On algebraic structure of soft sets

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Abstract. In this paper, we shall study the algebraic structure of the set of all soft sets defined on a fixed universe. We shall show that the set of all soft sets on a fixed set of parameters is actually a Boolean algebra. Properties of the set of all soft sets on a fixed set of parameters are studied.

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1. Introduction

The concept of soft sets was introduced by Molodtsov [13] in 1999, which is a new mathematical tool for dealing with uncertainties. Since the inception of this concept a large amount of papers devoted to development of this subject ([3, 6, 7, 8, 9, 10, 12, 15]) have appeared. Subsequently, various structures based on soft sets are developed. Some very recent works on soft sets can be found in [5, 18].

Maji et al. [12] defined various operations on soft set. Ali et al. [3] shows by counterexamples that various concepts defined in [12] are not true and they defined some new operations in soft set theory. In a subsequent paper Qin et al. [14] defined soft equality by two ways. In [14], it is proved that set of some soft sets with some suitable operations is a distributive bounded lattice. In this paper, we check which of these structures form a Boolean algebra. Sezgin et al. [16] studied several soft set operations. In a very recent paper, Rehman et al. [15] also discussed on some operations of soft sets. Also in a recent paper Zhu et al. [19] revisited operations on soft sets.


The organization of the paper is as follows:

Section 2 is the preliminary part where soft set and some operations of soft sets are defined. In section 3, we have studied whether a soft algebraic structure is a Boolean algebra or not. Also in this section we discuss the properties of all soft sets with a fixed set of parameters. In section 4, we define a new equivalence relation on the set of all soft sets on a fixed universe. The quotient algebra formed by this relation will become a Boolean algebra.

2. Preliminaries

In this section, we recall some basic notions in soft set theory. Let $U$ be an initial universe set and $E$ the set of all possible parameters under consideration with respect to $U$. The power set of $U$ is denoted by $P(U)$. Molodtsov [13] defined the notion of a soft set in the following way:

**Definition 2.1.** [13] A pair $(F, A)$ is called a soft set over $U$, where $A \subseteq E$ and $F$ is a mapping given by $F : A \rightarrow P(U)$.

If $A = \emptyset$ then also we consider $(F, \emptyset)$ as a soft set. In fact, in this case, all functions from an empty set to $P(U)$ are same and taken as empty function, so, all soft sets $(F, \emptyset)$ are same. If in stead of $F$, we write $G$, it does not matter. Qin et al. already considered such soft sets in [14]. If we do not consider such soft sets, restricted union and intersection cannot be defined for any two soft sets.

For $A \subseteq E$, $\mid A$ is the set of all not $e$’s, where $e \in A$. But in this case, we have to consider ‘not e’ also as a member of $E$, which may not hold in general. Since in the definition of the complement of a soft set $(F, A)$ is taken as $(F^c, \mid A)$, we do not consider this complement. In stead we consider only relative complement.

**Definition 2.2.** [3] The relative complement of a soft set $(F, A)$ is denoted by $(F, A)^r$ and is defined by $(F, A)^r = (F^c, A)$ where $F^c : A \rightarrow P(U)$ is defined as $F^c(e) = U - F(e)$ for all $e \in A$.

**Definition 2.3.** [12] The union of two soft sets $(F, A)$ and $(G, B)$ over a common universe $U$ is the soft set $(H, C)$, where $C = A \cup B$ and for all $e \in C$, $H(e) = F(e)$, if $e \in A - B$, $H(e) = G(e)$, if $e \in B - A$ and $H(e) = F(e) \cup G(e)$, otherwise.

We shall denote this soft set as $(F, A) \cup (G, B)$.

**Definition 2.4.** [14] The restricted intersection of two soft sets $(F, A)$ and $(G, B)$ over a common universe $U$ is the soft set $(H, C)$, where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cap G(e)$.
Here $C$ may be $\emptyset$, even when $A$ and $B$ are nonempty sets. In this paper, we call this operation as intersection and denote this soft set as $(F, A) \cap(G, B)$.

**Definition 2.5.** [14] The restricted union of two soft sets $(F, A)$ and $(G, B)$ over a common universe $U$ is the soft set $(H, C)$, where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cup G(e)$.

Here also $C$ may be $\emptyset$, even when $A$ and $B$ are nonempty sets. We shall denote it by $(F, A) \cup_e (G, B)$.

**Definition 2.6.** [14] The extended intersection of two soft sets $(F, A)$ and $(G, B)$ over a common universe $U$ is the soft set $(H, C)$, where $C = A \cup B$ and for all $e \in C$, $H(e) = F(e)$, if $e \in A - B$, $H(e) = G(e)$, if $e \in B - A$ and $H(e) = F(e) \cap G(e)$, otherwise.

We shall denote it by $(F, A) \cap_e (G, B)$.

**Definition 2.7.** [14] $(F, A)$ is called a relative null soft set (with respect to the parameter set $A$), denoted by $\emptyset_A$, if $F(e) = \emptyset$ for all $e \in A$.

$(F, A)$ is called a relative whole soft set (with respect to the parameter set $A$), denoted by $U_A$, if $F(e) = U$ for all $e \in A$.

If $A = \emptyset$ then any soft set $(F, A)$ is a relative null soft set and this set will be denoted by $\emptyset_\emptyset$. Here it may be noted that the relative whole soft set with respect to the parameter set $\emptyset$ is undefined.

**Definition 2.8.** The relative null soft set $(F, E)$ is called null soft set and the relative whole soft set $(F, E)$ is called whole soft set.

3. **Soft sets on fixed parameters**

In [14] the lattice structures of soft sets are discussed. It is proved soft sets form bounded distributive lattice under suitable operations. Here we shall further investigate on these structures.

**Theorem 3.1** ([14]). $(S(U, E), \subseteq, \cap)$ is a bounded distributive lattice, where $S(U, E) = \{(F, A) : A \subseteq U, F : A \to P(U)\}$ and $U_E$, $\emptyset$ are the greatest and least elements of the lattice respectively.

**Theorem 3.2** ([14]). $S_A$ is a sublattice of $(S(U, E), \subseteq, \cap)$, where $S_A = \{(F, A) : F : A \to P(U)\}$, i.e., $S_A$ is the set of all soft sets where the parameter set $A$ is fixed. In this lattice, $U_A$, $\emptyset_A$ are the greatest and least elements of the lattice respectively.

**Theorem 3.3.** $(S_A, \subseteq, \cap, \lor)$ is a complemented distributive lattice. In other words, $(S_A, \subseteq, \cap, \lor)$ is a Boolean algebra.

**Proof.** As $S(U, E)$ is a distributive lattice, by hereditary property, the lattice of $S_A$ is also distributive.

Now from the definition 2.2 of $(F, A)^c$, it follows that $(F, A)^c \in S_A$, as $(F, A) \in S_A$. On the set $S_A$, $(F, A) \cup (F, A)^c = (F, A) \cup (F^c, A) = (G, B)$, say. Then $B = A \cup A = A$, and for all $e \in B$, $G(e) = F(e) \cup F^c(e) = F(e) \cup (U - F(e)) = U$. Again, $(F, A)^c \cap (F, A)^c = (F, A)^c \cap (F^c, A) = (H, C)$, say. Then $C = A \cap A = A$ and for all $e \in C$, $H(e) = F(e) \cap F^c(e) = F(e) \cap (U - F(e)) = \emptyset$. Hence $S_A$ is a complemented lattice. So, $(S_A, \subseteq, \cap, \lor)$ is a Boolean algebra.  

□
With respect to other union and intersection, $S(U, E)$ is also a distributive lattice. In fact,

**Theorem 3.4** ([14]). $(S(U, E), \cup_r, \cap_e)$ is a distributive lattice.

**Theorem 3.5.** The lattice $(S(U, E), \cup_r, \cap_e)$ has a least element $\emptyset_E$ but does not have any greatest element.

Proof. Let us consider the soft set $(G, E)$, where $G(e) = \emptyset$ for all $e \in E$. So, $(G, E) = \emptyset_E$. Now, for any soft set $(F, A) \in S(U, E)$, $(F, A) \cup_r (G, E) = (H, C)$, say. Then $C = A \cap E = A$ and for all $e \in A$, $H(e) = F(e) \cup G(e) = F(e) \cup \emptyset = F(e)$. So, in this lattice, $\emptyset_E$ is the least element.

Let $(G, B)$ be the greatest element of this lattice. Let $(F, A) \in S(U, E)$. Then $(F, A) \cup_r (G, B) = (G, B)$. So, $A \cap B = B$, i.e., $B \subseteq A$. This has to be true for any $A$, consequently $B = \emptyset$. Also, for all $e \in B$, $G(e) = F(e) \cup G(e)$. So, $F(e) \subseteq G(e)$. But $G(e) = \emptyset$ as $B = \emptyset$. So, $F(e) = \emptyset$, for all $e \in A$ and for all $F$ — which is absurd for nonempty $E$. \hfill \Box

**Theorem 3.6** ([14]). $S_A$ is a sublattice of $(S(U, E), \cup_r, \cap_e)$.

Although the lattice $(S(U, E), \cup_r, \cap_e)$ is not bounded, the sublattice $S_A$ is bounded. In fact, $U_A$, $\emptyset_A$ are the greatest and least elements of this sublattice respectively.

**Theorem 3.7.** $(S_A, \cup_r, \cap_e, c)$ is a Boolean algebra.

Proof. On the set $S_A$, $(F, A) \cup_r (F, A)^c = (F, A) \cup_r (F^c, A) = (G, B)$, say. Then $B = A \cap A = A$, and for all $e \in B$, $G(e) = F(e) \cup F^c(e) = F(e) \cup (U - F(e)) = U$. Again, $(F, A) \cap_e (F, A)^c = (F, A) \cap_e (F^c, A) = (H, C)$, say. Then $C = A \cup A = A$ and for all $e \in C$, $H(e) = F(e) \cap F^c(e) = F(e) \cap (U - F(e)) = \emptyset$. \hfill \Box

This specialty of $S_A$ motivates us to study such collection.

Henceforth, in this section, we consider all soft sets taken from $S_A$, unless otherwise stated.

**Theorem 3.8.** In $S_A$, $(F, A) \cup_e (G, A) = (F, A) \cup_r (G, A)$ and $(F, A) \cap_e (G, A) = (F, A) \cap_r (G, A)$.

Proof. We already know that, $S_A$ is closed under both unions and both intersections. Now, let $(F, A) \cup_e (G, A) = (H, A)$ and $(F, A) \cup_r (G, A) = (I, A)$. Then for all $e \in A$, $H(e) = F(e) \cup G(e) = I(e)$.

Also if $(F, A) \cap_e (G, A) = (H, A)$ and $(F, A) \cap_r (G, A) = (I, A)$. Then for all $e \in A$, $H(e) = F(e) \cap G(e) = I(e)$. \hfill \Box

We already have the following De Morgan’s laws [3, 14].

**Theorem 3.9.** [3] Let $(F, A)$ and $(G, B)$ be two soft sets over the same universe $U$. Then

- $((F, A) \cup_r (G, B))^c = (F, A)^c \cap_e (G, B)^c$.
- $((F, A) \cap_e (G, B))^c = (F, A)^c \cup_r (G, B)^c$.

**Theorem 3.10.** [14] Let $(F, A)$ and $(G, B)$ be two soft sets over the same universe $U$. Then

- $((F, A) \cup_e (G, B))^c = (F, A)^c \cap_r (G, B)^c$.
Theorem 3.11. Let \((F, A)\) and \((G, A)\) be two soft sets over the same universe \(U\). Then

\[
((F, A) \cup_r (G, A))^c = (F, A)^c \cap_r (G, A)^c.
\]

\[
((F, A) \cap_r (G, A))^c = (F, A)^c \cap (G, A)^c.
\]

\[
((F, A) \cap (G, A))^c = (F, A)^c \cup (G, A)^c.
\]

Proof. It follows from Theorem 3.8. \(\square\)

In literature, we get three types of soft subsets. Let \((F, A)\) and \((G, B)\) be two soft sets over the same universe \(U\). Here \(A\) and \(B\) are not necessarily same. Then

- \((F, A)\) is called a soft \(M\)-subset of \((G, B)\) if and only if \(A \subseteq B\) and for all \(e \in A\), \(F(e) = G(e)\). This is denoted by \((F, A) \subseteq_M (G, B)\) [12].
- \((F, A)\) is called a soft \(F\)-subset of \((G, B)\) if and only if \(A \subseteq B\) and for all \(e \in A\), \(F(e) \subseteq G(e)\). This is denoted by \((F, A) \subseteq_F (G, B)\) [7]. In fact, these relations are same with the lattice order of \((S(U, E), \cup, \cap)\).
- Another type of soft subset relation also present in the lattice \((S(U, E), \cup, \cap)\) [14].
- \((F, A)\) is called a soft \(Q\)-subset of \((G, B)\) if and only if \(B \subseteq A\) and for all \(e \in B\), \(F(e) \subseteq G(e)\). This is denoted by \((F, A) \subseteq_Q (G, B)\). In fact, these relations are same with the lattice order of \((S(U, E), \cup, \cap)\).

It is easy to see that the concepts of soft \(F\)-subset and soft \(Q\)-subset coincide in \(S_A\). Also in \(S_A\), if \((F, A)\) is a soft \(M\)-subset of \((G, A)\) then \((F, A)\) is also a soft \(F\)-subset as well as soft \(Q\)-subset of \((G, A)\). But the converse may not be true as illustrated by the following example.

Example 3.12. Suppose there are five houses under consideration, which constitutes the universe \(U = \{h_1, h_2, h_3, h_4, h_5\}\). Also we have a universal parameter set \(E = \{e_1, e_2, e_3, e_4\}\), where \(e_i (i = 1, 2, 3, 4)\) stand for “beautiful”, “attractive”, “expensive” and “in good repair” respectively. For subset \(A = \{e_1, e_3\}\) of \(E\), let \((F, A)\) and \((G, A)\) be two soft sets over \(U\), where \(F(e_1) = G(e_1) = \{h_1, h_3, h_4\}\), \(F(e_3) = \{h_3, h_4, h_5\}\) and \(G(e_3) = \{h_1, h_3, h_4, h_5\}\). Then \((F, A)\) is a soft \(F\)-subset as well as soft \(Q\)-subset of \((G, A)\) but \((F, A)\) is not a soft \(M\)-subset of \((G, A)\).

Also we get three types of soft equality relation in literature. Let \((F, A)\) and \((G, B)\) be two soft sets over the same universe \(U\). Here \(A\) and \(B\) are not necessarily same. Then

- \((F, A)\) and \((G, B)\) are soft \(M\)-equal if and only if \((F, A) \subseteq_M (G, B)\) and \((G, B) \subseteq_M (F, A)\) [12].
- \((F, A)\) and \((G, B)\) are soft \(F\)-equal if and only if \((F, A) \subseteq_F (G, B)\) and \((G, B) \subseteq_F (F, A)\) [7].
- \((F, A)\) and \((G, B)\) are soft \(Q\)-equal if and only if \((F, A) \subseteq_Q (G, B)\) and \((G, B) \subseteq_Q (F, A)\) [14].

Theorem 3.13. Let \((F, A)\) and \((G, B)\) be two soft sets over \(U\). Then the following conditions are equivalent.
• \((F, A)\) and \((G, B)\) are soft \(Q\)-equal.
• \((F, A)\) and \((G, B)\) are soft \(M\)-equal.
• \((F, A)\) and \((G, B)\) are soft \(F\)-equal.
• \(A = B\) and \(F = G\).

Proof. It is easy to observe that all three concepts of soft equality relations coincide not only in \(S_A\) but also in \(S(U, E)\). In fact, the equivalence of last three are already proved in [7]. □

Another two types of soft equality relations are present in [14]. Let \((F, A)\) and \((G, B)\) be two soft sets over the same universe \(U\). Here \(A\) and \(B\) are not necessarily same. Then

• \((F, A) \approx_S (G, B)\) if and only if for all \(e \in A \cap B\), \(F(e) = G(e)\), for \(e \in A - B\), \(F(e) = \emptyset\) and for \(e \in B - A\), \(G(e) = \emptyset\).
• \((F, A) \approx^S (G, B)\) if and only if for all \(e \in A \cap B\), \(F(e) = G(e)\), for \(e \in A - B\), \(F(e) = U\) and for \(e \in B - A\), \(G(e) = U\).

Theorem 3.14. In \(S_A\), \((F, A) \approx_S (G, A)\) if and only if \((F, A) \approx^S (G, A)\).

Proof. It follows from the definitions of these soft equality relations. □

In fact in \(S_A\), all five types of soft equality coincide. So we simply call it soft equality and denote it by \(=\), i.e., if \((F, A)\) is soft equal to \((G, A)\), then we write \((F, A) = (G, A)\).

In [4] soft sets on a fixed set of parameters are also discussed. Here we compare various types of soft subsets and soft equalities which are not covered in [4]. It is trivial that \(=\) is an equivalence relation on \(S_A\). In fact, all the equivalent class of the quotient set \(S_A/=\) is singleton.

4. AN EQUIVALENCE RELATION ON \(S(U, E)\)

In this section, we shall discuss another equivalence relation on \(S(U, E)\) and study the properties of the corresponding quotient algebra. Here we can observe the structure \(S(U, E)\) very rigorously. In fact, the equivalence class we shall discuss in this section are precisely the \(S_A\) corresponding to every \(A \subseteq E\). This type of study is also not under consideration in [4].

Definition 4.1. A relation \(\rho\) on \(S(U, E)\) is a subset of \(S(U, E) \times S(U, E)\). \(\rho\) is said to be an equivalence relation on \(S(U, E)\) if it is reflexive, symmetric and transitive, in other words,

• \((F, A)\rho(F, A)\) for all \((F, A) \in S(U, E)\)
• If \((F, A)\rho(G, B)\) then \((G, B)\rho(F, A)\) for all \((F, A), (G, B) \in S(U, E)\) and
• If \((F, A)\rho(G, B)\) and \((G, B)\rho(H, C)\) then \((F, A)\rho(H, C)\) for all \((F, A), (G, B), (H, C) \in S(U, E)\).

Let \((F, A)\) and \((G, B)\) be two soft sets over the same universe \(U\) and let the parameter set is \(E\). We define \((F, A)\) and \((G, B)\) are \(\rho\)-related and write as \((F, A)\rho(G, B)\) if and only if \(A = B\). So the relation \(\rho\) depends on the parameter set. It is easy to verify that \(\rho\) is an equivalence relation. The quotient set is denoted by...
$S(U, E)/\rho$. We shall denote the equivalence class corresponding to $(F, A)$ by $[(F, A)]$. So $[(F, A)] = \{(G, B) : A = B\} = S_A$. It may be noted that $S_\emptyset$ is singleton.

We define $[(F, A)] \cup [(G, B)] := [(F, A) \cup (G, B)]$. It is routine and also easy to check that $\cup$ is well-defined. Actually, the operation depends on the parameter set of the class. So, $[(F, A)] \cup [(G, B)] = [(H, A \cup B)]$. If one define $[(F, A)] \cup [(G, B)]$ as $[(F, A) \cap_e (G, B)]$ then also we get the same $\cup$. Hence, here $S_A \cup S_B = S_{A \cup B}$.

In a similar way, we define $[(F, A)] \cap [(G, B)] := [(F, A) \cap (G, B)]$. It is well-defined too. For a similar reason, $[(F, A)] \cap [(G, B)] = [(F, A) \cap_r (G, B)]$. Hence, here $S_A \cap S_B = S_{A \cap B}$.

**Theorem 4.2.** $(S(U, E)/\rho, \cup, \cap, S_\emptyset, S_E)$ is a bounded distributive lattice.

**Proof.** Here we shall sketch a proof. The operations $\cup$ and $\cap$ defined on $S(U, E)$ are actually depend on the ordinary set theoretic union and intersection of the corresponding parameter sets respectively. Since the power set $P(E)$ of the mother parameter set $E$ is a distributive lattice with respect to set theoretic union and intersection, $(S(U, E)/\rho, \cup, \cap)$ is a distributive lattice. As $\emptyset \subseteq A \subseteq E$, $S_\emptyset$ and $S_E$ are the least element and greatest elements of this lattice respectively. \hfill $\Box$

Now we shall define the complement $c$ on $S(U, E)/\rho$ as follows. $[(F, A)]^c := [(G, E - A)]$. It is also easy to check that the unary operation is well-defined.

**Theorem 4.3.** $(S(U, E)/\rho, \cup, \cap, c, S_\emptyset, S_E)$ is a Boolean algebra.

**Proof.** It is clear that $(S_A)^c = S_{A^c}$. From this definition it follows that $(S_\emptyset)^c = S_E$. Also, $(S_A)^c = S_A$, $S_A \cup (S_A)^c = S_E$ and $S_A \cap (S_A)^c = S_\emptyset$. Thus $(S(U, E)/\rho, \cup, \cap, c, S_\emptyset, S_E)$ is a Boolean algebra. \hfill $\Box$

In this section we have studied $S(U, E)$ in a different way. $S(U, E)$ is partitioned into $S_A$'s for all different $A \subseteq E$. After that it is observed that $S(U, E)$ with respect to this partition form a Boolean algebra. $S_\emptyset$ is the least element and $S_E$ is the greatest. So here the importance of studying soft sets relative to a fixed parameter set gets a new height.

5. Conclusions

Algebraic structures of soft sets are investigated thoroughly in this paper. It is shown that soft sets on a fixed parameter set is a Boolean algebra. Here many other interesting properties of these types of sets are discussed. Soft sets on a fixed universe are partitioned in such a way that each partition is a set of soft sets on fixed parameter. Also under this partition a Boolean algebra is formed. Investigation in these directions may be a good area of research. We shall study in future the connection between soft sets and Boolean algebra that revealed in this paper.

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**References**


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