Annals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (Month 201y), pp. 1–xx
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Doubt intuitionistic fuzzy H-ideals in BCK/BCI-algebras

TRIPTI BEJ, MADHUMANGAL PAL

Received 18 December 2013; Revised 24 March 2014; Accepted 02 April 2014

Abstract. The purpose of this paper is to define the notion of a doubt intuitionistic fuzzy H-ideals in BCK/BCI-algebras, and to study some related properties of it. Eventually we found that an intuitionistic fuzzy subset of BCK/BCI-algebras is an intuitionistic fuzzy H-ideal if and only if the complement of this intuitionistic fuzzy subset is a doubt intuitionistic fuzzy H-ideal.

2010 AMS Classification: 06F35, 03G25, 94D05

Keywords: BCK/BCI-algebra, intuitionistic fuzzy ideal, intuitionistic fuzzy H-ideal, doubt fuzzy H-ideal, doubt intuitionistic fuzzy H-ideal.

Corresponding Author: Tripti Bej (tapubej@gmail.com)

1. Introduction

The study of BCK/BCI-algebra was initiated by Imai and Iseki [7, 8, 9] as a generalisation of the concept of set-theoretic difference and proportional calculi. It is known that the class of BCK-algebra is a proper subclass of the class of BCI-algebra. In [5], Huang gave another definition of fuzzy BCI-algebras and some results about it. After the introduction of fuzzy sets by Zadeh [28], there has been a number of generalisation of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [1, 2], is one among them.

To develope the theory of BCK/BCI-algebras the ideal theory plays an important role. Several researchers investigated properties of fuzzy subalgebra and ideals in BCK/BCI-algebras [6, 10, 11, 12, 13, 14, 16, 17, 18, 19, 27, 30]. In 1999, KhalIid and Ahmad [15] introduced fuzzy H-ideals in BCI-algebras. Also, Senapati et al. have presented several results on BCK/BCI-algebras, BG-algebra and B-algebra [22, 23, 24, 25, 26].

In [3], the authors have studied doubt intuitionistic fuzzy subalgebra and doubt intuitionistic fuzzy ideals in BCK/BCI-algebras.

In 2003, Zhan and Tan [29] introduced doubt fuzzy H-ideals in BCK-algebras and in the recent past in 2010, Satyanarayan et al. [20, 21] introduced intuitionistic fuzzy H-ideals in BCK-algebras respectively and also several interesting properties of these concepts are studied.

Following [29] and [20, 21], we are going to introduce the concept of doubt intuitionistic fuzzy H-ideals in BCK/BCI-algebras. After a detailed study of its properties, we come to this conclusion that in BCK/BCI-algebras, an intuitionistic fuzzy subset is a doubt intuitionistic fuzzy H-ideal if and only if the complement of this intuitionistic fuzzy subset is an intuitionistic fuzzy H-ideal. Relations among doubt intuitionistic fuzzy ideals and doubt intuinistic fuzzy H-ideals are also finally investigated.

2. Preliminaries

In this section, some elementary aspects that are necessary for this paper are included.

An algebra $(X;*,0)$ of type (2, 0) is called a BCI-algebra if it satisfies the following axioms for all $x, y, z \in X$:
(A1) $((x*y)*(x*z))*(z*y)=0$
(A2) $(x*(x*y))*y=0$
(A3) $x*x=0$
(A4) $x*y=0$ and $y*x=0$ imply $x=y$.
If a BCI-algebra X satisfies $0*x=0$. Then X is called a BCK-algebra.

In a BCK/BCI-algebra, $x*0=x$ hold. A partial ordering "$\leq$" on a BCK/BCI-algebra X can be defined by $x \leq y$ if and only if $x*y=0$.
Any BCK-algebra X satisfies the following axioms for all $x, y, z \in X$:
(i) $(x*y)*z=(x*z)*y$
(ii) $x*y \leq x$
(iii) $(x*y)*z \leq (x*z)*(y*z)$
(iv) $x \leq y \Rightarrow x*z \leq y*z, z*y \leq z*x$.

Definition 2.1 ([4])**.** If a BCK-algebra satisfies $(x*y)*z=x*(y*z)$ for all $x, y, z \in X$, then it is called associative.

Throughout this paper, X always means a BCK/BCI-algebra without any specification.

Definition 2.2. A non-empty subset I of a BCK/BCI-algebra X is called an ideal of X if
(i) $0 \in I$
(ii) $x*y \in I$ and $y \in I$ then $x \in I$, for all $x, y \in X$.

Definition 2.3 ([15])**.** A non-empty subset I of a BCK/BCI-algebra X is said to be a H-ideal of X if
(i) $0 \in I$

(ii) $x * (y * z) \in I$ and $y \in I$ then $x * z \in I$, for all $x, y, z \in X$.

Definition 2.4 ([15, 29])**.** A fuzzy set $A = \{\langle x, \mu_A(x)\rangle : x \in X\}$ in X is called a fuzzy H-ideal of X if
(i) $\mu_A(0) \geq \mu_A(x)$
(ii) $\mu_A(x * z) \geq \mu_A(x * (y * z)) \bigwedge \mu_A(y)$, for all $x, y, z \in X$.

The propose work is done on intuitionistic fuzzy set. The formal definition of intuitionistic fuzzy set is given below:

An intuitionistic fuzzy set A in a non-empty set X is an object having the form $A = \{x, \mu_A(x), \lambda_A(x)/x \in X\}$, where the function $\mu_A : X \to [0, 1]$ and $\lambda_A : X \to [0, 1]$, denoted the degree of membership and the degree of non-membership of each element $x \in X$ to the set A respectively and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$, for all $x \in X$.

For the sake of simplicity, we use the symbol form $A = (X, \mu_A, \lambda_A)$ or (μ_A, λ_A) for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \lambda_A(x)\rangle : x \in X\}$.
The two operators used in this paper are defined as:

If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy set then,

$$\Pi A = \{(x, \mu_A(x), \bar{\mu}_A(x))/x \in X\}$$

$$\Diamond A = \{(x, \bar{\lambda}_A(x), \lambda_A(x))/x \in X\}.$$

For the sake of simplicity, we also use $x \bigvee y$ for $max(x, y)$, and $x \bigwedge y$ for $min(x, y)$.

Definition 2.5 ([12])**.** An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in X is called an intuitionistic fuzzy ideal of X, if it satisfies the following axioms:
(i) $\mu_A(0) \geq \mu_A(x), \lambda_A(0) \leq \lambda_A(x)$,
(ii) $\mu_A(x) \geq \mu_A(x * y) \bigwedge \mu_A(y)$,
(iii) $\lambda_A(x) \leq \lambda_A(x * y) \bigvee \lambda_A(y)$, for all $x, y \in X$.

Definition 2.6 ([21])**.** An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in X is called an intuitionistic fuzzy H-ideal of X, if it satisfies the following axioms:
(i) $\mu_A(0) \geq \mu_A(x), \lambda_A(0) \leq \lambda_A(x)$,
(ii) $\mu_A(x * z) \geq \mu_A(x * (y * z)) \bigwedge \mu_A(y)$,
(iii) $\lambda_A(x * z) \leq \lambda_A(x * (y * z)) \bigvee \lambda_A(y)$, for all $x, y, z \in X$..

Jun [14] introduced the definition of doubt fuzzy subalgebra and doubt fuzzy ideals in BCK/BCI-algebras, which are as follows:

Definition 2.7 ([14])**.** A fuzzy set $A = \{\langle x, \mu_A(x)\rangle : x \in X\}$ in X is called a doubt fuzzy subalgebra of X if
$\mu_A(x * y) \leq \mu_A(x) \bigvee \mu_A(y)$, for all $x, y \in X$.

Definition 2.8 ([14])**.** A fuzzy set $A = \{\langle x, \mu_A(x)\rangle : x \in X\}$ in X is called a doubt fuzzy ideal of X if
(i) $\mu_A(0) \leq \mu_A(x)$
(ii) $\mu_A(x) \leq \mu_A(x * y) \bigvee \mu_A(y)$, for all $x, y \in X$.

Definition 2.9 ([3]). An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in a BCK/BCI-algebra X is called a doubt intuitionistic fuzzy ideal if
(i) $\mu_A(0) \leq \mu_A(x); \lambda_A(0) \geq \lambda_A(x)$
(ii) $\mu_A(x) \leq \mu_A(x * y) \bigvee \mu_A(y)$
(iii) $\lambda_A(x) \geq \lambda_A(x * y) \bigwedge \lambda_A(y)$, for all $x, y \in X$.

3. Major Section

In this section, we define doubt intuitionistic fuzzy H-ideals in BCK/BCI-algebras and investigate its properties.

Definition 3.1. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy subset of a BCK/BCI-algebra X, then A is called a **doubt intuitionistic fuzzy H-ideal** of X if
(i) $\mu_A(0) \leq \mu_A(x), \lambda_A(0) \geq \lambda_A(x)$
(ii) $\mu_A(x * z) \leq \mu_A(x * (y * z)) \bigvee \mu_A(y)$
(iii) $\lambda_A(x * z) \geq \lambda_A(x * (y * z)) \bigwedge \lambda_A(y)$, for all $x, y, z \in X$.

Theorem 3.2. *Let an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in X be a doubt intuitionistic fuzzy H-ideal of an associative BCK/BCI-algebra X. Then if the inequility $x * a \leq b$ holds in X, then*
(i) $\mu_A(x * a) \leq \mu_A(b)$
(ii) $\lambda_A(x * a) \geq \lambda_A(b)$.

Proof. Let $x, a, b \in X$ be such that $x * a \leq b$ then $(x * a) * b = 0$ and since A is a doubt intuitionistic fuzzy H-ideal of X, so

$$\begin{aligned} \mu_A(x * a) &\leq max\{\mu_A(x * (b * a)), \mu_A(b)\}, \\ &= max\{\mu_A((x * b) * a), \mu_A(b)\} \quad \text{[Since X is associative]} \\ &= max\{\mu_A((x * a) * b), \mu_A(b)\} \\ &= max\{\mu_A(0), \mu_A(b)\} \\ &= \mu_A(b) \text{ [because } \mu_A(0) \leq \mu_A(b) \text{]} \end{aligned}$$

Therefore, $\mu_A(x * a) \leq \mu_A(b)$. Again,

$$\begin{aligned} \lambda_A(x * a) &\geq min\{\lambda_A(x * (b * a)), \lambda_A(b)\}, \\ &= min\{\lambda_A((x * b) * a), \lambda_A(b)\} \quad \text{[Since X is associative]} \\ &= min\{\lambda_A((x * a) * b), \lambda_A(b)\} \\ &= min\{\lambda_A(0), \lambda_A(b)\} \\ &= \lambda_A(b) \text{ [because } \lambda_A(0) \geq \lambda_A(b) \text{]} \end{aligned}$$

Therefore, $\lambda_A(x * a) \geq \lambda_A(b)$. This completes the proof. □

Proposition 3.3. *Let an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ be a doubt intuitionistic fuzzy H-ideal of a BCK-algebra X. Then $\mu_A(0 * (0 * x)) \leq \mu_A(x)$ and $\lambda_A(0 * (0 * x)) \geq \lambda_A(x)$, for all $x \in X$.*

Proof.

$$\begin{aligned}
\mu_A(0*(0*x)) &\leq \mu_A(0*(x*(0*x))) \bigvee \mu_A(x) \\
&= \mu_A(0*(x*0) \bigvee \mu_A(x) \\
&= \mu_A(0*x) \bigvee \mu_A(x) \\
&= \mu_A(0) \bigvee \mu_A(x) \\
&= \mu_A(x), \text{ for all } x \in X.
\end{aligned}$$

Therefore, $\mu_A(0*(0*x)) \leq \mu_A(x)$, for all $x \in X$. Again,

$$\begin{aligned}
\lambda_A(0*(0*x)) &\geq \lambda_A(0*(x*(0*x))) \bigwedge \lambda_A(x) \\
&= \lambda_A(0*(x*0) \bigwedge \lambda_A(x) \\
&= \lambda_A(0*x) \bigwedge \lambda_A(x) \\
&= \lambda_A(0) \bigwedge \lambda_A(x) \\
&= \lambda_A(x), \text{ for all } x \in X
\end{aligned}$$

Therefore, $\lambda_A(0*(0*x)) \geq \lambda_A(x)$, for all $x \in X$.

□

Lemma 3.4. *If an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ be a doubt intuitionistic fuzzy H-ideal of a BCK/BCI-algebra X. Then we have the followings, $x \leq a$, then $\mu_A(x) \leq \mu_A(a)$ and $\lambda_A(x) \geq \lambda_A(a)$, for all $x, a \in X$.*

Proof. Let $x, a \in X$ such that $x \leq a$ then $x * a = 0$. Now, $\mu_A(x) = \mu_A(x*0) \leq max\{\mu_A(x*(a*0)), \mu_A(a)\} = max\{\mu_A(x*a), \mu_A(a)\} = max\{\mu_A(0), \mu_A(a)\} = \mu_A(a)$. Therefore, $\mu_A(x) \leq \mu_A(a)$.

Again, $\lambda_A(x) = \lambda_A(x*0) \geq min\{\lambda_A(x*(a*0)), \lambda_A(a)\} = min\{\lambda_A(x*a), \lambda_A(a)\} = min\{\lambda_A(0), \lambda_A(a)\} = \lambda_A(a)$. Therefore, $\lambda_A(x) \geq \lambda_A(a)$. □

Example 3.5. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set of X as defined by

X	0	1	2	3	4
μ_A	0.1	0.4	0.7	0.8	0.8
λ_A	0.9	0.6	0.2	0.2	0.2

Then $A = (\mu_A, \lambda_A)$ is a doubt intuitionistic fuzzy H-ideal of X.

Theorem 3.6. *Let $A = (\mu_A, \lambda_A)$ be a doubt intuitionistic fuzzy H-ideal of X. Then so is $\Pi A = \{\langle x, \mu_A(x), \bar{\mu}_A(x)\rangle / x \in X\}$.*

Proof. Since $A = (\mu_A, \lambda_A)$ is a doubt intuitionistic fuzzy H-ideal of X, then $\mu_A(0) \leq \mu_A(x)$ and $\mu_A(x * z) \leq \mu_A(x * (y * z)) \bigvee \mu_A(y)$.

Now, $\mu_A(0) \leq \mu_A(x)$, or $1 - \bar{\mu}_A(0) \leq 1 - \bar{\mu}_A(x)$, or $\bar{\mu}_A(0) \geq \bar{\mu}_A(x)$, for any $x \in X$. Now for any $x, y, z \in X$, $\mu_A(x * z) \leq max\{\mu_A(x * (y * z)), \mu_A(y)\}$. This gives, $1 - \bar{\mu}_A(x*z) \leq max\{1 - \bar{\mu}_A(x*(y*z)), 1 - \bar{\mu}_A(y)\}$ or, $\bar{\mu}_A(x*z) \geq 1 - max\{1 - \bar{\mu}_A(x * (y * z)), 1 - \bar{\mu}_A(y)\}$. Finally, $\bar{\mu}_A(x * z) \geq min\{\bar{\mu}_A(x * (y * z)), \bar{\mu}_A(y)\}$. Hence, $\Pi A = \{(x, \mu_A(x), \bar{\mu}_A(x))/x \in X\}$ is a doubt intuitionistic fuzzy H-ideal of X. □

Theorem 3.7. *Let $A = (\mu_A, \lambda_A)$ be a doubt intuitionistic fuzzy H-ideal of X. Then so is $\Diamond A = \{\langle x, \bar{\lambda}_A(x), \lambda_A(x)\rangle/x \in X\}$.*

Proof. Since $A = (\mu_A, \lambda_A)$ is a doubt intuitionistic fuzzy H-ideal of X, then $\lambda_A(0) \geq \lambda_A(x)$.
Also, $\lambda_A(x * z) \geq \lambda_A(x * (y * z) \bigwedge \lambda_A(y)$.

Again, we have, $\lambda_A(0) \geq \lambda_A(x)$, or $1 - \bar{\lambda}_A(0) \geq 1 - \bar{\lambda}_A(x)$, or $\bar{\lambda}_A(0) \leq \bar{\lambda}_A(x)$, for any $x \in X$. Also for any $x, y, z \in X$, $\lambda_A(x*z) \geq min\{\lambda_A(x*(y*z), \lambda_A(y)\}$ This implies, $1 - \bar{\lambda}_A(x * z) \geq min\{1 - \bar{\lambda}_A(x * (y * z), 1 - \bar{\lambda}_A(y)\}$. That is, $\bar{\lambda}_A(x * z) \leq 1 - min\{1 - \bar{\lambda}_A(x * (y * z), 1 - \bar{\lambda}_A(y)\}$ or, $\bar{\lambda}_A(x * z) \leq max\{\bar{\lambda}_A(x * (y * z), \bar{\lambda}_A(y)\}$. Hence, $\Diamond A = \{\langle x, \bar{\lambda}_A(x), \lambda_A(x)\rangle/x \in X\}$ is a doubt intuitionistic fuzzy H-ideal of X. □

Theorem 3.8. *Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set in X. Then $A = (\mu_A, \lambda_A)$ is a doubt intuitionistic fuzzy H-ideal of X if and only if $\Pi A = \{\langle x, \mu_A(x), \bar{\mu}_A(x)\rangle \ / \ x \in X\}$ and $\Diamond A = \{\langle x, \bar{\lambda}_A(x), \lambda_A(x)\rangle/x \in X\}$ are doubt intuitionistic fuzzy H-ideals of X.*

Proof. The proof is same as Theorem 3.6 and Theorem 3.7. □

Let us illustrate the Theorem 3.6, Theorem 3.7 and Theorem 3.8 using the following example.

Example 3.9. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

Let $A = (\mu_A, \lambda_A)$ be a doubt intuitionistic fuzzy H-ideal of X as defined by

X	0	1	2	3	4
μ_A	0.2	0.4	0.5	0.5	0.6
λ_A	0.8	0.6	0.4	0.5	0.4

Then $\Pi A = \{\langle x, \mu_A(x), \bar{\mu}_A(x)\rangle/x \in X\}$, where $\mu_A(x)$ and $\bar{\mu}_A(x)$ are defined as follows:

X	0	1	2	3	4
μ_A	0.2	0.4	0.5	0.5	0.6
$\bar{\mu}_A$	0.8	0.6	0.5	0.5	0.4

Also $\Diamond A = \{\langle x, \bar{\lambda}_A(x), \lambda_A(x)\rangle / x \in X\}$,whose $\lambda_A(x)$ and $\bar{\lambda}_A(x)$ are defined by

X	0	1	2	3	4
$\bar{\lambda}_A$	0.2	0.4	0.6	0.5	0.6
λ_A	0.8	0.6	0.4	0.5	0.4

So, it can be verified that $\Pi A = \{\langle x, \mu_A(x), \bar{\mu}_A(x)\rangle / x \in X\}$ and $\Diamond A = \{\langle x, \bar{\lambda}_A(x), \lambda_A(x)\rangle / x \in X\}$ are doubt intuitionistic fuzzy H-ideals of X.

Theorem 3.10. *An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ is a doubt intuitionistic fuzzy H-ideal of a BCK/BCI-algebra X if and only if the fuzzy sets μ_A and $\bar{\lambda}_A$ are doubt fuzzy H-ideals of X.*

Proof. Let $A = (\mu_A, \lambda_A)$ be a doubt intuitionistic fuzzy H-ideal of X. Then it is obvious that μ_A is a doubt fuzzy H-ideal of X, and from Theorem 3.8, we can prove that $\bar{\lambda}_A$ is a doubt fuzzy H-ideal of X.

Conversely, let μ_A be a doubt fuzzy H-ideal of X. Therefore $\mu_A(0) \leq \mu_A(x)$ and $\mu_A(x) \leq max\{\mu_A(x * (y * z)), \mu_A(y)\}$, for all $x, y, z \in X$. Again, since $\bar{\lambda}_A$ is a doubt fuzzy H-ideal of X, so, $\bar{\lambda}_A(0) \leq \bar{\lambda}_A(x)$, gives $1-\lambda_A(0) \leq 1-\lambda_A(x)$, implies $\lambda_A(0) \geq \lambda_A(x)$.

Also, $\bar{\lambda}_A(x*z) \leq max\{\bar{\lambda}_A(x*(y*z)), \bar{\lambda}_A(y)\}$ or, $1-\lambda_A(x*z) \leq max\{1-\lambda_A(x*(y*z)), 1-\lambda_A(y)\}$ or, $\lambda_A(x*z) \geq 1 - max\{1-\lambda_A(x*(y*z)), 1-\lambda_A(y)\}$. Finally, $\lambda_A(x*z) \geq min\{\lambda_A(x*(y*z)), \lambda_A(y)\}$, for all $x, y \in X$. Hence, $A = (\mu_A, \lambda_A)$ is a doubt intuitionistic fuzzy H-ideal of X. □

Corollary 3.11. *Let $A = (\mu_A, \lambda_A)$ be a doubt intuitionistic fuzzy H-ideal of a BCK/BCI-algebra X. Then the sets, $D_{\mu_A} = \{x \in X / \mu_A(x) = \mu_A(0)\}$, and $D_{\lambda_A} = \{x \in X / \lambda_A(x) = \lambda_A(0)\}$ are H-ideals of X.*

Proof. Let $A = (\mu_A, \lambda_A)$ be a doubt intuitionistic fuzzy H-ideal of X. Obviously, $0 \in D_{\mu_A}$ and D_{λ_A}. Now, let $x, y, z \in X$, such that $x * (y * z)$, $y \in D_{\mu_A}$. Then $\mu_A(x * (y * z)) = \mu_A(0) = \mu_A(y)$. Now, $\mu_A(x * z) \leq max\{\mu_A(x * (y * z)), \mu_A(y)\} = \mu_A(0)$.

Again, since μ_A is a doubt fuzzy H-ideal of X, $\mu_A(0) \leq \mu_A(x * z)$. Therefore, $\mu_A(0) = \mu_A(x*z)$. It follows that, $x*z \in D_{\mu_A}$, for all $x, y, z \in X$. Therefore, D_{μ_A} is an H-ideal of X. Following the same way we can prove that D_{λ_A} is also an H-ideal of X. □

Definition 3.12. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set of X, and $t, s \in [0, 1]$, then μ level t-cut and λ level s-cut of A, is as followes:

$$\mu^{\leq}_{A,t} = \{x \in X / \mu_A(x) \leq t\}$$
$$\text{and } \lambda^{\geq}_{A,s} = \{x \in X / \lambda_A(x) \geq s\}.$$

Theorem 3.13. *If $A = (\mu_A, \lambda_A)$ be a doubt intuitionistic fuzzy H-ideal of X, then $\mu^{\leq}_{A,t}$ and $\lambda^{\geq}_{A,s}$ are H-ideals of X for any $t, s \in [0, 1]$.*

Proof. Let $A = (\mu_A, \lambda_A)$ be a doubt intuitionistic fuzzy H-ideal of X, and let $t \in [0, 1]$ with $\mu_A(0) \leq t$. Then we have, $\mu_A(0) \leq \mu_A(x)$, for all $x \in X$, but $\mu_A(x) \leq t$, for all $x \in \mu^{\leq}_{A,t}$. So, $0 \in \mu^{\leq}_{A,t}$. Let $x, y, z \in X$ be such that $x * (y * z) \in \mu^{\leq}_{A,t}$ and

$y \in \mu^{\leq}_{A,t}$, then, $\mu_A(x*(y*z)) \in \mu^{\leq}_{A,t}$ and $\mu_A(y) \in \mu^{\leq}_{A,t}$. Therefore, $\mu_A(x*(y*z)) \leq t$ and $\mu_A(y) \leq t$. Since μ_A is a doubt fuzzy H-ideal of X, it follows that, $\mu_A(x*z) \leq \mu_A((x*(y*z)) \bigvee \mu_A(y) \leq t$ and hence $x*z \in \mu^{\leq}_{A,t}$, for all $x, y, z \in X$. Therefore, $\mu^{\leq}_{A,t}$ is an H-ideal of X for $t \in [0,1]$. Similarly, we can prove that $\lambda^{\geq}_{A,s}$ is an H-ideal of X for $s \in [0,1]$. □

Theorem 3.14. *If $\mu^{\leq}_{A,t}$ and $\lambda^{\geq}_{A,s}$ are either empty or H-ideals of X for $t, s \in [0,1]$, then $A = [\mu_A, \lambda_A]$ is a doubt intuitionistic fuzzy H-ideal of X.*

Proof. Let $\mu^{\leq}_{A,t}$ and $\lambda^{\geq}_{A,s}$ be either empty or H-ideals of X for $t, s \in [0,1]$. For any $x \in X$, let $\mu_A(x) = t$ and $\lambda_A(x) = s$. Then $x \in \mu^{\leq}_{A,t} \bigwedge \lambda^{\geq}_{A,s}$, so $\mu^{\leq}_{A,t} \neq \phi \neq \lambda^{\geq}_{A,s}$. Since $\mu^{\leq}_{A,t}$ and $\lambda^{\geq}_{A,s}$ are H-ideals of X, therefore $0 \in \mu^{\leq}_{A,t} \bigwedge \lambda^{\geq}_{A,s}$. Hence, $\mu_A(0) \leq t = \mu_A(x)$ and $\lambda_A(0) \geq s = \lambda_A(x)$, where $x \in X$. If there exist $x', y', z' \in X$ such that $\mu_A(x' * z') > max\{\mu_A(x' * (y' * z')), \mu_A(y')\}$, then by taking, $t_0 = \frac{1}{2}(\mu_A(x' * z') + max\{\mu_A(x' * (y' * z')), \mu_A(y')\})$, We have, $\mu_A(x' * z') > t_0 > max\{\mu_A(x' * (y' * z')), \mu_A(y')\}$. Hence, $x' * z' \notin \mu^{\leq}_{A,t_0}, (x' * (y' * z')) \in \mu^{\leq}_{A,t_0}$ and $y' \in \mu^{\leq}_{A,t_0}$, that is $\mu^{\leq}_{A,t_0}$ is not an H-ideal of X, which is a contradiction. Therefore, $\mu_A(x*z) \leq \mu_A((x*(y*z)) \bigvee \mu_A(y)$, for any $x, y, z \in X$.

Finally, assume that there exist $p, q, r \in X$ such that $\lambda_A(p*r) < min\{\lambda_A(p*(q*r)), \lambda_A(q)\}$. Taking $s_0 = \frac{1}{2}(\lambda_A(p*r) + min\{\lambda_A(p*(q*r)), \lambda_A(q)\})$, then $min\{\lambda_A(p*(q*r)), \lambda_A(q)\} > s_0 > \lambda_A(p*r)$. Therefore, $p*(q*r) \in \lambda^{\geq}_{A,s}$ and $q \in \lambda^{\geq}_{A,s}$ but $p*r \notin \lambda^{\geq}_{A,s}$. Again a contradiction. This completes the proof. □

But, if an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$, is not a doubt intuitionistic fuzzy H-ideal of X, then $\mu^{\leq}_{A,t}$ and $\lambda^{\geq}_{A,s}$ are not H-ideals of X for $t, s \in [0,1]$, which is illustrated in the following example.

Example 3.15. Let $X = \{0, 1, 2, 3\}$ be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	1	0	0
3	3	1	3	0

Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set of X as defined by

X	0	1	2	3
μ_A	0.1	0.5	0.7	0.6
λ_A	0.8	0.4	0.2	0.4

which is not a doubt intuitionistic fuzzy H-ideal of X .

For $t = 0.67$ and $s = 0.25$, we get $\mu^{\leq}_{A,t} = \lambda^{\geq}_{A,s} = \{0, 1, 3\}$, which are not H-ideals of X, as $2*(1*0) = 2*1 = 1 \in \{0, 1, 3\}$, and $1 \in \{0, 1, 3\}$, but $2*0 \notin \{0, 1, 3\}$.

Theorem 3.16. *Union of any two doubt intuitionistic fuzzy H-ideals of X, is also a doubt intuitionistic fuzzy H-ideal of X.*

Proof. Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be two doubt intuitionistic fuzzy H-ideals of X. Again let, $C = A \cup B = (\mu_C, \lambda_C)$, where $\mu_C = \mu_A \vee \mu_B$ and $\lambda_C = \lambda_A \wedge \lambda_B$. Let $x \in X$, then, $\mu_C(0) = (\mu_A \vee \mu_B)(0) = max\{\mu_A(0), \mu_B(0)\} \leq max\{\mu_A(x), \mu_B(x)\} = (\mu_A \vee \mu_B)(x) = \mu_C(x)$ and $\lambda_C(0) = (\lambda_A \wedge \lambda_B)(0) = min\{\lambda_A(0), \lambda_B(0)\} \geq min\{\lambda_A(x), \lambda_B(x)\} = (\lambda_A \wedge \lambda_B)(x) = \lambda_C(x)$ Also,

$$\begin{aligned}
\mu_C(x * z) &= max\{\mu_A(x * z), \mu_B(x * z)\} \\
&\leq max\{max[\mu_A(x * (y * z), \mu_A(y)], max[\mu_B(x * (y * z), \mu_B(y)]\} \\
&= max\{max[\mu_A(x * (y * z), \mu_B(x * (y * z)], max[\mu_A(y), \mu_B(y)]\} \\
&= max[\mu_C(x * (y * z), \mu_C(y)].
\end{aligned}$$

Similarly, we can prove that, $\lambda_C(x * z) \geq min[\lambda_C(x * (y * z), \lambda_C(y)]$.
This completes the proof. □

Theorem 3.17. *Let A and B be two intuitionistic fuzzy subsets of X, such that one is contained another. Also A and B are two doubt intuitionistic fuzzy H-ideals of X. Then the intersection of A and B are also doubt intuitionistic fuzzy H-ideal of X .*

Proof. Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be two doubt intuitionistic fuzzy H-ideals of X. Again let, $D = A \cap B = (\mu_D, \lambda_D)$, where $\mu_D = \mu_A \wedge \mu_B$ and $\lambda_D = \lambda_A \vee \lambda_B$. Let $x, y, z \in X$, then $\mu_D(0) = \mu_A(0) \wedge \mu_B(0) \leq \mu_A(x) \wedge \mu_B(x) = \mu_D(x)$ and $\lambda_D(0) = \lambda_A(0) \vee \lambda_B(0) \geq \lambda_A(x) \vee \lambda_B(x) = \lambda_D(x)$. Also,

$$\begin{aligned}
\mu_D(x * z) &= \mu_A(x * z) \wedge \mu_B(x * z) \\
&\leq max[\mu_A(x * (y * z)), \mu_A(y)] \wedge max[\mu_B(x * (y * z)), \mu_B(y)] \\
&= max\{[\mu_A(x * (y * z)) \wedge \mu_B(x * (y * z))], [\mu_A(y) \wedge \mu_B(y)]\}, \\
&\quad \text{[because one is contained another]} \\
&= max[\mu_D(x * (y * z)), \mu_D(y)].
\end{aligned}$$

Similarly, we can prove that, $\lambda_D(x * z) \geq min[\lambda_D(x * (y * z)), \lambda_D(y)]$.
This completes the proof. □

Theorem 3.17 and Theorem 3.18 are verified by the following example.

Example 3.18. Let $X = \{0, 1, 2, 3\}$ be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set of X as defined by

X	0	1	2	3
μ_A	0	0.3	0.2	0.3
λ_A	1	0.7	0.8	0.7

Then $A = (\mu_A, \lambda_A)$ is a doubt intuitionistic fuzzy H-ideal of X.
Again, let $B = (\mu_B, \lambda_B)$ be an intuitionistic fuzzy set of X as defined by

X	0	1	2	3
μ_B	0.2	0.4	0.5	0.5
λ_B	0.8	0.6	0.5	0.5

Then $B = (\mu_B, \lambda_B)$ is a doubt intuitionistic fuzzy H-ideal of X.
We also assume that $P = A \cup B = (\mu_P, \lambda_P)$ where $\mu_P = \mu_A \vee \mu_B$ and $\lambda_P = \lambda_A \wedge \lambda_B$ and P is defined as:

X	0	1	2	3
μ_P	0.2	0.4	0.5	0.5
λ_P	0.8	0.6	0.5	0.5

Then $P = (\mu_P, \lambda_P)$ is a doubt intuitionistic fuzzy H-ideal of X.

Now let, $Q = A \cap B = (\mu_Q, \lambda_Q)$ where $\mu_Q = \mu_A \wedge \mu_B$ and $\lambda_Q = \lambda_A \vee \lambda_B$.
Then Q is an intuitionistic fuzzy set of X which can be defined as:

X	0	1	2	3
μ_Q	0	0.3	0.2	0.3
λ_Q	1	0.7	0.8	0.7

Then it is clear that $Q = (\mu_Q, \lambda_Q)$ is a doubt intuitionistic fuzzy H-ideal of X.

Theorem 3.19. *Every doubt intuitionistic fuzzy H-ideal of X is a doubt intuitionistic fuzzy ideal of X..*

Proof. Let $A = (\mu_A, \lambda_A)$ be a doubt intuitionistic fuzzy H-ideal of X, then (i) $\mu_A(0) \leq \mu_A(x); \lambda_A(0) \geq \lambda_A(x)$, (ii) $\mu_A(x * z) \leq \mu_A(x * (y * z)) \bigvee \mu_A(y)$, and (iii) $\lambda_A(x * z) \geq \lambda_A(x * (y * z)) \bigwedge \lambda_A(y)$, for all $x, y, z \in X$.
If we put $z = 0$, then from (ii) and (iii), we get $\mu_A(x) \leq \mu_A(x * y) \bigvee \mu_A(y)$ and $\lambda_A(x) \geq \lambda_A(x * y) \bigwedge \lambda_A(y)$, for all $x, y, z \in X$, since $x * 0 = x$, for all $x \in X$.

Hence, A is a doubt intuitionistic fuzzy ideal of X. □

But the converse may not be true. That is every doubt intuitionistic fuzzy ideal of X is not a doubt intuitionistic fuzzy H-ideal of X. It can be verified by the following example:

Example 3.20. Let $X = \{0, 1, 2\}$ be a BCI-algebra with the following Cayley table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set of X as defined by

X	0	1	2
μ_A	0	0.8	0.8
λ_A	1	0.2	0.2

Then $A = (\mu_A, \lambda_A)$ is a doubt intuitionistic fuzzy ideal of X. But A is not a doubt intuitionistic fuzzy H-ideal of X, as $\mu_A(1*2) \nleq max\{\mu_A(1*(0*2)), \mu_A(0)\}$. Because, $\mu_A(1 * 2) = 0.8$ and $max\{\mu_A(1 * (0 * 2)), \mu_A(0)\} = \mu_A(0) = 0$.

We now give a condition for the intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$, which is a doubt intuitionistic fuzzy ideal of X to be a doubt intuitionistic fuzzy H-ideal of X.

Theorem 3.21. *In an associative BCK/BCI-algebra X, every doubt intuitionistic fuzzy ideal is a doubt intuitionistic fuzzy H-ideal of X.*

Proof. Let $A = (\mu_A, \lambda_A)$ be a doubt intuitionistic fuzzy ideal of X. Then, $\mu_A(0) \leq \mu_A(x); \lambda_A(0) \geq \lambda_A(x)$. Now, since X is associative, then for $x, y, z \in X$, $x * (y * z) = (x * y) * z$. Now,

$$\begin{aligned} \mu_A(x * (y * z)) \bigvee \mu_A(y) &= \mu_A((x * y) * z) \bigvee \mu_A(y) \\ &= \mu_A((x * z) * y) \bigvee \mu_A(y) \\ &\geq \mu_A(x * z) \end{aligned}$$

[because A is a doubt intuitionistic fuzzy ideal.]

Therefore, $\mu_A(x*z) \leq \mu_A(x*(y*z)) \bigvee \mu_A(y)$. Similarly we can prove that, $\lambda_A(x*z) \geq \lambda_A(x * (y * z)) \bigwedge \lambda_A(y)$, for all $x, y, z \in X$. Hence, A is a doubt intuitionistic fuzzy H-ideal of X. This completes the proof. □

Let us illustrate the Theorem 3.21 using the following example.

Example 3.22. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra [from Example 3.9], with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

Here X is an associative BCK-algebra. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set of X as defined by

X	0	1	2	3	4
λ_A	0	0.6	0.4	0.8	0.9
λ_A	1	0.4	0.6	0.2	0.1

Hence, A is a doubt intuitionistic fuzzy ideal as well as doubt intuitionistic fuzzy H-ideal of X.

4. Conclusions

In the present paper, we have introduced the concept of doubt intuitionistic fuzzy H-ideals in BCK/BCI-algebras and investigated some of its essential properties. We think this work would enhance the scope for further study in this field of intuitionistic fuzzy sets.

It is our hope that this work is going to impact the upcoming research works in this field of BCK/BCI-algebras with a new horizon of interest and innovation.

Acknowledgements. The authors would like to thank all the anonymous referees for their valuable comments and recommendations on this work leading to the improvement in the presentation of the paper.

References

[1] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87-96.
[2] K. T. Atanassov, New operations defined over the intuitionistic fuzzy sets, Fuzzy Sets and Systems 61 (1994) 137-142.
[3] T. Bej and M. Pal, Doubt intuitionistic fuzzy ideals in BCK/BCI-algebras, Communicated.
[4] Q. P. Hu and K. Iseki, On BCI-algebra satisfying $(x*y)*z = x*(y*z)$, Math. Sem. Notes 8 (1980) 553-555.
[5] F. Y. Huang, Another definition of fuzzy BCI-algebras, Selected Papers on BCK/BCI-Algebras (China), **1** (1992), 91-92.
[6] K. Iseki, On ideals in BCK-algebras, Math. Sem. Notes 3 (1975) 1-12.
[7] Y. Imai and K. Iseki, On axiom systems of propositional calculi, Proc. Japan Academy 42 (1966) 19-22.
[8] K. Iseki, An algebra related with a propositional calculus, Proc. Japan Academy 42 (1966) 26-29.
[9] K. Iseki, On BCI-algebras, Math. Sem. Notes 8 (1980) 125-130.
[10] K. Iseki and S. Tanaka, An introduction to the theory of BCK-algebras, Math. Japonica 23 (1978) 1-26.
[11] Y. B. Jun, Closed fuzzy ideals in BCI-algebras, Math. Japonica 38 (1993) 199-202.
[12] Y. B. Jun and K. H. Kim, Intuitionistic fuzzy ideals of BCK-algebras, Int. J. Math and Math. Sci. 24 (2000) 839-849.
[13] Y. B. Jun, Traslations of fuzzy ideals in BCK/BCI-algebras, Hacettepe J. Math. and Stat. 38(3) (2011) 349-358.
[14] Y. B. Jun, Doubt fuzzy BCK/BCI-algebras, Soochow J. Math. 20(3) (1994) 351-358.
[15] H. M. Khalid and B. Ahmad, Fuzzy H-ideals in BCI-algebras, Fuzzy Sets and Systems 101 (1999) 153-158.
[16] Y.L. Liu and J. Meng, Fuzzy ideals in BCI-algebras, Fuzzy Sets and Systems 123 (2001) 227-237.
[17] J. Meng, On ideals in BCK-algebras, Math. Japonica 40 (1994) 143-154.
[18] J. Meng and Y.B. Jun, BCK-algebras, Kyung Moon Sa Co. (1994).
[19] J. Meng and X. Guo, On fuzzy ideals in BCK-algebras, Fuzzy Sets and Systems 149(3) (2005) 509-525.
[20] B. Satyanarayan, U. B. Madhavi and R. D. Prasad, On intuitionistic fuzzy H-ideals in BCK-algebras, Int. J. Algebra, 4(15) (2010) 743-749.
[21] B. Satyanarayan, U. B. Madhavi and R. D. Prasad, On foldness of intuitionistic fuzzy H-ideals in BCK-algebras, Int. Mathematical Forum 5(45) (2010) 2205-2221.
[22] T. Senapati, M. Bhowmik and M. Pal, Atanassov's intuitionistic fuzzy translations of intuitionistic fuzzy H-ideals in BCK/BCI-algebras, Notes on Intuitionistic Fuzzy Sets, 19(1) (2013) 32-47.
[23] T. Senapati, M. Bhowmik and M. Pal, Interval-valued intuitionistic fuzzy BG-subalgebras, The Journal of Fuzzy Mathematics, 20(3) (2012), 707-720.
[24] T. Senapati, M. Bhowmik and M. Pal, Interval-valued intuitionistic fuzzy closed ideals of BG-algebra and their products, Int. J. Fuzzy Logic Systems, 2(2) (2012), 27-44.
[25] T. Senapati, M. Bhowmik and M. Pal, Intuitionistic fuzzifications of ideals in BG-algebras, Mathematica Aeterna 2(9) (2012), 761-778.
[26] T. Senapati, M. Bhowmik and M. Pal, Fuzzy closed ideals of B-algebras, International Journal of Computer Science Engineering and Technology, 1(10) (2011), 669-673.
[27] T. Srinivas, T. Nagaiah and P. Narasimha Swamy, Anti fuzzy ideals of Γ-near-rings, Annals of Fuzzy Mathematics and Informatics 3(2) (2012) 255- 266.
[28] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338-353.

[29] J. Zhan and Z. Tan, Characterization of doubt fuzzy H-ideals in BCK-algebras, Soochow Journal of Mathematics 29(3) (2003) 293-298.

[30] M. Zhou, D. Xiang and J. Zhan, On anti fuzzy ideals of Γ-rings, Annals of Fuzzy Mathematics and Informatics 1(2) (2011) 197-205.

TRIPTI BEJ (tapubej@gmail.com)
Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore – 721102, India.

MADHUMANGAL PAL (mmpalvu@gmail.com)
Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore – 721102, India.