i_{M,\ N},\ \mathcal{T})$ is G-complete, that is, $(X,\ \mathcal{M}_{M,\ N},\ \mathcal{T})$ is G-bicomplete.

Corollary 4. 2. Each complete non-Archimedean modified intuitionistic fuzzy metric space is G-complete.

5 Application to the Domain of words

Let Σ be a non-empty alphabet. Let Σ^∞ be the set of all finite and infinite sequences ("words") over Σ, where we adopt the convention that the empty sequence ϕ is an element of Σ^∞. The symbol $\sqsubseteq$ denote the prefix order on Σ^∞, that is, $x \sqsubseteq y \iff x$ is a prefix of y. Now, for each $x \in \Sigma^\infty$, $l(x)$ denote the length of x. Then $l(x) \in [1,\ \infty)$ whenever $x \neq \phi$ and $l(\phi) = 0$. For each x, $y \in \Sigma^\infty$, let $x \sqcap y$ be the common prefix of x and y. Thus the function $d_\sqsubseteq$ defined on $\Sigma^\infty \times \Sigma^\infty$ by

$$d_\sqsubseteq(x,\ y) = \begin{cases} 0, \text{ if } x \sqsubseteq y, \\ 2^{-l\ (x \sqcap y)}, \text{ otherwise}, \end{cases}$$

is a quasi-metric on Σ^∞. (We adopt the convention that $2^{-\infty} = 0$). Actually $d_\sqsubseteq$ is a non-Archimedean quasi-metric on Σ^∞ and the non-Archimedean quasi-metric $(d_\sqsubseteq)^s$ is the Baire metric on Σ^∞, that is,

$$(d_\sqsubseteq)^s(x,\ x) = 0 \text{ and } (d_\sqsubseteq)^s(x,\ y) = 2^{-l(x \sqcap y)},$$

for all x, $y \in \Sigma^\infty$ such that $x \neq y$. It is well known that $(d_\sqsubseteq)^s$ is complete. From this fact it is clear that $d_\sqsubseteq$ is bicomplete. The quasi-metric $d_\sqsubseteq$, which was introduced by Smyth [42], will be called the Baire quasi-metric. Observe that condition $d_\sqsubseteq(x,\ y) = 0$ can be used to distinguish between the case that x is a prefix of y and the remaining cases.

Example 5. 1. Let $d_\sqsubseteq$ be a (non-Archimedean) quasi-metric on a set X and let $\mathcal{M}_{(M,\ N)_{d_\sqsubseteq}}$ in $X \times X \times [0,\ \infty)$ given by

$$\mathcal{M}_{(M,\ N)_{d_{\sqsubseteq}}}(x,\ y,\ t) = \left(\frac{t}{t + d_{\sqsubseteq}(x,\ y)},\ \frac{d_{\sqsubseteq}(x,\ y)}{t + d_{\sqsubseteq}(x,\ y)}\right),\ \forall x,\ y \in X \text{ and } t > 0.$$

Then $(\mathcal{M}_{(M,\ N)_{d_{\sqsubseteq}}}, \mathcal{T})$ is a (non-Archimedean) modified intuitionistic fuzzy quasi-metric on X, where $\mathcal{T}$ denotes the continuous t-representable given by $\mathcal{T} = (\min, \max)$.

Proposition 5. 1. $(\Sigma^{\infty}, \mathcal{M}_{(M,\ N)_{d_{\sqsubseteq}}}, \mathcal{T})$ is a G-bicomplete non-Archimedean intuitionistic fuzzy quasi-metric space.

Consequently, Theorem 3.1 can be applied to this useful space.

Proposition 5. 2. $(\Sigma^{\infty}, \mathcal{M}_{(M,\ N)_{d_{\sqsubseteq L^*}}}, \mathcal{T})$ is a G-bicomplete non-Archimedean modified intuitionistic fuzzy quasi-metric space. The modified intuitionistic fuzzy non-Archimedean quasi-metric $(\mathcal{M}_{(M,\ N)_{d_{\sqsubseteq L^*}}}, \mathcal{T})$ is given by

$\mathcal{M}_{(M,\ N)_{d_{\sqsubseteq L^*}}}(x, y, 0) = 0_{L^*}$ for all $x, y \in \Sigma^{\infty}$,

$\mathcal{M}_{(M,\ N)_{d_{\sqsubseteq L^*}}}(x, y, t) = 1_{L^*}$ if x is a prefix of y and $t > 0$,

$\mathcal{M}_{(M,\ N)_{d_{\sqsubseteq L^*}}}(x, y, t) = \left(1 - 2^{-l(x \sqcap y)},\ 2^{-l(x \sqcap y)}\right)$ if x is not a prefix of y and $t \in (0, 1)$,

$\mathcal{M}_{(M,\ N)_{d_{\sqsubseteq L^*}}}(x, y, t) = 1_{L^*}$ if x is not a prefix of y and $t > 1$.

Now we apply any of the Proposition 5.1 and Theorem 3.1 to the complexity analysis of quicksort algorithm, to show, in direct way, the existence and uniqueness of solution for the following recurrence equation:

$$T(1) = 0 \text{ and } T(n) = \frac{2(n-1)}{n} + \frac{n+1}{n}T(n-1),\ n \geq 2.$$

The average case analysis of Quicksort is discussed in [22] (see also [15]), where the above recurrence equation is obtained. Consider as an alphabet Σ the set of non-negative real numbers, that is, $\Sigma = [0, \infty)$. We associate to T the functional $\Phi : \Sigma^{\infty} \to \Sigma^{\infty}$ given by

$$(\Phi(x))_1 = T(1) \text{ and } (\Phi(x))_n = \frac{2(n-1)}{n} + \frac{n+1}{n}x_{n-1} \text{ for all } n \geq 2.$$

If $x \in \Sigma^{\infty}$ has length $n < \infty$, we write $x = x_1x_2x_3...x_n$, and if x is an infinite word we write $x = x_1x_2x_3...$

Next we show that Φ is a B-contraction on the G-bicomplete non-Archimedean modified intuitionistic fuzzy quasi-metric space $(\Sigma^{\infty}, \mathcal{M}_{(M,\ N)_{d_{\sqsubseteq}}}, \mathcal{T})$ with contraction constant $\frac{1}{2}$.

To this end, we first note that, by construction, we have $l(\Phi(x)) = l(x) + 1$ for all $x \in \Sigma^{\infty}$ (in particular $l(\Phi(x)) = \infty$ whenever $l(x) = \infty$).

Furthermore, it is clear that

$$x \sqsubseteq y \Longleftrightarrow \Phi(x) \sqsubseteq \Phi(y),$$

and consequently

$$\Phi(x \sqcap y) \sqsubseteq \Phi(x) \sqcap \Phi(y) \text{ for all } x,\ y \in \Sigma^{\infty}.$$

Hence

$$l(\Phi(x \sqcap y)) \leq l(\Phi(x) \sqcap \Phi(y)) \text{ for all } x,\ y \in \Sigma^{\infty}.$$

From the preceding observations we deduce that for all x, $y \in X$, if x is a prefix of y, then

$$\mathcal{M}_{(M,\ N)_{d_{\sqsubseteq}}}(\Phi(x),\ \Phi(y),\ \frac{t}{2}) = \mathcal{M}_{(M,\ N)_{d_{\sqsubseteq}}}(x,\ y,\ t) = 1_{L^*}.$$

and if x is not a prefix of y, then

$$\begin{aligned}
& \mathcal{M}_{(M,\ N)_{d_{\sqsubseteq}}}(\Phi(x),\ \Phi(y),\ \frac{t}{2}) \\
= & \left(\frac{\frac{t}{2}}{\frac{t}{2} + 2^{-l(\Phi(x) \sqcap \Phi(y))}}, \frac{2^{-l(\Phi(x) \sqcap \Phi(y))}}{\frac{t}{2} + 2^{-l(\Phi(x) \sqcap \Phi(y))}} \right) \\
\geq & _{L^*} \left(\frac{\frac{t}{2}}{\frac{t}{2} + 2^{-l(\Phi(x \sqcap y))}}, \frac{2^{-l(\Phi(x \sqcap y))}}{\frac{t}{2} + 2^{-l(\Phi(x \sqcap y))}} \right) \\
\geq & _{L^*} \left(\frac{\frac{t}{2}}{\frac{t}{2} + 2^{-(l(x \sqcap y)+1)}}, \frac{2^{-(l(x \sqcap y)+1)}}{\frac{t}{2} + 2^{-(l(x \sqcap y)+1)}} \right) \\
\geq & _{L^*} \left(\frac{t}{t + 2^{-l(x \sqcap y)}}, \frac{2^{-l(x \sqcap y)}}{t + 2^{-l(x \sqcap y)}} \right) \\
\geq & _{L^*} \mathcal{M}_{(M,\ N)_{d_{\sqsubseteq}}}(x,\ y,\ t),
\end{aligned}$$

for all $t > 0$. Therefore Φ is a B-contraction on $(\Sigma^{\infty},\ \mathcal{M}_{(M,\ N)_{d_{\sqsubseteq}}},\ \mathcal{T})$ with contraction constant $\frac{1}{2}$. So, by Theorem 3.1, Φ has a unique fixed point $z = z_1 z_2 z_3 ...$, which is obviously the unique solution to the recurrence equation T, that is, $z_1 = 0$ and $z_n = \frac{2(n-1)}{n} + \frac{n+1}{n} z_{n-1}$ for all $n \geq 2$.

6 Conclusions

We conclude the paper by applying our results to the complexity analysis of Divide and Conquer algorithm. Recall [10, 34] that Divide and Conquer algorithms solve a problem by recursively splitting it into subproblems each of which is solved separately by the same algorithm, after which the results are combined into a solution of the original problem. Thus, the complexity of a Divide and Conquer algorithm typically is the solution to the recurrence equation given by

$$T(1) = c \text{ and } T(n) = aT(\frac{n}{b}) + h(n),$$

where $a,\ b,\ c \in N$ with $a,\ b \geq 2$, n range over the set $\{b^p : p = 0,\ 1,\ 2, ...\}$, and $h(n) \geq 0$ for all $n \in N$. As in the case of Quicksort algorithm, take $\Sigma = [0,$

∞) and put $\Sigma^N = \{ x \in \Sigma^\infty : l(x) = \infty\}$. Clearly Σ^N is a closed subset of $(\Sigma^\infty, \mathcal{M}^i_{(M,\ N)_{d_\sqsubseteq}}, \mathcal{T})$, so $(\Sigma^N, \mathcal{M}_{(M,\ N)_{d_\sqsubseteq}}, \mathcal{T})$ is a non-Archimedean modified intuitionistic G-bicomplete fuzzy quasi-metric space by Proposition 5.1. Now we associate to T the functional $\Phi : \Sigma^N \to \Sigma^N$ given by $(\Phi(x))_1 = T(1)$, and

$$\begin{aligned} (\Phi(x))_n &= ax_{n/b} + h(n), \text{ if } n \in \{b^p : p = 1, 2, ...\} \\ \text{and } (\Phi(x))_n &= 0 \text{ otherwise, for all } x \in \Sigma^N. \end{aligned}$$

For our purposes here it suffices to observe that for each $x, y \in \Sigma^N$, the following inequality holds

$$l(\Phi(x) \sqcap \Phi(y)) \geq 1 + l(x \sqcap y).$$

In fact, If $l(x \sqcap y) = 0$, then $l(\Phi(x) \sqcap \Phi(y)) \geq 1$ and if $b^p > l(x \sqcap y) \geq b^{p-1}$, $p \geq 1$, then $b^{p+1} > l(\Phi(x) \sqcap \Phi(y)) \geq b^p$.

Hence, for each $x, y \in \Sigma^N$ and $t > 0$, we obtain

$$\begin{aligned} & \mathcal{M}_{(M,\ N)_{d_\sqsubseteq}}(\Phi(x),\ \Phi(y),\ \frac{t}{2}) \\ = & \left(\frac{\frac{t}{2}}{\frac{t}{2} + 2^{-l(\Phi(x)\sqcap\Phi(y))}},\ \frac{2^{-l(\Phi(x)\sqcap\Phi(y))}}{\frac{t}{2} + 2^{-l(\Phi(x)\sqcap\Phi(y))}} \right) \\ \geq_{L^*} & \left(\frac{\frac{t}{2}}{\frac{t}{2} + 2^{-l(\Phi(x\sqcap y))}},\ \frac{2^{-l(\Phi(x\sqcap y))}}{\frac{t}{2} + 2^{-l(\Phi(x\sqcap y))}} \right) \\ \geq_{L^*} & \left(\frac{\frac{t}{2}}{\frac{t}{2} + 2^{-(l(x\sqcap y)+1)}},\ \frac{2^{-(l(x\sqcap y)+1)}}{\frac{t}{2} + 2^{-(l\ (x\sqcap y)+1)}} \right) \\ \geq_{L^*} & \left(\frac{t}{t + 2^{-l(x\sqcap y)}},\ \frac{2^{-l(x\sqcap y)}}{t + 2^{-l(x\sqcap y)}} \right) \\ \geq_{L^*} & \mathcal{M}_{(M,\ N)_{d_\sqsubseteq}}(x,\ y,\ t). \end{aligned}$$

Therefore Φ is a B-contraction on $(\Sigma^N, \mathcal{M}_{(M,\ N)_{d_\sqsubseteq}}, \mathcal{T})$ with contraction constant $\frac{1}{2}$. So, by Theorem 3.1, Φ has a unique fixed point $z = z_1 z_2 z_3 ...$

Consequently, the function F defined on $\{b^p : p = 0, 1, 2, ...\}$ by $F(b^p) = z_{b^p}$ for all $p \geq 0$, is the unique solution to the recurrence equation of the given Divide and Conquer algorithm.

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