Lattice and topological structures of soft sets over the power set of a universe

GANGQIANG ZHANG, NINGHUA GAO

Received 29 October 2013; Revised 23 December 2013; Accepted 08 February 2014

ABSTRACT. In this paper, we introduce soft sets over $2^U$ where $2^U$ is the power set of the universe $U$, propose some operations on soft sets over $2^U$ and investigate some types of soft sets over $2^U$ such as keeping intersection and keeping union. We obtain lattice and topological structures of soft sets over $2^U$. We consider soft rough approximations and soft rough sets, and obtain structures of soft rough sets.

2010 AMS Classification: 03E72; 49J53; 54A40.

Keywords: Soft set; Power set; Soft rough approximation; Soft rough set; Lattice; Topology.

Corresponding Author: Gangqiang Zhang (gaoninghua1987513@126.com )

1. Introduction

Most of traditional methods for formal modeling, reasoning, and computing are crisp, deterministic, and precise in character. However, Many practical problems within fields such as economics, engineering, environmental science, medical science and social sciences involve data that contain uncertainties. We cannot use traditional methods because of various types of uncertainties present in these problems.

There are several theories: probability theory, theory of fuzzy sets [23] and theory of rough sets [18], which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. For example, theory of probabilities can deal only with stochastically stable phenomena. To overcome these kinds of difficulties, Molodtsov [16] proposed a completely new approach, which is called soft set theory, for modeling uncertainty.

Recently there has been a rapid growth in soft set theory and its applications. Maji et al. [14, 15] defined several operations on soft sets, made a theoretical study on soft set theory and defined fuzzy soft sets by combining soft sets with fuzzy sets. Aktas et al. [2] introduced the concept of soft groups. Jun [8, 9, 10] applied

The purpose of this paper is to investigate a soft set over \(2^U\) and give their lattice and topological.

2. Preliminaries

Throughout this paper, \(U\) denotes initial universe, \(E\) denotes the set of all possible parameters and \(2^U\) denotes the power set of \(U\). For \(A, B \subseteq 2^U\), denote

\[ A^* = \bigcup_{A \in A} A, \]

\[ A \cap B = \{ M : M \in A \text{ and } M \in B \}, \]

\[ A \cup B = \{ M : M \in A \text{ or } M \in B \}, \]

\[ A \land B = \{ A \cap B : A \in A \text{ and } B \in B \}, \]

\[ A \lor B = \{ A \cup B : A \in A \text{ and } B \in B \}, \]

\[ A - B = \{ M : M \in A \text{ and } M \notin B \}, \quad A^c = 2^U - A. \]

In this paper, we only consider the case where \(U\) and \(E\) are both nonempty finite sets.

2.1. Rough sets. Rough set theory was initiated by Pawlak [18] for dealing with vagueness and granularity in information systems.

Let \(R\) be an equivalence relation on \(U\). The pair \((U, R)\) is called a approximation space. The equivalence relation \(R\) is often called an indiscernibility relation. Using the indiscernibility relation \(R\), one can define the following two rough approximations:

\[ \underline{R}(X) = \{ x \in U : [x]_R \subseteq X \}, \]

\[ \overline{R}(X) = \{ x \in U : [x]_R \cap X \neq \emptyset \}. \]

\(\underline{R}(X)\) and \(\overline{R}(X)\) called the lower approximation and the upper approximation of \(X\), respectively. In general, we refer to \(\underline{R}(X)\) and \(\overline{R}(X)\) as rough approximations of \(X\).

The boundary region of \(X\), defined by the difference between these rough approximations, that is \(Bnd_R(X) = \overline{R}(X) - \underline{R}(X)\). It can easily be seen that \(\underline{R}(X) \subseteq X \subseteq \overline{R}(X)\).

A set is rough if its boundary region is not empty; otherwise, the set is crisp. Thus, \(X\) is rough if \(\underline{R}(X) \neq R(X)\).
In Example 2.2, it is easy to see from Table 1 that parameter $e_i$ ($i = 1, 2, 3, 4$) induces an equivalence relation on $U$, and we denote it by $\sigma_i$ ($i = 1, 2, 3, 4$). Thus, we get the equivalence classes as follows:

- For $\sigma_1$, the equivalence classes are $\{h_1, h_3\}$, $\{h_2, h_4, h_5, h_6\}$.
- For $\sigma_2$, the equivalence classes are $\{h_1, h_3, h_6\}$, $\{h_2, h_4, h_5\}$.
- For $\sigma_3$, the equivalence classes are $\{h_1, h_3, h_4, h_5\}$, $\{h_2, h_6\}$.
- For $\sigma_4$, the equivalence classes are $\{h_1, h_2, h_3\}$, $\{h_4, h_5, h_6\}$.

Put

$\sigma(e_1) = \{\{h_1, h_3\}, \{h_2, h_4, h_5, h_6\}\}$, $\sigma(e_2) = \{\{h_1, h_3, h_6\}, \{h_2, h_4, h_5\}\}$,
$\sigma(e_3) = \{\{h_1, h_3, h_4, h_5\}, \{h_2, h_6\}\}$, $\sigma(e_4) = \{\{h_1, h_2, h_3\}, \{h_4, h_5, h_6\}\}$.

### 3. Soft sets over $2^U$

#### 3.1. The concept of soft sets over $2^U$.

**Definition 3.1.** Let $A \subset E$. A pair $(\sigma, A)$ is called a soft set over $U$, if $\sigma$ is a mapping given by $\sigma: A \to 2^U$. We denote $(\sigma, A)$ by $f_A$.

To illustrate the background of Definition 3.1, let us consider the following example.

**Example 3.2.** Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be a set of houses under consideration, where $A = \{e_1, e_2, e_3, e_4\}$ is a set of parameters for selection of the house. Let $e_1$ stands for expensive houses, $e_2$ stands for wooden houses, $e_3$ stands for houses located in green surroundings, $e_4$ stands for houses located in the urban area, $e_5$ stands for houses located in the urban area, $e_6$ stands for houses located in the green area.

We define $f_A$ as follows:

$\sigma(e_1) = \{h_1, h_3\}$, $\sigma(e_2) = \{h_1, h_3, h_6\}$, $\sigma(e_3) = \{h_1, h_3, h_4, h_5\}$, $\sigma(e_4) = \{h_1, h_2, h_3\}$.

### 2.2. Soft sets over $U$.

**Definition 2.1 ([16]).** Let $A \subseteq E$. A pair $(f, A)$ is called a soft set over $U$, if $f$ is a mapping given by $f: A \to 2^U$. We denote $(f, A)$ by $f_A$.

In other words, a soft set over $U$ is a parameterized family of subsets of the universe $U$. For $e \in A$, $f(e)$ may be considered as the set of $e$-approximate elements of $f_A$.

To illustrate this idea, let us consider the following example.

**Example 2.2.** Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be a set of houses under consideration, where $A = \{e_1, e_2, e_3, e_4\}$ is a set of parameters for selection of the house. Let $e_1$ stands for expensive houses, $e_2$ stands for wooden houses, $e_3$ stands for houses located in green surroundings, $e_4$ stands for houses located in the urban area, $e_5$ stands for houses located in the urban area, $e_6$ stands for houses located in the green area.

We define $f_A$ as follows:

$\sigma(e_1) = \{h_1, h_3\}$, $\sigma(e_2) = \{h_1, h_3, h_6\}$, $\sigma(e_3) = \{h_1, h_3, h_4, h_5\}$, $\sigma(e_4) = \{h_1, h_2, h_3\}$.

$f_A$ can be described as the following Table 1. If $h_i \in f(a_j)$, then $h_{ij} = 1$; otherwise $h_{ij} = 0$, where $h_{ij}$ are the entries in Table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$h_3$</th>
<th>$h_4$</th>
<th>$h_5$</th>
<th>$h_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$e_2$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$e_3$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$e_4$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Then we obtain a soft set $\sigma_A$ over $2^U$. That is, a soft set $f_A$ over $U$ induces a soft set $\sigma_A$ over $2^U$.

**Proposition 3.3.** Every soft set $f_A$ over $U$ may be considered as a soft set $\sigma_A$ over $2^U$.

**Proof.** Let $f_A$ be a soft set over $U$. For $a \in A$, put $\sigma(a) = \{f(a)\}$. Then $\sigma_A$ is a soft set over $2^U$. Thus $f_A$ may be considered as a soft set $\sigma_A$ over $2^U$.

**Definition 3.4.** Let $A, B \subseteq E$ and let $\sigma_A$ and $\delta_B$ be two soft sets over $2^U$.

1. $\sigma_A$ and $\delta_B$ are called soft equal, if $A = B$ and $\sigma(e) = \delta(e)$ for each $e \in A$. We write $\sigma_A = \delta_B$.

2. $\sigma_A$ is called a soft subset of $\delta_B$, if $A \subseteq B$ and $\sigma(e) = \delta(e)$ for each $e \in A$. We write $\sigma_A \subset \delta_B$.

Obviously, $\sigma_A = \delta_B$ if and only if $\sigma_A \subset \delta_B$ and $\delta_B \supset \sigma_A$.

### 3.2. Some types of soft sets over $2^U$.

**Definition 3.5.** Let $\sigma_A$ be a soft set over $2^U$.

1. $\sigma_A$ is called full, if $\bigcup_{a \in A} \sigma(a)^* = U$.

2. $\sigma_A$ is called keeping intersection, if for any $a, b \in A$, $M \in \sigma(a)$ and $N \in \sigma(b)$, there exist $c \in A$ and $Q \in \sigma(c)$ such that $M \cap N = Q$.

3. $\sigma_A$ is called keeping union, if for any $a, b \in A$, $M \in \sigma(a)$ and $N \in \sigma(b)$, there exist $c \in A$ and $Q \in \sigma(c)$ such that $M \cup N = Q$.

4. $\sigma_A$ is called uniform keeping intersection, if for any $a, b \in A$, there exists $c \in A$ such that $\sigma(a) \cap \sigma(b) \subseteq \sigma(c)$.

5. $\sigma_A$ is called uniform keeping union, if for any $a, b \in A$, there exists $c \in A$ such that $\sigma(a) \cup \sigma(b) \subseteq \sigma(c)$.

Obviously, $\sigma_A$ is uniform keeping intersection $\implies \sigma_A$ is keeping intersection,

$\sigma_A$ is uniform keeping union $\implies \sigma_A$ is keeping union.

**Example 3.6.** Let $U = \{h_1, h_2, h_3, h_4, h_5\}$, let $A = \{a_1, a_2, a_3, a_4\}$ and let $\sigma_A$ be a soft set over $2^U$, defined as follows:

\[
\sigma(a_1) = \{\emptyset, \{h_1\}, \{h_1, h_2\}\}, \sigma(a_2) = \{\{h_1, h_2\}, \{h_1, h_3\}\}, \sigma(a_3) = \{\{h_1, h_3\}\}, \sigma(a_4) = \{\{h_4, h_5\}\}.
\]

Then $\sigma_A$ is keeping intersection.

\[\sigma(a_2) \cap \sigma(a_3) = \{\{h_1\}, \{h_1, h_3\}\} \not\subseteq \sigma(a) \text{ for any } a \in A.\]

Thus $\sigma_A$ is not uniform keeping intersection.

This example illustrates that

$\sigma_A$ is keeping intersection $\not\implies \sigma_A$ is uniform keeping intersection.

**Example 3.7.** Let $U = \{h_1, h_2, h_3, h_4, h_5\}$, $A = \{a_1, a_2, a_3, a_4\}$ and let $\sigma_A$ be a soft set over $2^U$, defined as follows:

\[
\sigma(a_1) = \{\{h_1, h_3, h_4\}, \{h_1, h_2\}\}, \sigma(a_2) = \{\{h_1, h_2\}, \{h_1, h_3\}\}, \sigma(a_3) = \{\{h_1, h_3\}\}, \{h_3, h_4\}\}, \sigma(a_4) = \{\{h_1, h_2, h_3, h_4\}\}.
\]
Then $\sigma_A$ is keeping union.

$$\sigma(a_1) \lor \sigma(a_2) = \{\{h_1, h_2, h_3, h_4\}, \{h_1, h_3, h_4\}, \{h_1, h_2\}, \{h_1, h_2, h_3\}\} \not\subseteq \sigma(a)$$

for any $a \in A$. Thus $\sigma_A$ is not uniform keeping union.

This example illustrates that $\sigma_A$ is keeping union $\neq \Rightarrow \sigma_A$ is uniform keeping union.

**Definition 3.8.** Let $\sigma_A$ be a soft set over $2^U$. $\sigma_A$ is bijective, if $\sigma_A$ satisfies the following conditions:

(i) For any $a \in A$, $\sigma(a)^* = U$;

(ii) If $M, N \in \sigma(a)$ and $M \neq N$ for any $a \in A$, then $M \cap N = \emptyset$.

In order to elaborate this concept, we consider the following example.

**Example 3.9.** Let $U = \{h_1, h_2, h_3, h_4, h_5\}$ be a universe consisting of five houses as possible alternatives, and $A = \{a_1, a_2, a_3, a_4\} \subseteq E$ be a set of parameters considered by the decision makers, where

- $a_1$ represents the parameter "beauty", we divided it into three grades: "prettyish", "beautiful" and "wonderful";
- $a_2$ represents the parameter "modernization", we divided it into two grades: "plain" and "modern";
- $a_3$ represents the parameter "price", we divided it into two grades: "cheap" and "expensive";
- $a_4$ represents the parameter "in the green surroundings", we divided it into three grades: "a little green", "green" and "much more green".

Now, we consider a bijective soft set $\sigma_A$, which describes the "attractiveness of the houses" that Mr.X is going to buy. In this case, to define the soft set $\sigma_A$ means to point out classification based on the parameters beauty, modernization and so on. Consider the mapping $\sigma$ given by classification based on one of the parameters $a_i \in A$. For instance, $\sigma(a_1)$ means the classification based on the parameter $a_1$. Let $U = \{h_1, h_2, h_3, h_4, h_5\}$, $A = \{a_1, a_2, a_3, a_4\}$ and let $\sigma_A$ be a bijective soft set over $2^U$, defined as follows

$$
\sigma(a_1) = \{\{h_1, h_2\}, \{h_3, h_4\}, \{h_5\}\}, \sigma(a_2) = \{\{h_1, h_3\}, \{h_2, h_4, h_5\}\},
\sigma(a_3) = \{\{h_1, h_2, h_4\}, \{h_3, h_5\}\}, \sigma(a_4) = \{\{h_1, h_2, h_3, h_4\}, \{h_5\}\}.
$$

Then a bijective soft set $\sigma_A$ is described as the following Table 2.

**Table 2.** Tabular representation of the bijective soft set $\sigma_A$

<table>
<thead>
<tr>
<th></th>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$h_3$</th>
<th>$h_4$</th>
<th>$h_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>prettyish</td>
<td>prettyish</td>
<td>beautiful</td>
<td>beautiful</td>
<td>wonderful</td>
</tr>
<tr>
<td>$a_2$</td>
<td>plain</td>
<td>modern</td>
<td>plain</td>
<td>modern</td>
<td>modern</td>
</tr>
<tr>
<td>$a_3$</td>
<td>cheap</td>
<td>cheap</td>
<td>expensive</td>
<td>cheap</td>
<td>expensive</td>
</tr>
<tr>
<td>$a_4$</td>
<td>a little green</td>
<td>a little green</td>
<td>a little green</td>
<td>green</td>
<td>much more green</td>
</tr>
</tbody>
</table>
3.3. Some operations on soft sets over \(2^U\).

**Definition 3.10.** Let \(A, B \subseteq E\) and let \(\sigma_A\) and \(\delta_B\) be two soft sets over \(2^U\).

1. \(h_C\) is called the intersection of \(\sigma_A\) and \(\delta_B\), if \(C = A \cap B \) and \(h(e) = \sigma(e) \cap \delta(e)\) for each \(e \in C\). We write \(\sigma_A \cap \delta_B = h_C\).
2. \(h_C\) is called the union of \(\sigma_A\) and \(\delta_B\), if \(C = A \cup B\) and

\[
h(e) = \begin{cases} 
\sigma(e), & \text{if } e \in A - B, \\
\delta(e), & \text{if } e \in B - A, \\
\sigma(e) \cup \delta(e), & \text{if } e \in A \cap B.
\end{cases}
\]

We write \(\sigma_A \cup \delta_B = h_C\).

**Example 3.11.** Let \(U = \{h_1, h_2, h_3\}\), \(A = \{a_1, a_2\}\), \(B = \{a_2, a_3\}\) and let \(\sigma_A\) and \(\delta_B\) be two soft sets over \(2^U\), defined as follows:

\[
\sigma(a_1) = \{\{h_1, h_2, h_3\}, \{h_1, h_2\}\}, \quad \sigma(a_2) = \{\{h_1, h_2\}, \{h_1, h_3\}\}.
\]

\[
\delta(a_2) = \{\{h_2, h_3\}, \{h_1, h_2\}\}, \quad \delta(a_3) = \{\{h_2, h_3\}, \emptyset\}.
\]

1. Put \(\sigma_A \cap \delta_B = h_C\). Then \(C = A \cap B = \{a_2\}\) and \(h(a_2) = \sigma(a_2) \cap \delta(a_2) = \{\{h_1, h_2\}\}\).
2. Put \(\sigma_A \cup \delta_B = k_D\). Then \(D = A \cup B = \{a_1, a_2, a_3\}\) and

\[
\begin{align*}
k(a_1) &= \sigma(a_1) = \{\{h_1, h_2, h_3\}, \{h_1, h_2\}\}, \\
k(a_2) &= \sigma(a_2) \cup \delta(a_2) = \{\{h_1, h_2\}, \{h_1, h_3\}, \{h_2, h_3\}\}, \\
k(a_3) &= \delta(a_3) = \{\{h_2, h_3\}, \emptyset\}.
\end{align*}
\]

**Definition 3.12.** Let \(A \subseteq E\) and let \(\sigma_A\) be an soft set over \(2^U\). The complement of \(\sigma_A\) is denoted by \((\sigma_A)^c\) and is defined by \((\sigma_A)^c = (\sigma^c, A)\), where \(\sigma^c : A \to 2^U\) is a mapping given by \(\sigma^c(a) = 2^U - \sigma(a)\) for each \(a \in A\).

**Example 3.13.** In Example 3.11, we obtained \(\sigma^c_A\) as follows:

\[
\sigma^c(a_1) = \{\emptyset, \{h_1\}, \{h_2\}, \{h_3\}, \{h_2, h_3\}, \{h_1, h_3\}\},
\]

\[
\sigma^c(a_2) = \{\emptyset, \{h_1\}, \{h_2\}, \{h_3\}, \{h_2, h_3\}, \{h_1, h_2, h_3\}\}.
\]

**Definition 3.14.** Let \(A, B \subseteq E\) and let \(\sigma_A\) and \(\delta_B\) be two soft sets over \(2^U\).

1. \(h_C\) is called the intersection of \(\sigma_A\) and \(\delta_B\), if \(C = A \cap B\) and \(h(e) = \sigma(e) \land \delta(e)\) for each \(e \in C\). We write \(\sigma_A \cap \delta_B = h_C\).
2. \(h_C\) is called the union of \(\sigma_A\) and \(\delta_B\), if \(C = A \cup B\) and

\[
h(e) = \begin{cases} 
\sigma(e), & \text{if } e \in A - B, \\
\delta(e), & \text{if } e \in B - A, \\
\sigma(e) \lor \delta(e), & \text{if } e \in A \cap B.
\end{cases}
\]

We write \(\sigma_A \lor \delta_B = h_C\).
Example 3.15. In Example 3.11, we have
(1) Put
\[ \sigma_A \cup \delta_B = h_C. \]
Then \( C = A \cap B = \{a_2\} \) and \( h(a_2) = \sigma(a_2) \cup \delta(a_2) = \{\{h_1\}, \{h_2\}, \{h_3\}, \{h_1, h_2\}\}. \)
(2) Put
\[ \sigma_A \bar{\cup} \delta_B = k_D. \]
Then \( D = A \cup B = \{a_1, a_2, a_3\} \)
\[ k(a_1) = \sigma(a_1) = \{\{h_1, h_2, h_3\}, \{h_1, h_2\}\}. \]
\[ k(a_2) = \sigma(a_2) \cup \delta(a_2) = \{\{h_1, h_2\}, \{h_1, h_2, h_3\}\}. \]
\[ k(a_3) = \delta(a_3) = \{\{h_2, h_3\}, \emptyset\}. \]

Proposition 3.16. Let \( A, B, C \subseteq E \) and let \( \sigma_A, \delta_B \) and \( h_C \) be three soft sets over \( 2^U \). Then
(1) \( \delta_A \cup \delta_A = \delta_A \).
(2) \( \delta_A \cup \delta_B = \delta_B \cup \delta_A \).
(3) \( \delta_A \cup \delta_B \cap h_C = \delta_A \cup (\delta_B \cap h_C) \).
(4) \( \delta_A \cap \delta_A = \delta_A \).
(5) \( \delta_A \cap \delta_B = \delta_B \cap \delta_A \).
(6) \( \delta_A \cap \delta_B \cap h_C = \delta_A \cap (\delta_B \cap h_C) \).

Proof. (1) and (2) are obvious.
(3) Put
\[ (\sigma_A \cup \delta_B) \cup h_C = k_{A \cup B \cup C}, \quad \sigma_A \cup (\delta_B \cup h_C) = l_{A \cup B \cup C}; \]
\[ \sigma_A \cup \delta_B = s_{A \cup B}, \quad \delta_B \cup h_C = t_{B \cup C}. \]
For any \( e \in A \cup B \cup C \), it follows that \( e \in A \), \( e \in B \), or \( e \in C \).

Case 1 \( e \in C \).
(a) If \( e \notin A \) and \( e \notin B \), then \( k(e) = h(e) = t(e) = l(e) \).
(b) If \( e \notin A \) and \( e \in B \), then \( k(e) = s(e) \cup h(e) = \delta(e) \cup h(e) = t(e) = l(e) \).
(c) If \( e \in A \) and \( e \notin B \), then \( k(e) = s(e) \cup h(e) = \sigma(e) \cup h(e) = \sigma(e) \cup t(e) = l(e) \).
(d) If \( e \in A \) and \( e \in B \), \( k(e) = s(e) \cup h(e) = (\sigma(e) \cup \delta(e)) \cup h(e) = \sigma(e) \cup (\delta(e) \cup h(e)) = \sigma(e) \cup t(e) = l(e) \).

Case 2 \( e \notin C \).
(a) If \( e \notin A \) and \( e \in B \), then \( k(e) = s(e) = \delta(e) = t(e) = l(e) \).
(b) If \( e \in A \) and \( e \notin B \), then \( k(e) = s(e) = \sigma(e) = t(e) = l(e) \).
(c) If \( e \in A \) and \( e \in B \), then \( k(e) = s(e) = \sigma(e) \cup \delta(e) = \sigma(e) \cup t(e) = l(e) \).
Thus \( (\sigma_A \cup \delta_B) \cup h_C = \sigma_A \cup (\delta_B \cup h_C) \).

(4) and (5) are obvious.
(6) Put
\[ (\sigma_A \bar{\cup} \delta_B) \bar{\cup} h_C = k'_{A \cup B \cup C}, \quad \sigma_A \bar{\cup} (\delta_B \bar{\cup} h_C) = l'_{A \cup B \cup C}; \]
\[ \sigma_A \bar{\cup} \delta_B = s'_{A \cup B}, \quad \delta_B \bar{\cup} h_C = t'_{B \cup C}. \]
For any \( e \in A \cup B \cup C \), it follows that \( e \in A \), \( e \in B \), or \( e \in C \).

Case 1 \( e \in C \).
Proposition 3.17. Let $A, B, C \subseteq E$ and let $\sigma_A$, $\delta_B$ and $h_C$ be three soft sets over $2^U$. Then

\begin{enumerate}
  \item $(\sigma_A \cap \delta_B) \cap h_C = k_{A \cap B \cap C}$; \quad $(\sigma_A \cap \delta_B) \cap h_C = l_{A \cap B \cap C}$.
  \item $(\sigma_A \cap \delta_B) \cap h_C = \sigma_A \cap (\delta_B \cap h_C)$.
  \item $(\sigma_A \cap \delta_B) \cap h_C = \sigma_A \cap (\delta_B \cap h_C)$.
  \item $(\sigma_A \cap \delta_B) \cap h_C = \sigma_A \cap (\delta_B \cap h_C)$.
  \item $(\sigma_A \cap \delta_B) \cap h_C = \sigma_A \cap (\delta_B \cap h_C)$.
  \item $(\sigma_A \cap \delta_B) \cap h_C = \sigma_A \cap (\delta_B \cap h_C)$.
\end{enumerate}

Proof. (1) and (2) are obvious.

(3) Put

$$(\sigma_A \cap \delta_B) \cap h_C = k_{A \cap B \cap C}; \quad (\sigma_A \cap (\delta_B \cap h_C)) = l_{A \cap B \cap C}.$$ 

For any $e \in A \cap B \cap C$, it follows that $e \in A$, $e \in B$ and $e \in C$. $k(e) = (\sigma(e) \cap \delta(e)) \cap h(e) = \sigma(e) \cap (\delta(e) \cap h(e)) = l(e)$, then $(\sigma_A \cap \delta_B) \cap h_C = \sigma_A \cap (\delta_B \cap h_C)$.

(4) and (5) are obvious.

(6) Put

$$(\sigma_A \cap \delta_B) \cap h_C = k'_{A \cap B \cap C}; \quad (\sigma_A \cap (\delta_B \cap h_C)) = l'_{A \cap B \cap C}.$$ 

For any $e \in A \cap B \cap C$, it follows that $e \in A$, $e \in B$ and $e \in C$. Then $k'(e) = (\sigma(e) \cap \delta(e)) \cap h(e) = \sigma(e) \cap (\delta(e) \cap h(e)) = l'(e)$. So $(\sigma_A \cap \delta_B) \cap h_C = \sigma_A \cap (\delta_B \cap h_C)$.

Proposition 3.18. Let $A, B, C \subseteq E$ and let $\sigma_A$, $\delta_B$ and $h_C$ be three soft sets over $2^U$. Then

\begin{enumerate}
  \item $(\sigma_A \cup \delta_B) \cap h_C = (\sigma_A \cap h_C) \cup (\delta_B \cap h_C)$.
  \item $(\sigma_A \cap \delta_B) \cup h_C = (\sigma_A \cup h_C) \cap (\delta_B \cup h_C)$.
  \item $(\sigma_A \cup \delta_B) \cap h_C = (\sigma_A \cap h_C) \cup (\delta_B \cup h_C)$.
  \item $(\sigma_A \cap \delta_B) \cup h_C = (\sigma_A \cup h_C) \cap (\delta_B \cup h_C)$.
\end{enumerate}

Proof. (1) Put

$$(\sigma_A \cup \delta_B) \cap h_C = k'_{(A \cup B) \cap C}; \quad (\sigma_A \cap h_C) \cup (\delta_B \cap h_C) = l'_{(A \cap C) \cup (B \cap C)}.$$ 

Obviously, $(A \cup B) \cap C = (A \cup B) \cap C$. For any $e \in (A \cup B) \cap C$, it follows that $e \in A \cap C$, or $e \in B \cap C$.

1) If $e \notin A \cap C$ and $e \in B \cap C$, then $e \notin A, e \in B$ and $e \in C$. So $k'(e) = \delta(e) \cup h(e) = l'(e)$.
2) If \( e \in A \cap C \) and \( e \notin B \cap C \), then \( e \in A, e \notin B \) and \( e \in C \). So \( k'(e) = \sigma(e) \cup h(e) = l'(e) \).

3) If \( e \in A \cap C \) and \( e \in B \cap C \), then \( e \in A, e \in B \) and \( e \in C \). \( k'(e) = (\sigma(e) \cup \delta(e)) \cap h(e) = (\sigma(e) \cap h(e)) \cup (\delta(e) \cap h(e)) = l'(e) \).

Thus \( (\sigma_A \cup \delta_B) \bar{\wedge} h_C = (\sigma_A \bar{\wedge} h_C) \cup (\delta_B \bar{\wedge} h_C) \).

(2) This is similar to the proof of (1).

(3) Put \( (\sigma_A \bar{\vee} \delta_B) \bar{\wedge} h_C = k_{(A \cup B) \cap C}, \ (\sigma_A \bar{\wedge} h_C) \vee (\delta_B \bar{\wedge} h_C) = l_{(A \cap C) \cup (B \cap C)} \).

Obviously, \( (A \cup B) \cap C = (A \cap C) \cup (B \cap C) \). For any \( e \in (A \cup B) \cap C \), it follows that \( e \in A \cap C \), or \( e \in B \cap C \).

1) If \( e \notin A \cap C \) and \( e \notin B \cap C \), then \( e \notin A, e \notin B \) and \( e \in C \). So \( k(e) = \delta(e) \vee h(e) = l(e) \).

2) If \( e \in A \cap C \) and \( e \notin B \cap C \), then \( e \in A, e \notin B \) and \( e \in C \). So \( k(e) = \sigma(e) \vee h(e) = l(e) \).

3) If \( e \in A \cap C \) and \( e \in B \cap C \), then \( e \in A, e \in B \) and \( e \in C \). So \( k(e) = (\sigma(e) \vee \delta(e)) \wedge h(e) \leq (\sigma(e) \wedge h(e)) \vee (\delta(e) \wedge h(e)) = l(e) \).

Thus \( (\sigma_A \bar{\vee} \delta_B) \bar{\wedge} h_C \subseteq (\sigma_A \bar{\wedge} h_C) \bar{\vee} (\delta_B \bar{\wedge} h_C) \).

(4) This is similar to the proof of (3).

Example 3.19. Let \( U = \{h_1, h_2, h_3\}, A = \{a_1, a_2\}, B = \{a_2, a_3\}, C = \{a_2, a_4\} \) and let \( \sigma_A, \delta_B \) and \( h_C \) be three soft sets over \( 2^U \), defined as follows:

\[
\sigma(a_1) = \{\{h_1, h_2, h_3\}, \{h_1, h_2\}\}, \ \sigma(a_2) = \{\{h_1\}\}.
\]

\[
\delta(a_2) = \{\emptyset, \{h_2, h_3\}\}, \ \delta(a_3) = \{\{h_2, h_3\}, \emptyset\}.
\]

\[
h(a_2) = \{\{h_1\}, \{h_3\}\}, \ h(a_4) = \{\{h_1, h_2\}\}.
\]

(1) Put \( (\sigma_A \bar{\vee} \delta_B) \bar{\wedge} h_C = k_D, \ (\sigma_A \bar{\wedge} h_C) \bar{\vee} (\delta_B \bar{\wedge} h_C) = l_N \).

Then \( D = (A \cup B) \cap C = (A \cap C) \cup (B \cap C) = N = \{a_2\} \).

\[
k(a_2) = (\sigma(a_2) \vee \delta(a_2)) \wedge h(a_2) = \{\{h_1\}, \emptyset, \{h_3\}\},
\]

\[
l(a_2) = (\sigma(a_2) \wedge h(a_2)) \vee (\delta(a_2) \wedge h(a_2)) = \{\{h_1\}, \{h_1, h_3\}, \{h_3\}, \emptyset\} \neq k(a_2).
\]

Thus \( (\sigma_A \bar{\vee} \delta_B) \bar{\wedge} h_C \neq (\sigma_A \bar{\wedge} h_C) \bar{\vee} (\delta_B \bar{\wedge} h_C) \).

(2) Put \( (\sigma_A \bar{\wedge} \delta_B) \bar{\vee} h_C = k_D', \ (\sigma_A \bar{\vee} h_C) \bar{\wedge} (\delta_B \bar{\vee} h_C) = l_N' \).

Then \( D' = (A \cap B) \cup C = (A \cup C) \cap (B \cup C) = N' = \{a_2, a_4\} \).

\[
k'(a_2) = (\sigma(a_2) \wedge \delta(a_2)) \vee h(a_2) = \{\{h_1\}, \{h_3\}\},
\]

\[
l'(a_2) = (\sigma(a_2) \vee h(a_2)) \wedge (\delta(a_2) \vee h(a_2)) = \{\{h_1\}, \{h_1, h_3\}, \{h_3\}, \emptyset\} \neq k'(a_2).
\]
Example 3.21. Let $U = \{h_1, h_2, h_3\}$, $A = \{a_1, a_2\}$ and let $\sigma_A$ and $\delta_A$ be two soft sets over $2^U$, defined as follows:

\[
\sigma(a_1) = \{\{h_1, h_2, h_3\}, \{h_1, h_2\}\}, \quad \sigma(a_2) = \{\{h_1, h_2\}, \{h_1, h_3\}\},
\]

\[
\delta(a_1) = \{\{h_1, h_3\}, \{h_1, h_2\}\}, \quad \delta(a_2) = \{\{h_2, h_3\}, \emptyset\}.
\]

(1) \[
\sigma(a_1) = \{\emptyset, \{h_1\}, \{h_2\}, \{h_3\}, \{h_2, h_3\}, \{h_1, h_3\}\},
\]

\[
\sigma(a_2) = \{\emptyset, \{h_1\}, \{h_2\}, \{h_3\}, \{h_2, h_3\}, \{h_1, h_2, h_3\}\},
\]

so \[
\sigma(a_1) \cup \sigma(a_1) = \{\{h_1, h_2\}, \{h_1, h_2, h_3\}\} = 2^U.
\]
\[ \sigma(a_2) \vee \sigma(a_2)^c = \{h_1, h_2, h_1, h_3, h_2, h_3\} \neq 2^U. \]

Thus
\[ \sigma_A \bar{\vee} (\sigma_A)^c \neq U_A. \]

(2) Put
\[ (a_1 \wedge (a_1)^c = \{\varnothing, \{h_1\}, \{h_2\}, \{h_3\}, \{h_1, h_3\}, \{h_2, h_3\}\}, \{h_1, h_2, h_3\}\} \neq \{\varnothing\}, \]
\[ \sigma(a_2) \wedge \sigma(a_2)^c = \{\varnothing, \{h_1\}, \{h_2\}, \{h_3\}, \{h_1, h_2, h_3\}\} \neq \{\varnothing\}. \]

Thus
\[ \sigma_A \bar{\wedge} (\sigma_A)^c \neq \varnothing_A. \]

(3) Put
\[ (\sigma_A \bar{\vee} \delta_A)^c = h_A, \quad (\sigma_A)^c \bar{\wedge} (\delta_A)^c = l_A. \]
\[ h(a_1) = \{\varnothing, \{h_1\}, \{h_2\}, \{h_3\}, \{h_1, h_3\}, \{h_2, h_3\}\}, \]
\[ h(a_2) = \{\varnothing, \{h_1\}, \{h_2\}, \{h_3\}, \{h_1, h_2, h_3\}\}. \]
\[ l(a_1) = \{\varnothing, \{h_1\}, \{h_2\}, \{h_3\}, \{h_1, h_2, h_3\}\} \neq h(a_1), \]
\[ l(a_2) = \{\varnothing, \{h_1\}, \{h_2\}, \{h_3\}, \{h_1, h_2, h_3\}\} \neq h(a_2). \]

Thus
\[ (\sigma_A \bar{\vee} \delta_A)^c \neq (\sigma_A)^c \bar{\wedge} (\delta_A)^c. \]

(4) Put
\[ (\sigma_A \bar{\wedge} \delta_A)^c = k_A, \quad (\sigma_A)^c \bar{\vee} (\delta_A)^c = t_A. \]

Then
\[ k(a_1) = \{\varnothing, \{h_2\}, \{h_3\}, \{h_1, h_2, h_3\}\}, \]
\[ k(a_2) = \{\{h_1\}, \{h_1, h_2\}, \{h_1, h_3\}, \{h_2, h_3\}, \{h_1, h_2, h_3\}\}. \]
\[ t(a_1) = \{\varnothing, \{h_1\}, \{h_2\}, \{h_3\}, \{h_1, h_2, h_3\}\} \neq k(a_1), \]
\[ t(a_2) = \{\{h_1\}, \{h_2\}, \{h_3\}, \{h_1, h_2, h_3\}\} \neq k(a_2). \]

Thus
\[ (\sigma_A \bar{\vee} \delta_A)^c \neq (\sigma_A)^c \bar{\wedge} (\delta_A)^c. \]

4. Lattice structures of soft sets over \(2^U\)

We denote
\[ S(U, E) = \{\sigma_A : A \subseteq E \text{ and } \sigma_A \text{ is a soft set over } 2^U\}, \]
\[ S_1(U, E) = \{\sigma_E : \sigma_E \text{ is a soft set over } 2^U\}. \]

Obviously,
\[ S_1(U, E) \subseteq S(U, E). \]

**Theorem 4.1.** For any \(\sigma_A, \delta_B \in S(U, E)\), define
\[ \sigma_A \leq \delta_B \iff \sigma_A \bar{\subseteq} \delta_B, \]
\[ \sigma_A \vee \delta_B = \sigma_A \bar{\cup} \delta_B, \]
\[ \sigma_A \wedge \delta_B = \sigma_A \bar{\cap} \delta_B. \]

Then \((S(U, E), \bar{\cup}, \bar{\cap})\) is a lattice.

**Proof.** This is obvious. \(\square\)
Theorem 4.2. For any $\sigma_A, \delta_B \in S(U, E)$, define
\[
\sigma_A \leq \delta_B \Leftrightarrow \sigma_A \subseteq \delta_B,
\]
\[
\sigma_A \lor \delta_B = \sigma_A \lor \delta_B,
\]
\[
\sigma_A \land \delta_B = \sigma_A \land \delta_B.
\]
Then $(S(U, E), \lor, \land)$ is a lattice.

Proof. This is obvious. □

Theorem 4.3. For any $\sigma_E, g_E \in S_1(U, E)$, define
\[
\sigma_E \leq g_E \Leftrightarrow \sigma_E \subseteq g_E,
\]
\[
\sigma_E \lor g_E = \sigma_E \lor g_E,
\]
\[
\sigma_E \land g_E = \sigma_E \land g_E.
\]
Then $(S_1(U, E), \lor, \land)$ is a Boolean lattice.

Proof. Denote $\sum_1 = S_1(U, E)$. It is easily proved that
\[
0_{\sum_1} = \emptyset_{E} \text{ and } 1_{\sum_1} = U_{E}.
\]
By Proposition 3.18, $S_1(U, E)$ is a distributive lattice with $1_{\sum_1}$ and $0_{\sum_1}$.
For any $\sigma_E \in S_1(U, E)$, $(\sigma_E)' = f_\sigma_E$.
Hence $(S_1(U, E), \lor, \land)$ is a Boolean lattice. □

Theorem 4.4. For any $\sigma_E, \delta_E \in S_1(U, E)$, define
\[
\sigma_E \leq \delta_E \Leftrightarrow \sigma_E \subseteq \delta_E,
\]
\[
\sigma_E \lor \delta_E = \sigma_E \lor \delta_E,
\]
\[
\sigma_E \land \delta_E = \sigma_E \land \delta_E.
\]
Then $(S_1(U, E), \lor, \land)$ is a lattice.

Proof. This is obvious. □

Corollary 4.5. (1) $(S_1(U, E), \lor, \land)$ is a sublattice of $(S(U, E), \lor, \land)$
(2) $(S_1(U, E), \lor, \land)$ is a sublattice of $(S(U, E), \lor, \land)$. 
5. Soft rough approximations and soft rough sets

5.1. Soft rough approximations.

Definition 5.1. Let \( \sigma_A \) be a soft set over \( 2^U \). Then the pair \( P = (U, \sigma_A) \) is called a soft approximation space. Based on the soft approximation space \( P \), we define a pair of operations \( \text{apr}_P, \overline{\text{apr}}_P : 2^U \rightarrow 2^U \) as follows:

\[
\text{apr}_P(X) = \{ x \in U : \exists a \in A, M \in \sigma(a) \text{ s.t. } x \in M \in \sigma(a) \text{ and } M \subseteq X \},
\]

\[
\overline{\text{apr}}_P(X) = \{ x \in U : \exists a \in A, M \in \sigma(a) \text{ s.t. } x \in M \in \sigma(a) \text{ and } M \cap X \neq \emptyset \}.
\]

\( \text{apr}_P(X) \) and \( \overline{\text{apr}}_P(X) \) are called the soft \( P \)-lower approximation and the soft \( P \)-upper approximation of \( X \), respectively.

In general, we refer to \( \text{apr}_P(X) \) and \( \overline{\text{apr}}_P(X) \) as soft rough approximations of \( X \) with respect to \( P \).

Proposition 5.2. Let \( \sigma_A \) be a soft set over \( 2^U \) and let \( P = (U, \sigma_A) \) be a soft approximation space. Then for any \( X, Y \in 2^U \),

1. \( \text{apr}_P(X) = \bigcup \{ M : a \in A, M \in \sigma(a) \text{ and } M \subseteq X \} \subseteq X \);
2. \( \overline{\text{apr}}_P(X) = \bigcup \{ M : a \in A, M \in \sigma(a) \text{ and } M \cap X \neq \emptyset \} \).
3. \( X \subseteq Y \Rightarrow \text{apr}_P(X) \subseteq \text{apr}_P(Y) \); \( X \subseteq Y \Rightarrow \overline{\text{apr}}_P(X) \subseteq \overline{\text{apr}}_P(Y) \).
4. \( \overline{\text{apr}}_P(X \cup Y) = \overline{\text{apr}}_P(X) \cup \overline{\text{apr}}_P(Y) \).
5. \( \text{apr}_P(\overline{\text{apr}}_P(X)) = \text{apr}_P(X) \); \( \overline{\text{apr}}_P(\overline{\text{apr}}_P(X)) = \overline{\text{apr}}_P(X) \).

Proof. (1) and (2) are obvious.

(3) (i) Suppose that \( \text{apr}_P(X) - \text{apr}_P(Y) \neq \emptyset \). Pick \( x \in \text{apr}_P(X) - \text{apr}_P(Y) \). Then there exist \( a \in A \) and \( M \in \sigma(a) \) such that \( x \in M \in \sigma(a) \) and \( M \subseteq X \). Since \( X \subseteq Y \), we have \( M \subseteq Y \) and \( x \in \text{apr}_P(Y) \). This is a contradiction.

Thus \( \text{apr}_P(X) \subseteq \text{apr}_P(Y) \).

(ii) Suppose that \( \overline{\text{apr}}_P(X) - \overline{\text{apr}}_P(Y) \neq \emptyset \). Pick \( x \in \overline{\text{apr}}_P(X) - \overline{\text{apr}}_P(Y) \). Then there exist \( a \in A \) and \( M \in \sigma(a) \) such that \( x \in M \in \sigma(a) \) and \( M \cap X \neq \emptyset \). Since \( X \subseteq Y \), \( M \cap Y \neq \emptyset \). This implies that \( x \in \overline{\text{apr}}_P(Y) \). This is a contradiction.

Thus \( \overline{\text{apr}}_P(X) \subseteq \overline{\text{apr}}_P(Y) \).

(4) By (3),

\( \overline{\text{apr}}_P(X \cup Y) \supseteq \overline{\text{apr}}_P(X) \cup \overline{\text{apr}}_P(Y) \).

Suppose that \( \overline{\text{apr}}_P(X \cup Y) - (\overline{\text{apr}}_P(X) \cup \overline{\text{apr}}_P(Y)) \neq \emptyset \). Pick \( x \in \overline{\text{apr}}_P(X \cup Y) - (\overline{\text{apr}}_P(X) \cup \overline{\text{apr}}_P(Y)) \). Then there exist \( a \in A \) and \( M \in \sigma(a) \) such that \( x \in M \in \sigma(a) \) and \( M \cap X \neq \emptyset \). Since \( X \subseteq Y \), \( M \cap Y \neq \emptyset \). This implies that \( x \in \overline{\text{apr}}_P(Y) \). This is a contradiction.

Thus \( \overline{\text{apr}}_P(X \cup Y) \subseteq \overline{\text{apr}}_P(X) \cup \overline{\text{apr}}_P(Y) \).
and \( M \cap (X \cup Y) \neq \emptyset \). This implies that \( M \cap X \neq \emptyset \) or \( M \cap Y \neq \emptyset \). So \( x \in \overline{\text{apr}_p}(X) \) or \( x \in \overline{\text{apr}_p}(Y) \), then \( x \in \overline{\text{apr}_p}(X) \cup \overline{\text{apr}_p}(Y) \). This is a contradiction. Thus
\[
\overline{\text{apr}_p}(X \cup Y) \subseteq \overline{\text{apr}_p}(X) \cup \overline{\text{apr}_p}(Y).
\]
Hence
\[
\overline{\text{apr}_p}(X \cup Y) = \overline{\text{apr}_p}(X) \cup \overline{\text{apr}_p}(Y).
\]
(5) (i) By (1), \( \text{apr}_p(X) \subseteq X \). By (3),
\[
\text{apr}_p(\text{apr}_p(X)) \subseteq \text{apr}_p(X).
\]
Suppose that \( x \in \overline{\text{apr}_p}(X) \). Then there exist \( a \in A \) and \( N \in \sigma(a) \) such that \( x \in N \in \sigma(a) \) and \( N \subseteq X \). Since \( \overline{\text{apr}_p}(X) = \bigcup \{M : a \in A, M \in \sigma(a) \text{ and } M \subseteq X\} \), we have \( x \in N \subseteq \overline{\text{apr}_p}(X) \). This implies that \( x \in \text{apr}_p(\text{apr}_p(X)) \). Thus
\[
\text{apr}_p(\text{apr}_p(X)) \supseteq \text{apr}_p(X).
\]
Hence
\[
\text{apr}_p(\text{apr}_p(X)) = \text{apr}_p(X).
\]
(ii) Suppose that \( x \in \overline{\text{apr}_p}(X) \), then there exist \( a \in A \) and \( N \in \sigma(a) \) such that \( x \in N \in \sigma(a) \) and \( N \cap X \neq \emptyset \). Since \( \overline{\text{apr}_p}(X) = \bigcup \{M : a \in A, M \in \sigma(a) \text{ and } M \cap X \neq \emptyset\} \), we have \( x \in N \subseteq \overline{\text{apr}_p}(X) \). This implies that \( x \in \text{apr}_p(\overline{\text{apr}_p}(X)) \). Thus
\[
\text{apr}_p(\overline{\text{apr}_p}(X)) \supseteq \overline{\text{apr}_p}(X).
\]
By (3),
\[
\text{apr}_p(\overline{\text{apr}_p}(X)) \subseteq \overline{\text{apr}_p}(X).
\]
Hence
\[
\text{apr}_p(\overline{\text{apr}_p}(X)) = \overline{\text{apr}_p}(X).
\]
Proposition 5.3. Let \( \sigma_A \) be a soft set over \( 2^U \) and let \( P = (U, \sigma_A) \) be a soft approximation space. Then

1. If \( \sigma_A \) is full, then \( \text{apr}_p(X) \subseteq X \subseteq \overline{\text{apr}_p}(X) \) for any \( X \in 2^U \).
2. If \( \sigma_A \) is full, then \( \text{apr}_p(U) = \overline{\text{apr}_p}(U) = U \).
3. If \( \sigma_A \) is keeping intersection, then \( \overline{\text{apr}_p}(X \cap Y) = \text{apr}_p(X) \cap \text{apr}_p(Y) \) for any \( X, Y \in 2^U \).
4. If \( \sigma_A \) is full and keeping union, then \( \overline{\text{apr}_p}(X) = U \) for any \( X \in 2^U \setminus \emptyset \).

Proof. (1) By Proposition 5.2, \( \overline{\text{apr}_p}(X) \subseteq X \). Suppose that \( X \setminus \overline{\text{apr}_p}(X) \neq \emptyset \). Pick
\[
x \in X \setminus \overline{\text{apr}_p}(X) \neq \emptyset.
\]
Since \( \sigma_A \) is full, \( \bigcup \{\sigma(a)^* : a \in A\} = U \). So \( x \in M \in \sigma(a) \) for some \( a \in A \). Note that \( x \in X \). Then \( M \cap X \neq \emptyset \). Thus \( x \in \overline{\text{apr}_p}(X) \neq \emptyset \). This is a contradiction. Hence
\[
X \subseteq \overline{\text{apr}_p}(X).
\]
(2) This holds by Proposition 5.2.
(3) By Proposition 5.2, $\text{apr}_p(X \cap Y) \subseteq \overline{\text{apr}}_p(X) \cap \overline{\text{apr}}_p(Y)$. Suppose that $\text{apr}_p(X) \cap \text{apr}_p(Y) - \overline{\text{apr}}_p(X \cap Y) \neq \emptyset$. Pick $x \in \text{apr}_p(X) \cap \text{apr}_p(Y) - \overline{\text{apr}}_p(X \cap Y)$.

Then there exist $a, b \in A$. $M \in \sigma(a)$ and $N \in \tau(b)$ such that $M \subseteq X$, $N \subseteq Y$. Since $\sigma$ is keeping intersection, $M \cap N = Q$ for some $c \in A$ and $Q \in \tau(c)$. This implies that $x \in Q \subseteq X \cap Y$. Thus $x \in \text{apr}_p(X \cap Y)$. This is a contradiction. Hence

$$\text{apr}_p(X \cap Y) \subseteq \text{apr}_p(X) \cap \text{apr}_p(Y).$$

Hence

$$\text{apr}_p(X \cap Y) = \text{apr}_p(X) \cap \text{apr}_p(Y).$$

(4) Suppose that $X \in \mathcal{U} \setminus \emptyset$. Obviously, $\overline{\text{apr}}_p(X) \subseteq U$.

Since $\sigma$ is full and keeping union, we claim that there exist $a \in A$ and $N \in \sigma(a)$ such that $N = U$.

Otherwise. Suppose that there is not $a \in A$ such that exists $N \in \sigma(a)$ and $N = U$.

Since $\sigma$ is full, there exists $M_i(\ i \in \tau)$ such that $\cup_{i \in \tau}M_i = U$. But $\sigma$ is keeping union. This is a contradiction.

Since $\overline{\text{apr}}_p(X) = \bigcup\{M : a \in A, M \in \sigma(a) \text{ and } M \cap X \neq \emptyset\}$, $U \subseteq \overline{\text{apr}}_p(X)$.

Hence $\overline{\text{apr}}_p(X) = U$. $\square$

5.2. Structures of soft rough sets.

**Definition 5.4.** Let $\sigma$ be a soft set over $\mathcal{U}$ and let $P = (U, \sigma)$ be a soft approximation space. $X \in \mathcal{U}$ is called a soft $P$-definable set if $\text{apr}_p(X) = \overline{\text{apr}}_p(X)$; $X$ is called a soft $P$-rough set if $\text{apr}_p(X) \neq \overline{\text{apr}}_p(X)$.

Denote

$$\mathcal{R} = \{X \in \mathcal{U} : X \text{ is a soft } P\text{-rough set}\},$$

$$\mathcal{D} = \{X \in \mathcal{U} : X \text{ is a soft } P\text{-definable set}\},$$

$$\tau_f = \{X \in \mathcal{U} : \text{apr}_p(X) = X\},$$

$$\sigma_f = \{X \in \mathcal{U} : \overline{\text{apr}}_p(X) = X\}.$$

**Proposition 5.5.** Let $\sigma$ be a soft set over $U$ and let $P = (U, \sigma)$ be a soft approximation space. Then for each $X \in \mathcal{U}$,

$$X \in \mathcal{D} \iff \overline{\text{apr}}_p(X) \subseteq X.$$

*Proof.* “$\Rightarrow$”. This is obvious.

“$\Leftarrow$”. Obviously, $\text{apr}_p(X) \subseteq \overline{\text{apr}}_p(X)$.

Suppose that $x \in \overline{\text{apr}}_p(X)$. Then there exist $a \in A$ and $N \in \sigma(a)$ such that $x \in N \in \sigma(a)$ and $N \cap X \neq \emptyset$. $\overline{\text{apr}}_p(X) = \bigcup\{M : a \in A, M \in \sigma(a) \text{ and } M \cap X \neq \emptyset\}$ and $\overline{\text{apr}}_p(X) \subseteq X$ imply $N \subseteq X$. So $x \in \text{apr}_p(X)$. Thus $\text{apr}_p(X) \supseteq \overline{\text{apr}}_p(X)$. Hence $\text{apr}_p(X) = \overline{\text{apr}}_p(X)$ and $X \in \mathcal{D}$. $\square$
Corollary 5.6. Let $\sigma_A$ be a soft set over $U$ and let $P = (U, \sigma_A)$ be a soft approximation space. Then for each $X \in 2^U$,
$$X \in \mathcal{R} \iff \overline{\text{apr}}_P(X) \not\subseteq X.$$  

The following theorem gives structures of soft rough sets.

Theorem 5.7. Let $\sigma_A$ be a soft set over $U$ and let $P = (U, \sigma_A)$ be a soft approximation space.

(1) $\mathcal{R} \cup \mathcal{D} = 2^U$, $\mathcal{R} \cap \mathcal{D} = \emptyset$ and $\sigma_f \subseteq \mathcal{D}$.

(2) If $\sigma_A$ is full, then $\mathcal{R} = 2^U - \sigma_f$ and $\mathcal{D} = \sigma_f \subseteq \tau_f$.

(3) If $\sigma_A$ is full and keeping union, then $\mathcal{R} = 2^U - \{\emptyset, U\}$ and $\mathcal{D} = \{\emptyset, U\} = \sigma_f \subseteq \tau_f$.

Proof. This holds by Propositions 5.2, 5.3 and 5.5. □

6. Topological structures of soft sets over $2^U$

Theorem 6.1. Let $\sigma_A$ be a soft set over $2^U$ and let $P = (U, \sigma_A)$ be a soft approximation space. If $\sigma_A$ is full and keeping intersection or bijective, then $\tau_f$ is a topology on $U$.

Proof. This holds by Propositions 5.2 and 5.3. □

Definition 6.2. Let $\sigma_A$ be a full and keeping intersection soft set over $2^U$ and let $P = (U, \sigma_A)$ be a soft approximation space. Then $\tau_f$ is called the topology induced by $\sigma_A$ on $U$.

Theorem 6.3. Let $\sigma_A$ be a full and keeping intersection over $2^U$, let $P = (U, \sigma_A)$ be a soft approximation space and let $\tau_f$ be the topology induced by $\sigma_A$ on $U$. Then

(1) $\{\overline{\text{apr}}_P(X) : X \in 2^U\} \subseteq \tau_f = \{\overline{\text{apr}}_P(X) : X \in 2^U\}$.

(2) $\sigma(a) \subseteq \tau_f$ for any $a \in A$.

(3) $\overline{\text{apr}}_P$ is an interior operator of $\tau_f$.

Proof. (1) By Proposition 5.2, $\{\overline{\text{apr}}_P(X) : X \in 2^U\} \subseteq \tau_f$.

Obviously,
$$\tau_f \subseteq \{\overline{\text{apr}}_P(X) : X \subseteq U\}.$$  

Let $Y \in \{\overline{\text{apr}}_P(X) : X \in 2^U\}$. Then $Y = \overline{\text{apr}}_P(X)$ for some $X \in 2^U$. By Proposition 5.2, $\overline{\text{apr}}_P(\overline{\text{apr}}_P(X)) = \overline{\text{apr}}_P(X)$. This implies that $Y \in \tau_f$. Thus
$$\tau_f \supseteq \{\overline{\text{apr}}_P(X) : X \in 2^U\}.$$  

Hence
$$\{\overline{\text{apr}}_P(X) : X \in 2^U\} \subseteq \tau_f = \{\overline{\text{apr}}_P(X) : X \in 2^U\}.$$  

(2) Let $a \in A$. Suppose that $M \in \sigma(a)$. Obviously, $\overline{\text{apr}}_P(M) \subseteq M$.  

16
Let \( x \in M \). Then \( M \in \sigma(a) \) and \( x \in M \subseteq M \). This implies that \( x \in \text{apr}_\rho(M) \).
So \( \text{apr}_\rho(M) \supseteq M \). Thus \( \text{apr}_\rho(M) = M \).

Hence \( \sigma(a) \subseteq \tau_f \) for any \( a \in A \).

(3) It suffices to show that
\[
\text{apr}_\rho(X) = \text{int}(X) \text{ for each } X \in 2^U.
\]

By (1), \( \text{apr}_\rho(X) \in \tau_f \). By Proposition 5.2, \( \text{apr}_\rho(X) \subseteq \text{int}(X) \).

Conversely, for each \( Y \in \tau_f \) with \( Y \subseteq X \), by Proposition 5.2, \( Y = \text{apr}_\rho(Y) \subseteq \text{apr}_\rho(X) \). Then
\[
\text{int}(X) = \bigcup \{ Y : Y \in \tau_f \text{ and } Y \subseteq X \} \subseteq \text{apr}_\rho(X).
\]
Thus \( \text{apr}_\rho(X) = \text{int}(X) \). \( \square \)

7. An application in decision making problems

In this section, we illustrate an application of soft sets over \( 2^U \) in decision making problems by Example 7.1.

Example 7.1. In Example 3.9, if the house is "wonderful", "modern", "cheap" and "much more green surroundings", then it is said to be satisfied. Let the score of satisfied houses be 1, the weight of any \( a \in A \) be 0.25. And

- if the house is prettyish, then for \( a_1 \), the score of it is 0.15;
- if the house is beautiful, then for \( a_1 \), the score of it is 0.20;
- if the house is wonderful, then for \( a_1 \), the score of it is 0.25;
- if the house is plain, then for \( a_2 \), the score of it is 0.15;
- if the house is modern, then for \( a_2 \), the score of it is 0.25;
- if the house is expensive, then for \( a_3 \), the score of it is 0.15;
- if the house is cheap, then for \( a_3 \), the score of it is 0.25;
- if the house is in a little green surroundings, then for \( a_4 \), the score of it is 0.15;
- if the house is in green surroundings, then for \( a_4 \), the score of it is 0.20;
- if the house is in much more green surroundings, then for \( a_4 \), the score of it is 0.25.

Since
\[
\sigma(a_1) \land \sigma(a_2) \land \sigma(a_3) \land \sigma(a_4) = \{ h_1 \}, \{ h_2 \}, \{ h_3 \}, \{ h_4 \}, \{ h_5 \},
\]
we have
\[
D = \{ \{ h_1 \} : 0.70, \{ h_2 \} : 0.75, \{ h_3 \} : 0.65, \{ h_4 \} : 0.90, \{ h_5 \} : 0.90 \}
\]
where \( X \) represents the score of houses in \( X \).

Thus, we can conclude that \( h_4 \) or \( h_5 \) is the best choice for Mr. X.
8. Conclusions

In this paper, we considered soft sets over $2^U$ and obtained their lattice and topological structures. Moreover, we introduced soft rough approximations and soft rough sets, and gave structures of soft rough sets. We will study applications of soft sets over $2^U$ in future papers.

Acknowledgements. This work is supported by the National Natural Science Foundation of China (No. 11061004), Guangxi University Science and Technology Research Project (No. 2013ZD020) and the Science Research Project 2013 of the China-ASEAN Study Center (Guangxi Science Experiment Center) of Guangxi University for Nationalities (No. KT201310).

References


Gangqiang Zhang (zhanggangqiang100@126.com)
College of Information Science and Engineering, Guangxi University for Nationalities, Nanning, Guangxi 530006, P.R.China

Ninghua Gao (gaoninghua1987513@126.com)
College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, P.R.China