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Hesitant fuzzy rough sets through hesitant fuzzy relations

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Abstract. Introducing rough sets in hesitant fuzzy set domain and using it for the various applications would open up new possibilities in rough set theory. For this purpose the notion of hesitant fuzzy relations is introduced. The foundation of equivalence hesitant fuzzy relation is laid. Definition of anti-reflexive kernel, symmetric kernel etc. is proposed and the formulae to evaluate them are derived. Hesitant Fuzzy Rough sets are introduced using the lower approximation and upper approximation of a hesitant fuzzy set. A relation between the approximation operators is proved.

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1. Introduction

With the increase in data flowing through the internet and our management systems, storage and extraction of valuable information has attained the utmost importance when it comes to Information systems. Of the many techniques of knowledge discovery, rough set based systems have emerged as a good alternative and research in this area is booming. Rough sets were introduced by Zdzislaw Pawlak [16]. When compared to Fuzzy set theory, Rough set theory approaches uncertainty in a different manner. It characterizes a set with the help of its lower and upper approximations which makes it suitable for its use in information systems.

Fuzzy Set Theory [30] added a new branch to set theory by allowing us to give membership values to the elements in a set. This new idea gave fuzzy set theory scope for application in a wide range of fields, opening new areas of research for researchers all over the world. Atanassov's intuitionistic fuzzy sets [1] took this area one step further by allotting membership and non-membership values for the elements in a set. Equivalence relations in these cases have also been studied [3]. Yang Hai-long and Li Sheng-gang [10] discuss the intuitionistic fuzzy relations in great detail.

One of the latest development to this area would be the introduction of "Hesitant fuzzy sets" by Vicenc Torra [21, 22, 26]. As the name suggests this allows scope for hesitancy. Hesitant fuzzy sets (HFS) allow us to give room for imprecision in assigning the membership values by considering all the possible membership values. Correlation coefficient formulas for HFSs [6] have been applied to clustering analysis under hesitant fuzzy environment. In [18] hesitant fuzzy sets are extended by intuitionistic fuzzy sets and their application in decision support system are studied. Hesitant fuzzy sets are used in multiple attribute decision making problems [24] in which the attributes are in different priority level. Verma and Sharma [23] introduced new Operations over Hesitant Fuzzy Sets. New types of fuzzy preference structures have been introduced to describe uncertain evaluation in group decision making processes [5]. Xia et al. [27] have studied the aggregation of the hesitant fuzzy information. Similarity measures for HFS and their applications are discussed in [2, 28, 32] and Farhadinia [9] discusses information measures. Zhu and Xu [33] gives a Hesitant fuzzy linguistic preference relation and its consistency measures based on hesitant fuzzy linguistic terms. The geometric bonferroni mean is extended to Hesitant fuzzy environment in [36] and the bonferroni means in Hesitant fuzzy environment for multi-criteria decision making is discussed in [34]. In [35] Zhu et al. have introduced the dual hesitant fuzzy sets which consists of the membership hesitancy function and the nonmembership hesitancy function. Ranking methods with hesitant fuzzy preference relations in group decision making environments is explored in [37].

Rough sets and their hybrid structures (including those with fuzzy sets) have evolved into an area which has immense applications in diverse fields like decision-making, pattern recognition, image processing, medical diagnosis, and data mining. They deal with uncertainty arising from imprecision and ambiguity of information. Dubois and Prade [8] introduced fuzzy rough sets as a fuzzy generalization of rough sets. Wei et al. [25] have studied the relationship among the generalized rough set models for hybrid data. Kandil et al. [13] constructed a new rough set structure for a given ideal, with a topology finer than that of the earlier methods. Variable precision fuzzy rough set model is one of the suitable tools for analyzing information systems with crisp or fuzzy attributes [19]. Hassanien [11] introduced a hybrid scheme that combines the advantages of fuzzy sets and rough sets in conjunction with statistical feature extraction techniques and used them for breast cancer detection (by classifying the breast cancer images). Jensen and Shen [12] have presented a fuzzy-rough method for attribute reduction which alleviates important problems encountered by traditional rough set attribute reduction such as dealing with noise and real-valued attributes. Intuitionistic fuzzy rough relations and their properties have been studied in detail in [15]. Anitha and Sunil introduced the concept of multi-fuzzy rough sets by combining the multi-fuzzy set and rough set models thereby studying multi-fuzzy rough relations. Degang and Suyun [7] emphasised the use of local reduction with fuzzy rough sets for decision systems. Petrosino and Ferone [17] introduced a new coding/decoding scheme based on the properties and operations of rough fuzzy sets for image compression. Zhang et al. [31] defined a basic vector $H(X)$ and four cut matrices of $H(X)$ which are used to derive the approximations, positive, boundary and negative regions intuitively. Kozae et al. [14] have succeeded in reducing the boundary region by increasing the lower approximation and decreasing the upper approximation . They have studied the applications of these current methods of rough set theory in network connectivity devices, network cables, network topologies and viruses. Yang et al. [29] have further generalized the multi-granulation rough set approach into fuzzy environment using a family of relations.

In an Information system [16] the attribute values pertaining to an element in the universe can take multiple values. This can be scaled down to form a hesitant fuzzy set pertaining to each element in the universe. This fact highlights the scope of introducing Hesitant fuzzy rough sets(HFRS). HFRS's would lay the foundations for using Rough sets into information systems where there is hesitancy in the data values. One of the main applications of HFRS's would be to discover knowledge from hybrid data using rough sets. In set valued information systems HFRS's can deal with decision tables with real valued conditional attributes taking multiple real values for a single attribute. Moreover applications involving fuzzy rough sets can be revisited with this new idea of HFRS's giving them more flexibility.

Although the major aspect of this paper is the formulation of hesitant fuzzy rough sets, the major part of the paper is about hesitant fuzzy relations as it is the building block of hesitant fuzzy rough sets. So the paper first attempts at creating a theoretical framework in Hesitant Fuzzy Relations. The core of the paper begins with the section which discusses some preliminaries on Hesitant fuzzy sets and goes on to basic operations on Hesitant fuzzy sets. The third section introduces relation on hesitant fuzzy sets and gives the conditions for it to be an equivalence relation. The paper then goes on to reflexive and symmetric kernel and gives the formulae to find them. The last section introduces the notion of a hesitant fuzzy rough sets and then moves on to prove that the Hesitant fuzzy rough approximation operators are dual to each other.

2. Preliminaries

In this section we discuss some of the basic definitions regarding hesitant fuzzy sets. Some new definitions of hesitant fuzzy subset and compliment are proposed. Some properties of operations on Hesitant fuzzy sets are studied to lay the theoritical framework for further studies in this area.

Definition 2.1 ([21])**.** Let X be a reference set then a Hesitant fuzzy set(HFS) on X is defined in terms of a function h that when applied to X returns a subset of $[0, 1]$ $h : X \to P[0, 1]$ where $P[0, 1]$ denotes power set of $[0, 1]$.

The empty hesitant set, the full hesitant set, the set to represent complete ignorance for x and the nonsense set are defined as follows:
empty set :$h_0(x) = 0$
full set : $h_X(x) = 1$
complete ignorance $h(x) = [0, 1]$

set for a nonsense $x : h(x) = \phi$

Given an hesitant fuzzy set h, its lower and upper bound are defined as follows :
$h^-(x) = \min h(x)$
$h^+(x) = \max h(x)$

Remark 2.2. For convenience we call $h(x)$ a hesitant fuzzy element (HFE) [26]. Let $l(h(x))$ be the number of values in $h(x)$.

Example 2.3. $X = \{a, b, c\}$
$h(a) = \{0.8, 0.5, 0.3\}$
$h(b) = \{0.9, 0.6, 0.4, 0.1\}$
$h(c) = \{0.3, 0.5, 0.7\}$.

Definition 2.4 ([26])**.** Score for a HFE , $s(h) = \dfrac{1}{l(h)} \sum\limits_{\gamma \in h} \gamma$ is called the score function of h.

Definition 2.5. Let h_1 and h_2 be two HFS's on X. Then we say that h_1 is a subset of h_2 denoted by, $h_1 \subseteq h_2, \Leftrightarrow h_1 \{x\} \subseteq h_2 \{x\} \forall x \in X$.
$h_1 = h_2$ iff $h_1 \subseteq h_2$ and $h_2 \subseteq h_1$.

Definition 2.6. Proper subset : $(h_1 \subset h_2)$
if $h_1(x) \subseteq h_2(x) \forall x \in X$ and $h_2(x) \neq h_1(x)$ for some $x \in X$
i.e., $h_1(x) \subseteq h_2(x) \forall x \in X$ and $h_2(x) \subset h_1(x)$ for some $x \in X$.

Definition 2.7. Hesitant Equality :
$(h_1 \approx h_2)$ iff $s(h_1(x)) = s(h_2(x) \forall x \in X$.

Definition 2.8. Hesitant subset :
Let h_1 and h_2 be two hesitant fuzzy sets on X, then we say that h_1 is a hesitant subset of h_2 (denoted by $h_1 \preceq h_2$) iff $s(h_1(x)) \leq s(h_2(x)) \forall x \in X$.

Definition 2.9. Hesitant proper subset :
$h_1 \prec h_2$ if $s(h_1(x)) \leq s(h_2(x) \forall x \in X$ and $s(h_1(x)) < s(h_2(x)$ for atleast one $x \in X$.

Remark 2.10. The usual or crisp subset notation defined above becomes a special case of the hesitant subset case. $h_1 \subseteq h_2 \Rightarrow h_1 \preceq h_2$ but $h_1 \preceq h_2 \nRightarrow h_1 \subseteq h_2$.

Definition 2.11 ([21])**.** Complement :
Given a hesitant fuzzy set represented by its membership function h, its complement is defined as follows :
$h^C : X \to P[0, 1]$
$h^C(x) = \bigcup\limits_{\gamma \in h(x)} \{1 - \gamma\}$

Definition 2.12. Relative complement :
Let h and h_1 be two hesitant fuzzy sets on X, then the relative complement of h_1 w.r.t h is defined as $(h \backslash h_1)(x) = \bigcup\limits_{\gamma \in h_1(x)} \{h^+ - \gamma\}$.

Definition 2.13 ([21]). Given two hesitant fuzzy sets represented by their membership functions h_1 and h_2, their union represented by $h_1 \bigcup h_2$ is defined as $(h_1 \bigcup h_2)(x) = \{\gamma \in (h_1(x) \bigcup h_2(x)/\gamma \geq \max(h_1^-, h_2^-)\} = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\}$.

Definition 2.14 ([21]). Given two hesitant fuzzy sets represented by their membership functions h_1 and h_2, their intersection represented by $h_1 \bigcap h_2$ is defined as $(h_1 \bigcap h_2)(x) = \{\gamma \in (h_1(x) \bigcup h_2(x)/\gamma \leq min(h_1^+, h_2^+)\}. = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} min\{\gamma_1, \gamma_2\}$.

The following lemma studies the properties of Operations on Hesitant Fuzzy Sets.

Lemma 2.15. *Let h be a HFS on X. h_X denotes the full set and h_0 denotes the empty set.*

(1) $(h \bigcup h_X)(x) = h_X(x)$
(2) $(h \bigcup h_0)(x) = h(x)$
(3) (a) $(h \bigcup h^C(x)) \neq h_X(x)$
 (b) $(h \bigcap h^C(x)) \neq h_0(x)$
(4) (a) $(h \bigcap h_X(x)) = h(x)$
 (b) $(h \bigcup h_0(x)) = h_0(x)$
(5) $(h^C)^C = h$ *(Involution)*
(6) (a) $h_1 \bigcup h_2 = h_2 \bigcup h_1$ *(Commutativity)*
 (b) $h_1 \bigcap h_2 = h_2 \bigcap h_1$
(7) (a) $(h_1 \bigcup h_2) \bigcup h_3 = h_1 \bigcup (h_2 \bigcup h_3)$ *(Associativity)*
 (b) $(h_1 \bigcap h_2) \bigcap h_3 = h_1 \bigcap (h_2 \bigcap h_3)$
(8) $h_1 \bigcap (h_2 \bigcup h_3) = (h_1 \bigcap h_2) \bigcup (h_1 \bigcap h_3)$ *(Distributivity)*
(9) (a) $(h_1 \bigcup h_2)^C = h_1^C \bigcap h_2^C$
 (b) $(h_1 \bigcap h_2)^C = h_1^C \bigcup h_2^C$

Proof. (1)

$$\begin{aligned}
(h \bigcup h_X(x)) &= \left\{\gamma \in \left\{h(x) \bigcup h_X(x)\right\} / \gamma \geq max(h^-, h_X^-)\right\} \\
&= \left\{\gamma \in \left\{h(x) \bigcup \{1\}\right\} / \gamma \geq max(h^-, 1)\right\} \\
&= \left\{\gamma \in \left\{h(x) \bigcup \{1\}\right\} / \gamma \geq 1\right\} \\
&= \{1\} \\
&= h_X(x)
\end{aligned}$$

(2) Similar to (1)
(3) Eg : Consider a hesitant fuzzy set with a hesitant fuzzy element $h(x) = \{0.5, 0.9\}$ Then, $h^C(x) = \{0.5, 0.1\}$
 (a) $(h \bigcup h^C(x)) = \{0.5, 0.9\} \neq h_X(x)$
 (b) $(h \bigcap h^C(x)) = \{0.1, 0.5\} \neq h_0(x)$

(4) (a)

$$\begin{aligned}(h\bigcap h_X(x)) &= \left\{\gamma\in\left\{h(x)\bigcup h_X(x)\right\}/\gamma\leq min(h^+,h_X^+)\right\}\\ &= \left\{\gamma\in\left\{h(x)\bigcup\{1\}\right\}/\gamma\leq min(h^+,1)\right\}\\ &= \left\{\gamma\in\left\{h(x)\bigcup\{1\}\right\}/\gamma\leq h^+\right\}\\ &= h(x)\end{aligned}$$

(b) Similar to (a)

(5) Clearly $1-(1-\gamma)=\gamma \quad \forall\gamma\in h(x)$

(6) Follows from the definition of hesitant union and hesitant intersection.

(7) Proof follows because

$$\begin{aligned}max(max(h_1^-,h_2^-),h_3^-) &= max(h_1^-,max(h_2^-,h_3^-))\\ &= max(h_1^-,h_2^-,h_3^-)\end{aligned}$$

(8) Let $h_L=(h_1\bigcap(h_2\bigcup h_3))(x)$ for any $x\in X$ and $h_R=((h_1\bigcap h_2)\bigcup(h_1\bigcap h_3))(x)$ for any $x\in X$

(a) Let $\gamma\in h_L$ (i.e., $h\in(h_1\bigcap(h_2\bigcup h_3))(x)$
$\Rightarrow\gamma\geq max(h_2^-,h_3^-)$ $[\because h\in(h_2\bigcup h_3)]$ and
$\gamma\leq min(h_1^+,max(h_2^+,h_3^+))[\because h\in h_1\bigcap(h_2\bigcup h_3)]$
$h_L^-=min(h_1^-,max(h_2^-,h_3^-))$ and $h_L^+=min(h_1^+,max(h_2^+,h_3^+))$
and above all this by the definition of hesitant fuzzy union and intersection we have
$\gamma\in h_L\Rightarrow h\in(h_1\bigcup h_2\bigcup h_3)(x)$

(b) Let $\gamma\in h_R$ (i.e., $h\in((h_1\bigcap h_2)\bigcup(h_1\bigcap h_3))(x)$
$\gamma\in h_1\bigcap h_2\Rightarrow h\leq min(h_1^+,h_2^+)$ and $h\in h_1\bigcap h_3\Rightarrow h\leq min(h_1^+,h_3^+)$
$((h\in h_1\bigcap h_2))\bigcup((h_1\bigcap h_3))(x)\Rightarrow h\geq max(min(h_1^-,h_2^-),min(h_1^-,h_3^-))$
$\Rightarrow h_R^-=max(min(h_1^-,h_2^-),min(h_1^-,h_3^-))$ and $h_R^+=max(min(h_1^+,h_2^+),min(h_1^+,h_3^+))$
Clearly, $h\in h_R\Rightarrow h\in(h_1\bigcup h_2\bigcup h_3)(x)$ by the definition of hesitant union.
Now from (a) and (b) it is enough to show that $h_L^-=h_R^-$ and $h_L^+=h_R^+$
$\because\forall h\in h_L,h_R;h$ is taken from $(h_1\bigcup h_2\bigcup h_3)(x)$
There are 6 possibilities:

(i) $h_1\leq h_2\leq h_3$
(ii) $h_1\leq h_3\leq h_2$
(iii) $h_2\leq h_1\leq h_3$
(iv) $h_2\leq h_3\leq h_1$
(v) $h_3\leq h_1\leq h_2$
(vi) $h_3\leq h_2\leq h_1$

Here h_1,h_2,h_3 signifies either h_i^- or h_i^+. But it is enough to prove that $h_L^-=h_R^-$ and $h_L^+=h_R^+$ when h_1^-,h_2^-,h_3^- and h_1^+,h_2^+,h_3^+ takes these six possibilities.
(i)$h_1\leq h_2\leq h_3$
$h_L^-=min(h_1^-,h_3^-)=h_1^-,\quad h_R^-=max(h_1^-,h_1^-)=h_1^-$. Hence $h_L^-=h_R^-$.
$h_L^+=min(h_1^+,h_3^+)=h_1^+,\quad h_R^+=max(h_1^+,h_1^+)=h_1^+$. Hence $h_L^+=h_R^+$.

(vi)$h_3 \leq h_2 \leq h_1$
$h_L^- = min(h_1^-, h_2^-) = h_2^-, \quad h_R^- = max(h_2^-, h_3^-) = h_2^-$. Hence $h_L^- = h_R^-$.
$h_L^+ = min(h_1^+, h_2^+) = h_2^+, \quad h_R^+ = max(h_2^+, h_3^+) = h_2^+$. Hence $h_L^+ = h_R^+$.
All the other cases can also be proved similarly.

(9) (a) To prove $(h_1 \bigcup h_2)^C = h_1^C \bigcap h_2^C$
$(h_1 \bigcup h_2)(x) = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1, \gamma_2\}$
LHS= $(h_1 \bigcup h_2)^C = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} (1 - max\{\gamma_1, \gamma_2\})$.
RHS= $h_1^C \bigcap h_2^C = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} min\{1-\gamma_1, 1-\gamma_2\}$
$= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} (1 - max\{\gamma_1, \gamma_2\})$ Hence proved.

(b) Proof similar to (a)

□

3. Hesitant Fuzzy Relations

This section introduces the notion of a hesitant fuzzy relation and studies some of its theoretical properties. We propose the definition of the complement and compostion of Hesitant fuzzy relations, moving on to the definition of an equivalence hesitant fuzzy relation. Some properties of hesitant fuzzy relations are discussed. Some results to arrive at the anti-reflexive kernel and symmetric kernel of any hesitant fuzzy relation is proved. The definition of the $\alpha-$ level cut set of a hesitant fuzzy set is proposed and some of its properties are proved. The $\alpha-$ level cut set of a hesitant fuzzy relation is also stated.

Definition 3.1. (Relation) : A Hesitant fuzzy subset R of $X \times Y$ is called a Hesitant fuzzy relation R from X to Y. i.e.,

$$R : X \times Y \to P[0, 1]$$

Note: $HF(X, Y)$ denotes the family of all Hesitant fuzzy relations from X to Y. Here X and Y are crisp sets.

Definition 3.2. Let I be a hesitant fuzzy relation from X to X.
i.e., $I : X \times X \to P[0, 1]$. If I satisfies

$$I(x, y) = \begin{cases} \{1\}, & \text{for } x = y \\ \{0\}, & \text{for } x \neq y \end{cases}$$

Then I is called a Identity hesitant fuzzy relation.

Definition 3.3. Let R be a hesitant fuzzy relation from X to Y. Then complement of R is a hesitant fuzzy relation satisfying $R^C(x, y) = [R(x, y)]^C = \bigcup_{\gamma \in R(x,y)} \{1 - \gamma\}$ and inverse of R is a hesitant fuzzy relation from Y to X satisfying $R^{-1}(x, y) = R(x, y)$.

Definition 3.4. Composition : Let $R_1 \in HF(X, Y)$ and $R_2 \in HF(Y, Z)$ then $R_1 \circ R_2 \in HF(X, Z)$ is defined as $R_1 \circ R_2(x, z) = \bigcup_{y \in Y} (R_1(x, y) \bigcap R_2(y, z))$. Clearly this will be a subset of $[0,1]$ $\forall (x,z) \in X \times Z$, Hence $R_1 \circ R_2$ will be a hesitant fuzzy relation.

Definition 3.5. Let $R \in HF(X,X)$, then

(1) R is reflexive iff $R(x,x) = \{1\}\, \forall x \in X$
(2) R is symmetric iff $R(x,y) = R(y,x) \quad \forall x \in X$
(3) R is transitive iff $h^+_{R(x,z)} \geq \sup\limits_{y \in X} min(h^+_{R(x,y)}, h^+_{R(y,z)})$ and
$$h^-_{R(x,z)} \geq \sup_{y \in X} max(h^-_{R(x,y)}, h^-_{R(y,z)})$$
(4) R is anti reflexive iff $R(x,x) = \{0\} \quad \forall x \in X$

Definition 3.6. R is an **Equivalence Hesitant Fuzzy relation** iff it is reflexive, symmetric and transitive.

Note : When R is a fuzzy relation our definitions are consistent with those of the fuzzy case, only the set notation has to be dropped.

Lemma 3.7. *Let P, Q, R be three hesitant fuzzy relations from X to Y then,*

(1) $P \preccurlyeq Q \Rightarrow P^{-1} \preccurlyeq Q^{-1}$
(2) $(P^{-1})^{-1} = P$
(3) (a) $h^+_P \leq h^+_{P \bigcup Q}$
    (b) $h^+_Q \leq h^+_{P \bigcup Q}$
(4) (a) $h^-_{P \bigcap Q} \leq h^-_P$
    (b) $h^-_{P \bigcap Q} \leq h^-_Q$
(5) (a) $(R \bigcup Q)^{-1} = R^{-1} \bigcup Q^{-1}$
    (b) $(R \bigcap Q)^{-1} = R^{-1} \bigcap Q^{-1}$
(6) $(R^{-1})^C = (R^C)^{-1}$

Proof. (1)
$$\begin{aligned} P \preccurlyeq Q \quad &\Rightarrow \quad s(P(x,y)) \leq s(Q(x,y)) \quad \forall x, y \in X \\ &\Rightarrow \quad s(P(y,x)) \leq s(Q(y,x)) \quad \text{as x and y are arbitrary} \\ &\Rightarrow \quad s(P^{-1}(x,y)) \leq s(Q^{-1}(x,y)) \quad [\because P^{-1}(x,y) = P(x,y)] \\ &\Rightarrow \quad P^{-1} \preccurlyeq Q^{-1} \end{aligned}$$

(2) $(P^{-1})^{-1}(x,y) = P^{-1}(y,x) = P(x,y) \Rightarrow (P^{-1})^{-1} = P$
(3) (a) $(P \bigcup Q)(x,y) = \Big\{ h \in (P(x,y) \bigcup Q(x,y))/h \geq max(h^-_P, h^-_Q) \Big\}$
where $h^-_P = h^-_{P(x,y)}$
$h^+_{P \bigcup Q} = \max\limits_{\gamma \in (P(x,y) \bigcup Q(x,y))} \Big\{ \gamma / \gamma \geq max(h^-_P, h^-_Q) \Big\}$
now, $h^+_P \in (P \bigcup Q)(x,y)$ and $h^+_Q \in (P \bigcup Q)(x,y)$ and
$\gamma \in (P \bigcup Q)(x,y) \Rightarrow \gamma \in P(x,y) \bigcup Q(x,y)$
$\Rightarrow h^+_{P \bigcup Q} = max(h^+_P, h^+_Q)$
$\Rightarrow h^+_P \leq h^+_{P \bigcup Q}$
(b) Similar to (a)

(4) (a) $(P\bigcap Q)(x,y) = \left\{h \in (P(x,y)\bigcup Q(x,y))/h \leq min(h_P^+, h_Q^+)\right\}$

$\Rightarrow h^-_{P\bigcap Q} = \min_{\gamma\in P\bigcup Q}\left\{\gamma/\gamma \leq min(h_P^+, h_Q^+)\right\}$

$\Rightarrow h_P^- \in (P\bigcap Q)(x,y)$ and $\Rightarrow h_Q^- \in (P\bigcap Q)(x,y)$

$\Rightarrow h^-_{P\bigcap Q} \leq h_P^-$

(b) similar to (a)

(5) (a)

$$(R\bigcup Q)^{-1}(x,y) = (R\bigcup Q)(y,x)$$

$$= \left\{h \in ((R(y,x)\bigcup Q(y,x))/h \geq max(h^-_{R(y,x)}, h^-_{Q(y,x)})\right\} \tag{3.1}$$

$$R^{-1}\bigcup Q^{-1}(x,y) = \left\{h \in ((R^{-1}(x,y)\bigcup Q^{-1}(x,y))/h \geq max(h^-_{R^{-1}(x,y)}, h^-_{Q^{-1}(x,y)})\right\}$$

$$= \left\{h \in ((R(y,x)\bigcup Q(y,x))/h \geq max(h^-_{R(y,x)}, h^-_{Q(y,x)})\right\} \tag{3.2}$$

From (3.1) and (3.2) we have $(R\bigcup Q)^{-1} = R^{-1}\bigcup Q^{-1}$

(b) Similar to (a)

(6)

$$(R^{-1}(x,y))^C = (R(y,x))^C = \{1-\gamma/\gamma \in R(y,x)\} \tag{3.3}$$

$$(R^C(x,y))^{-1} = (R^C(y,x)) = \{1-\gamma/\gamma \in R(y,x)\} \tag{3.4}$$

From (3.3) and (3.4) we have $(R^{-1})^C = (R^C)^{-1}$

. □

Definition 3.8. Let R be a Hesitant fuzzy relation on X. The maximal anti-reflexive hesitant fuzzy relation contained in R is called **anti-reflexive kernel** of R, denoted by $ar(R)$.

Definition 3.9. Let R be a Hesitant fuzzy relation on X. If $R = R^{-1}$ then R is called a symmetric hesitant fuzzy relation.

Theorem 3.10. *Let R be a Hesitant fuzzy relation on X. Then $ar(R) = R\bigcap I^C$.*

Proof. $\forall x, y \in X \times X$

$$\begin{aligned}(R\bigcap I^C)(x,x) &= \left\{h \in R(x,x)\bigcup I^C(x,x)/h \leq min(R^+, (I^C)^+)\right\}\\ &= \left\{h \in R(x,x)\bigcup I^C(x,x)/h \leq min(R^+, 0)\right\}\\ &= \left\{h \in R(x,x)\bigcup \{0\}/h \leq 0\right\}\\ &= \{0\}\end{aligned}$$

so clearly $R\bigcap I^C$ is anti-reflexive.Now

$$(R\bigcap I^C)(x,y) = \begin{cases}\{0\}, & \text{for } x = y\\ R(x,y), & \text{for } x \neq y\end{cases} \tag{3.5}$$

$s(R\bigcap I^C) \leq s(R) \quad \forall x, y \in X.$
Now let T be an anti-reflexive relation and $T \preccurlyeq R$.

To prove $s(T(x,y)) \leq s(R\bigcap I^C(x,y))$
when $x = y$;

$$T(x,x) = 0 \text{ and } (R\bigcap I^C)(x,x) = 0. \tag{3.6}$$

(3.7)

when $x \neq y$

$$\begin{aligned} T(x,y) &\preccurlyeq R(x,y) \quad \text{(by assumption)} \\ &= R\bigcap I^C(x,y) \quad \text{(by (3.5))} \end{aligned} \tag{3.8}$$

$\Rightarrow T \preccurlyeq R\bigcap I^C$ (from 3.6 and 3.7)
So $R\bigcap I^C$ is maximal. Hence $ar(R) = R\bigcap I^C$.

□

Definition 3.11. Let R be a Hesitant fuzzy relation from X to X. The maximal symmetric hesitant fuzzy relation contained in R is called **symmetric kernel** of R, denoted by $sy(R)$.

Theorem 3.12. *Let R be a Hesitant fuzzy relation from X to X. Then $sy(R) = \tilde{R}^+\bigcap(\tilde{R}^{-1})^+$, where $\tilde{R}^+(x,y) = \{R^+(x,y)\}$ $\forall x,y \in X$ and $(\tilde{R}^{-1})^+(x,y) = \{(R^{-1})^+(x,y)\}$ $\forall x,y \in X$.*

Proof. By lemma 3.7(5) $(R\bigcap Q)^{-1} = R^{-1}\bigcap Q^{-1}$
Now,

$$\begin{aligned} (R\bigcap R^{-1})^{-1} &= (R^{-1})\bigcap(R^{-1})^{-1} \\ &= R^{-1}\bigcap R \\ &= R\bigcap R^{-1} \end{aligned}$$

$\Rightarrow R\bigcap R^{-1}$ is a symmetric hesitant fuzzy relation.
$R^+\bigcap(R^{-1})^+ \preccurlyeq R^+$ If T is a symmetric hesitant fuzzy relation on X and $T \preccurlyeq R$ then by lemma 3.7(1) $T^{-1} \preccurlyeq R^{-1}$. To prove $T \preccurlyeq \tilde{R}^+\bigcap(\tilde{R}^{-1})^+$

$$(\tilde{R}^+\bigcap(\tilde{R}^{-1})^+)(x,y) = \max(R^+,(R^{-1})^+) \tag{3.9}$$

$T \preccurlyeq R^+$ and $T^{-1} \preccurlyeq R^-$
$T(x,y) = T^{-1}(x,y)$ ($\because$ T is symmetric)
$\Rightarrow T \preccurlyeq R^{-1}$
$T(x,y) \preccurlyeq R(x,y) \quad \forall x,y$
$\Rightarrow T(x,y) \preccurlyeq R^+(x,y)$

$$\Rightarrow s(T(x,y)) \preccurlyeq s(R^+(x,y)) \tag{3.10}$$

$T(x,y) \preccurlyeq R^{-1}(x,y) \quad \forall x,y$
$T(x,y) \preccurlyeq (R^{-1})^+(x,y)$

$$\Rightarrow s(T(x,y)) \preccurlyeq s((R^{-1})^{+})(x,y) \tag{3.11}$$

from (3.9), (3.10) and (3.11) we have
$\therefore s(T(x,y)) \preccurlyeq s((\tilde{R}^{+} \bigcap (\tilde{R}^{-1})^{+})(x,y))$

□

Definition 3.13. $A \in HF(U)$ and $\alpha \in [0,1]$ the $\alpha - level$ cut set of hesitant fuzzy set A , denoted by $h_{A\alpha}$ is $h_{A\alpha} = \{x \in U/s(h_A(x)) \geq \alpha\}$.
$h_{A\alpha+} = \{x \in U/s(h_A(x)) > \alpha\}$ is called strong $\alpha - level$ cut set of A.
We can define

$$\begin{aligned} h^1_{A\alpha} &= \left\{x \in U/h^+_A(x) \geq \alpha\right\} and \\ h^2_{A\alpha} &= \left\{x \in U/h^-_A(x) \geq \alpha\right\}. \end{aligned}$$

Then clearly $h^2_{A\alpha} \subseteq h_{A\alpha} \subseteq h^1_{A\alpha} \quad \forall \alpha \in [0,1]$.

Theorem 3.14. *The cut sets of hesitant fuzzy sets satisfy the following properties* $\forall h_A, h_B \in HF(U), \alpha \in [0,1]$

(1) $h_A \preccurlyeq h_B \Rightarrow h_{A\alpha} \subseteq h_{B\alpha}$
(2) $(h_A \bigcap h_B)_\alpha \subseteq h_{A\alpha} \bigcap h_{B\alpha}$
(3) $(h_A \bigcup h_B)_\alpha \subseteq h_{A\alpha} \bigcup h_{B\alpha}$
(4) $\alpha 1 \geq \alpha 2 \Rightarrow h_{A\alpha 1} \subseteq h_{A\alpha 2}$

Proof. (1) To prove

$$h_A \preccurlyeq h_B \quad \Rightarrow \quad h_{A\alpha} \subseteq h_{B\alpha}$$

$$\text{we have, } h_A \preccurlyeq h_B \quad \Rightarrow \quad s(h_A(x)) \leq s(h_B(x)) \quad \forall x \in U. \tag{3.12}$$

$$x \in h_{A\alpha} \Rightarrow s(h_A(x)) \geq \alpha \tag{3.13}$$

$i.e., \alpha \leq s(h_A(x)) \leq s(h_B(x))$ from (3.12) and (3.13)
$\Rightarrow s(h_B(x)) \geq \alpha$
$\Rightarrow x \in h_{A\alpha}$
$\Rightarrow h_{A\alpha} \subseteq h_{B\alpha}$

(2) $x \in (h_A \bigcap h_B)_\alpha \Rightarrow s(h_A \bigcap h_B)(x) \geq \alpha$
$\alpha \leq s((h_A \bigcap h_B)(x)) \leq s(h_A(x)) \Rightarrow x \in h_{A\alpha}$
$\alpha \leq s((h_A \bigcap h_B)(x)) \leq s(h_B(x)) \Rightarrow x \in h_{B\alpha}$
$\Rightarrow x \in h_{A\alpha} \bigcap h_{B\alpha}$
Hence proved.
(3) Similar to (2)
(4) $x \in h_{A\alpha 1} \Rightarrow s(h_{A\alpha 1}(x)) \geq \alpha 1 \geq \alpha 2$
$\alpha 2 \leq \alpha 1 \Rightarrow x \in h_{A\alpha 2}$
$\Rightarrow h_{A\alpha 1} \subseteq h_{A\alpha 2}$

□

Corollary 3.15. *R is a HF relation on U then*
$R_\alpha = \{(x,y) \in U \times U / s(h_R(x,y)) \geq \alpha\}$
$R_\alpha(x) = \{y \in U / s(h_R(x,y)) \geq \alpha\}, \quad \forall \alpha \in [0,1]$
$R_{\alpha+} = \{(x,y) \in U \times U / s(h_R(x,y)) > \alpha\}$
$R_{\alpha+}(x) = \{y \in U / s(h_R(x,y)) > \alpha\}, \quad \forall \alpha \in [0,1]$

4. Hesitant Fuzzy Rough Sets

In this section we introduce the notion of hesitant fuzzy rough sets. The main challenge in introducing this notion is that the hesitancy factor in the membership is involved . We thus have to deal with a set of possible membership values.

Definition 4.1. Let U be a non-empty and finite universe of discourse and $R \in HF(U \times U)$. The pair (U, R) is called a Hesitant Fuzzy approximation space. For any $F \in HF(U)$, the upper and lower approximations of F w.r.t (U, R) denoted by $R_*(F)$ and $R^*(F)$, are two Hesitant Fuzzy sets as defined below

$$R_*(F)(x) = \bigcup_{y_x} \delta(x, y_x) \quad \text{s.t} \quad s(\delta(x, y_x)) \leq s(\delta(x, y)) \quad \forall y \in U$$

where $\delta(x,y) = \bigcup_{\gamma \in R^C(x,y), \phi \in F(y)} max\,\{\gamma, \phi\}$ and

$$R^*(F)(x) = \bigcup_{y_x} \Delta(x, y_x) \quad \text{s.t} \quad s(\Delta(x, y_x)) \geq s(\Delta(x, y)) \quad \forall y \in U$$

where $\Delta(x,y) = \bigcup_{\gamma \in R(x,y), \phi \in F(y)} min\,\{\gamma, \phi\}$

Here $\delta(x,y)$ and $\Delta(x,y)$ forms hesitant fuzzy sets when taken over different values of y.

$R_*(F)(x)(or R^*(F)(x))$ assigns those hesitant fuzzy elements among $\delta(x,y)(or \Delta(x,y) resp)$ which has least (or highest) scores.

$R_*(F)(x)$ and $R^*(F)(x)$ are respectively called the upper and lower approximations of F w.r.t (U, R). The pair $(R_*(F)(x), R^*(F)(x))$ is called the Hesitant Fuzzy Rough Set of F w.r.t (U, R) and $R_*, R^* : HF(U) \to HF(U)$ are referred to as upper and lower Hesitant fuzzy rough approximation operators respectively.

Theorem 4.2. *Let U be a nonempty and finite universe of discourse and $R \in HF(U \times U)$. Then the Hesitant fuzzy rough approximation operators $R_*(F)$ and $R^*(F)$ are dual to each other. i.e.,$R_*(F) = (R^*(F^C))^C$ and $R^*(F) = (R_*(F^C))^C$*

Proof. $R^*(F)(x) = \bigcup_{y_x} \Delta(x, y_x) \quad \text{s.t} \quad s(\Delta(x, y_x)) \geq s(\Delta(x, y)) \quad \forall y \in U$

where $\Delta(x,y) = \bigcup_{\gamma \in R(x,y), \phi \in F(y)} min\,\{\gamma, \phi\}$.

$(R^*(F^C))^C(x) = \bigcup_{\mu \in \Delta(x, y_x)} \{1 - \mu\}$

Now let $\mu_i \in \bigcup \Delta(x, y_x)$

$\Rightarrow \mu_i \in \bigcup_{\gamma \in R(x, y_x), \phi \in F^C(y_x)} \min\{\gamma, \phi\}$

$$
\begin{aligned}
\Rightarrow 1-\mu_i \in & \bigcup_{\gamma\in R(x,y_x),\phi\in F^C(y_x)} 1-\min\{\gamma,\phi\} \\
= & \bigcup_{\gamma\in R(x,y_x),\phi\in F^C(y_x)} \max\{1-\gamma,1-\phi\} \\
= & \bigcup_{\gamma\in R(x,y_x),\phi\in F^C(y_x)} \max\{1-\gamma,1-\phi\} \\
= & \bigcup_{\gamma'\in R^C(x,y_x),\phi'\in F(y_x)} \max\left\{\gamma',\phi'\right\} \quad where \quad \gamma'=1-\gamma \quad and \quad \phi'=1-\phi. \\
= & \delta(x,y_x)
\end{aligned}
$$

$\Rightarrow 1-\mu_i \in \delta(x,y_x)$
$\Rightarrow (R^*(F^C))^C(x) = \bigcup \delta(x,y_x)$
Because if $s(\{\mu_i\})$ is maximum then $s(\{1-\mu_i\})$ will be minimum which guarantees that $s(\delta(x,y_x)) \leq s(\Delta(x,y)) \quad \forall y \in U$. Hence proved.
Similarly $R^*(F) = (R_*(F^C))^C$. □

5. Conclusions

The introduction of Hesitant Fuzzy sets will greatly help decision making problems and will further develop other areas in which fuzzy set theory and intuitionistic fuzzy sets have already been used with great success. The study of relations on Hesitant fuzzy sets will set a theoretical base for the further development of this area. Hesitant fuzzy rough sets can be further developed in the case of information systems to widen its scope.

Relations on Hesitant fuzzy sets will enable us to introduce new hybrid structures in this framework. This will enhance scope for further research in this area.

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