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Completeness and compactness of finite dimensional fuzzy $n-$normed linear spaces

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Abstract. Extending a recent approach of Bag & Samanta [1] towards the study of fuzzy normed linear spaces with general $t-$norm, we have studied, in this paper, completeness and compactness of finite dimensional fuzzy n-normed linear spaces.

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1. Introduction

The concept of $2-$metric spaces was introduced by Gähler in 1963 [8], where the role of the distance function in a metric space representing linear separation of points is replaced by a function representing area-separation of three points. Subsequently in 1965, he extended the idea of $2-$metric spaces to $2-$normed spaces [9]. The subject has been studied by several mathematicians like A. White and Y. J. Cho [27], R. W. Freese [6], and others. Later, Gähler himself initiated (see [10]) the idea of an $n-$norm on a linear space. Afterwards, Gunawan and Mashadi [11], Kim and Cho [13], Malceski [17], and Misiak [18] enriched the theory of fuzzy $n-$spaces.

On the other hand, after Lotfi Zadeh [28] introduced the idea of fuzzy subset in 1965 in order to formulate a theoretical background of machine intelligence, its demand has enormously increased day-by-day for its applicability in real life problems involving uncertainty, inconsistency, and vagueness.

It is well-known that metric and norm structures play vital role in the subject of functional analysis. So, in order to develop fuzzy functional analysis, one has to take care about the suitable extension of these structures. Historically, the problem of the fuzzification of the metric structure came first. In 1975, Kramosil and Michalek [14] introduced an idea of a fuzzy metric on a nonempty set X by a mapping which assigns some grade $\alpha \in (0,1]$ associated with the distance of a pair of points x, $y \in X$ and a non negative real number t. Also Zi-Ke Deng [4] introduced an idea of distance in the set of all fuzzy points of X and defined fuzzy pseudometric space in 1982. In 1984, Kaleva and Seikkala [15] defined fuzzy metric on X as a mapping which associates a fuzzy real number for each pair of elements x, y of X. In the same year, studies on fuzzy normed linear spaces were introduced by A. K. Katasaras [16] while he was studying fuzzy topological vector spaces. In 1992, Felbin [5] introduced an idea of a fuzzy norm on a linear space by assigning a fuzzy real number to each element of the linear space so that the corresponding fuzzy metric associated to this fuzzy norm is of Kaleva and Seikkala type. In 1994, Cheng and Mordeson [3] introduced another idea of a fuzzy norm on a linear space in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type. Following Cheng and Mordeson, Bag and Samanta [1] introduced the concept of a fuzzy norm on a linear space in 2003. In 2005, Vijayabalaji and Narayanan [19] extended Bag and Samanta type fuzzy normed linear spaces to fuzzy $n-$normed linear spaces. Also there are some papers and book published by S. Vijayabalaji, N. Thillaigovindan, Young Bae Jun and S. Anita Shanthi which are relevant to our paper (for reference please see [21, 24, 25]). There are also some papers and books which are relevant to our paper (for reference please see [7, 12, 20, 22, 26]).

It is worth mentioning that in Bag-Samanta type fuzzy normed spaces a decomposition theorem is derived, which actually expresses a fuzzy norm into a family of crisp norms in a unique manner and subsequently, these decomposition theorem was instrumental in developing the theory of fuzzy normed linear spaces and their duals with applications in stability theory of functional equations and fixed point theory. But ironically Bag and Samanta had to restrict the $t-$norm associated with the triangle inequality of fuzzy norm as $t_{\min}$. Recently in Bag and Samanta [2] tried to derive results on finite dimensional fuzzy normed linear spaces by waiving this restriction on the $t-$norm. Along this line of thought, in this paper, we have extended the results of Bag and Samanta in fuzzy $n-$normed linear spaces. The organisation of this paper is as follows:

Section 2 is the preliminary section. In Section 3 we first extend a crucial lemma to fuzzy $n-$normed linear space setting and afterwards derive the completeness and compactness of finite dimensional fuzzy $n-$normed linear spaces.

2. Preliminaries

In this section we discuss definitions of fuzzy norms, examples of fuzzy normed linear spaces and we state some of the results on finite dimensional fuzzy normed linear spaces regarding its compactness, completeness. More over this section contains definitions of fuzzy $n-$norm and some known results involving them.

2.1. $n-$Norm and Fuzzy $n-$Norm.

Definition 2.1 ($n-$Normed Linear Space [19])**.** Let $n \in \mathbb{N}$ (The set of all Natural numbers) and X be a real linear space of dimension $d \geq n$. (d can be infinite). A real valued function $\parallel .,., \ldots\ldots, . \parallel$ on X^n satisfying the following four properties is called an $n-$norm on the linear space X.

(1) $\parallel x_1, x_2, \ldots\ldots., x_n \parallel = 0$ if and only if $x_1, x_2, \ldots\ldots., x_n$ are linearly dependent.

(2) $\| x_1, x_2, ..., x_n \|$ is invariant under any permutation of $x_1, x_2, ..., x_n$.
(3) $\| x_1, x_2, ..., cx_n \| = | c | \| x_1, x_2, ..., x_n \|$, for any real c.
(4) $\| x_1, x_2, ..., x_{n-1}, y+z \| \leq \| x_1, x_2, ..., x_{n-1}, y \| + \| x_1, x_2, ..., x_{n-1}, z \|$ $\forall y, z \in X$.
The pair $(X, \| ., ., ..., \|)$ is called an $n-$normed linear space.

The definition of fuzzy $n-$normed linear spaces as introduced by A. L. Narayanan and S. Vijayabalaji is given below.

Definition 2.2 (Fuzzy $n-$Normed Linear Space [19])**.** Let X be a linear space over a real field $\mathbf{F}$. A fuzzy subset N of $X \times X \times .... \times X \times \mathbb{R}$ is called a fuzzy $n-$norm on X iff,
(N1) For all $t \in \mathbb{R}$ with $t \leq 0$, $N(x_1, x_2, ..., x_n, t) = 0$;
(N2) For all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, ..., x_n, t) = 1$ iff $x_1, x_2, ..., x_n$ are linearly dependent.
(N3) N$(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.
(N4) For all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, ..., cx_n, t) = N(x_1, x_2, ..., x_n, \frac{t}{|c|})$, if $c \in \mathbf{F}$ and $c \neq 0$.
(N5) For all $s, t \in \mathbb{R}$, $N(x_1, x_2, ..., x_n + x_n^{/}, s+t) \geq \min\{N(x_1, x_2, ..., x_n, s), N(x_1, x_2, ..., x, t)\}$.
(N6) $N(x_1, x_2, ..., x, t)$ is a non decreasing function of $\mathbb{R}$ and $\lim\limits_{t \to \infty} N(x_1, x_2, ..., x_n, t) = 1$.
Then the pair (X, N) is called a fuzzy $n-$normed linear space.

In [19] S. Vijayabalaji and A. L. Narayanan have introduced fuzzy $n-$norm and they decomposed a fuzzy $n-$norm to a family of crisp $n-$norms.
Later in[23], S. Vijayabalaji and N. Thillaigovindan took the following definition of a fuzzy $n-$normed linear space with general t-norm.

Definition 2.3 (Fuzzy $n-$Normed Linear Space [23])**.** Let X be a linear space over a real field $\mathbf{F}$. A fuzzy subset N of $X^n \times [0, \infty)$ is called a fuzzy $n-norm$ on X iff,
$(N1)^{/}$ $N(x_1, x_2, ..., x_n, t) > 0$;
$(N2)^{/}$ $N(x_1, x_2, ..., x_n, t) = 1$ iff $x_1, x_2, ..., x_n$ are linearly dependent.
$(N3)^{/}$ $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.
$(N4)^{/}$ $N(x_1, x_2, ...cx_n, t) = N(x_1, x_2, ..., x_n, \frac{t}{|c|})$ if $c \in \mathbf{F}$,$c \neq 0$
$(N5)^{/}$ $N(x_1, x_2, ..., x_n + x_n', s+t) \geq N(x_1, x_2, ..., x_n, s) * N(x_1, x_2, ..., x_n', t)$
$(N6)^{/}$ $N(x_1, x_2, ..., x_n, t)$ is left continuous and non-decreasing such that $\lim\limits_{t \to \infty} N(x_1, x_2, ...x_n, t) = 1$.
Then the pair (X, N) is called a fuzzy $n-$normed linear space.

Definition 2.4 ([23])**.** A sequence $\{x_n\}$in a fuzzy $n-$normed linear space (X, N) is said to converge to x if for each $y_1, y_2, \ldots, y_{n-1} \in X$, $r \in (0,1)$, $t > 0$, $\exists$ a $n_0 \in \mathbb{N}$ s.t. $N(y_1, y_2, ..., y_{n-1}, x_n - x, t) > 1 - r$ $\forall n \geq n_0$.

Theorem 2.5 ([23])**.** *In a fuzzy $n-$normed linear space (X, N) a sequence $\{x_n\}$ converge to x iff for each $y_1, y_2, \ldots, y_{n-1} \in X$, $N(y_1, y_2, ..., y_{n-1}, x_n - x, t) \to 1$ as $n \to \infty$.*

Definition 2.6 ([23])**.** A sequence $\{x_n\}$ in a fuzzy $n-$normed linear space (X, N) is said to be a Cauchy sequence if for each $y_1, y_2, \ldots, y_{n-1} \in X$ and $\epsilon \in (0,1)$, $t > 0$, $\exists$ an integer $n_o \in \mathbb{N}$ such that $N(y_1, y_2, ..., y_{n-1}, x_n - x_k, t) > 1 - \epsilon$ $\forall n, k \geq n_0$.

Theorem 2.7 ([23])**.** *In a fuzzy $n-$normed linear space (X, N) every convergent sequence is Cauchy.*

Definition 2.8 ([23])**.** A fuzzy $n-$normed linear space is said to be complete if every Cauchy sequence in it is convergent.

Definition 2.9. A fuzzy $n-$normed linear space (X, N) is said to be compact if every sequence in (X, N) has a convergent subsequence.

Definition 2.10. Let (X, N) be a fuzzy $n-$normed linear space. $A \subseteq X$ is said to be bounded if for each $y_1,\, y_2,\, \ldots,\, y_{n-1} \in X$ and $r \in (0, 1)\, \exists t_0 > 0$ such that $N(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x, t_0) > 1 - r \quad \forall x \in A$.

3. Completeness and Compactness

Firstly we take the definition of fuzzy $n-$norm in our sense.

Definition 3.1. Let X be a linear space over a field F ($\mathbb{R}$ or $\mathbb{C}$) of dimension $d \geq n$, and$*$ be a $t-norm$. A fuzzy subset N of $X^n \times \mathbb{R}$ is called a $fuzzy\ n-norm$ on X if the following conditions are satisfied:

(N1) for all $t \in \mathbb{R}$ with $t \leq 0$, $N(x_1,\, x_2,\, \ldots, x_n, t) = 0$;

(N2) for all $t \in \mathbb{R}$ with $t > 0$, $N(x_1,\, x_2,\, \ldots, x_n, t) = 1$, if and only if $x_1,\, x_2,\, \ldots, x_n$ are linearly dependent;

(N3)$N(x_1,\, x_2,\, \ldots, x_n, t)$ is invariant under any permutation of $x_1,\, x_2,\, \ldots, x_n$;

(N4) for all $t \in \mathbb{R}$ with $t > 0$ and $c \in F\ \&\ c \neq 0$,

$$N(x_1,\, x_2,\, \ldots, cx_n, t) = N\left(x_1,\, x_2,\, \ldots, x_n, \frac{t}{\mid c \mid}\right);$$

(N5) for all $s, t \in \mathbb{R}$,

$$N(x_1,\, x_2,\, \ldots, x_n + x'_n, s + t) \geq N(x_1,\, x_2,\, \ldots, x_n, s) * N(x_1,\, x_2,\, \ldots, x'_n, t);$$

(N6)

$$\lim_{t \to \infty} N(x_1,\, x_2,\, \ldots, x_n, t) = 1.$$

Then (X, N) is called a fuzzy $n-$normed linear space.

Remark 3.2. The non-decreasing property of $N(x_1,\, x_2, \ldots,\, x_n,\, \circ)$ follows from (N2) and (N5). In fact, for $s > t$,

$$\begin{aligned} N(x_1,\, x_2,\, \ldots,\, x_{n-1},\, x,\, s) &= N(x_1,\, x_2,\, \ldots,\, x_{n-1},\, x + \underline{0},\, t + (s - t)) \\ &\geq N(x_1,\, x_2,\, \ldots,\, x_{n-1},\, x,\, t) * N(x_1,\, x_2,\, \ldots, x_{n-1}, \underline{0},\, (s - t)) \\ &= N(x_1,\, x_2,\, \ldots, x_{n-1},\, x,\, t) * 1 = N(x_1,\, x_2,\, \ldots, x_{n-1},\, t) \end{aligned}$$

Example 3.3 ([19])**.** Let $(X,\, \|\bullet, \bullet,\, ,\, ,\, ,\, \bullet\|)$ be an $n-$normed spaces as in Definition 2.1. Define

$$N\left(x_1, x_2, \ldots, x_n, t\right)=\begin{cases} \frac{t}{t+\|x_1, x_2, \ldots, x_n\|} & when\, t \in \mathbb{R}^+, (x_1,\, x_2,\, \ldots, x_n) \in X^n \\ 0 & when\ t \leq 0 \end{cases}$$

Then $(X, N)$is a fuzzy $n-$normed linear space.

Example 3.4. Let (X, N) be a fuzzy $n-$normed linear space with underlying $t-norm$ = min. Define,$N^/ :\ X^n \times \mathbb{R} \rightarrow [0, 1]$ by,

$$N^/\left(x_1,\, x_2,\, \ldots,\, x_n, t\right)=\begin{cases} \frac{N(x_1, x_2, \ldots, x_n, t)+1}{2} & t>0 \\ 0 & t \leq 0 \end{cases}$$

Then (X, N) is a fuzzy $n-$normed linear space.

Theorem 3.5. *In a fuzzy $n-$normed linear space with the continuity of the underlying t-norm at the point $(1, 1)$, limit of a convergent sequence is unique.*

Proof. Let $(U,\, N)$ be a fuzzy $n-$normed linear space. Let $\{x_p\}$be a convergent sequence in U. If possible, let for $x, y \in X\ (x \neq y)$, $\{x_p\} \rightarrow x\, and\, \{x_p\} \rightarrow y\ \ as\ \ p \rightarrow \infty$. As $x \neq y\, and\, dimU \geq n$, thus $\exists$ a linearly independent set of vectors $\{u_1, u_2, u_3, ..., u_{n-1}, x-y\}$ in U. Now,

$$N(u_1, u_2, ..., u_{n-1}, x-y, 2s)$$

$$=N(u_1, u_2, ..., u_{n-1}, x-x_p+x_p-y, 2s)$$

$$\geq N(u_1, u_2, ..., u_{n-1}, x-x_p, s) * N(u_1, u_2, ..., u_{n-1}, x_p-y, s)$$

$$\rightarrow 1*1=1\ \text{ as } p \rightarrow \infty\, \forall s \in \mathbb{R}^+\ (\text{since}\, t-\text{norm is continuous at}\, (1,1))$$

$$\Rightarrow \{u_1, u_2, ..., u_{n-1,} x-y\}$$

is a linearly dependent set of vectors, which is a contradiction. □

Theorem 3.6. *Let (U, N) be a fuzzy $n-$normed linear space with underlying $t-$norm being continuous at$(1, 1)$. Let $\{x_p\}$ be a Cauchy sequence in (U, N) having a convergent subsequence. Then$\{x_p\}$ is convergent.*

Proof. Let $\epsilon > 0$ be chosen arbitrarily. By the continuity of the $t-$norm at $(1, 1)$, it follows that $\exists\ \delta > 0$ s.t.

$$(1-\delta)*(1-\delta)>(1-\epsilon)\ \ldots..(1)$$

Let $\{x_{p_l}\}$ be a convergent subsequence of $\{x_p\}$ converging to x "say" in U. Since $x_{p_l} \rightarrow x$ as $l \rightarrow \infty$, for fixed $y_1,\, y_2,\, \ldots.,\, y_{n-1} \in U$ and for any given $t \in \mathbb{R}^+, \exists\, k_1 \in \mathbb{N}$ such that

$$N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x_{p_l}-x,\, t\right)>1-\delta\, \forall\, l \geq k_1$$

Since $\{x_p\}$is Cauchy, $\exists\, k_2 \in \mathbb{N}$ such that

$N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x_m-x_k,\, t\right)>1-\delta\, \forall\, m, k \in \mathbb{N} \geq k_1$ with $m \geq k_2, k \geq k_2, \forall t \in \mathbb{R}^+$.

Let $k' = \max\{k_1, k_2\}$.

$$\therefore N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x_p - x,\, 2t\right)$$

$$= N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x_p - x_{p_l} + x_{p_l} - x,\, t\right)$$

$$\geq N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x_p - x_{p_l},\, t\right)$$

$$*N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x_{p_l} - x,\, t\right)$$

$$\geq (1-\delta) * (1-\delta) > (1-\epsilon) \;\; (From \;\; (1))$$

$$\forall\, p, l \in \mathbb{N} \text{ with } p \geq k',\, l \geq k'.$$

Thus $\lim\limits_{p\to\infty} N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x_p - x,\, 2t\right) = 1$. Since this is true for any $y_1,\, y_2,\, \ldots, y_{n-1}$, and $t \in \mathbb{R}^+$, it follows that $\{x_p\} \to x$. □

Lemma 3.7. *Let (U, N) be a fuzzy $n-$normed linear space with underlying $t-$ norm associated with the fuzzy $n-$norm N be continuous at $(1,1)$. If $\{x_1, x_2, ..., x_k\}$ be a linearly independent set of vectors in U, then $\exists c > 0,\, \delta > 0$ such that for each set of scalars $\{\alpha_1, \alpha_2, ..., \alpha_k\}$ $\exists\, y_1,\, y_2, ...,\, y_{n-1} \in U$ such that*

$$N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, \sum_{i=1}^{k} \alpha_i x_i,\, c\sum_{i=1}^{k} \mid \alpha_i \mid\right) < 1 - \delta \ldots.. (1)$$

Proof. Let $S = \sum\limits_{i=1}^{k} \mid \alpha_i \mid$. If $S = 0$, then $\mid \alpha_i \mid = 0$, $\forall i = 1,\, 2,\, \ldots,\, k$. Now $\mathrm{N}(y_1,\, y_2,\, \ldots,\, y_{n-1},\, \theta,\, 0) = 0$ and the result holds for any $c > 0$ and $\delta \in (0,\, 1)$. Next let $S \neq 0$. Then (1) is equivalent to

$$N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, \sum_{i=1}^{k} \beta_i x_i,\, c\right) < 1 - \delta \ldots.. (2).$$

where $\beta_i = \frac{\alpha_i}{S}, i = 1, 2, ..., k$, so that $\sum\limits_{i=1}^{k} \mid \beta_i \mid = 1$. If possible, let (2) does not hold.

Then for any $c > 0$ and $\delta \in (0, 1), \exists$ a set of scalars $\beta_1, \beta_2, ..., \beta_k$ with $\sum\limits_{i=1}^{k} \mid \beta_i \mid = 1$ such that for any $y_1,\, y_2,\, \ldots.,\, y_{n-1} \in U$

$$N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, \sum_{i=1}^{k} \beta_i x_i,\, c\right) \geq 1 - \delta \ldots.. (3).$$

In particular, for each positive integer m, if we choose $c = \delta = \frac{1}{m}$ then $\exists$ a set of scalars $\left\{\beta_1^{(m)}, \beta_2^{(m)}, \ldots \beta_k^{(m)}\right\}$ with $\sum\limits_{i=1}^{k} \mid \beta_i^{(m)} \mid = 1$ such that for each $y_1,\, y_2,\, \ldots.,\, y_{n-1} \in U$

$$N\left(y_1, y_2, \ldots y_{n-1}, z_m, \frac{1}{m}\right) \geq 1 - \frac{1}{m}$$

where,

$$z_m = \beta_1^{(m)} x_1 + \beta_2^{(m)} x_2 + \beta_3^{(m)} x_3 + \ldots + \beta_k^{(m)} x_k,\ m = 1, 2, ..$$

Since,

$$\sum_{i=1}^{k} \mid \beta_i^{(m)} \mid = 1,\ \ m = 1, 2, ..$$

we have,

$$0 \leq \mid \beta_i^{(m)} \mid \leq 1,\ i = 1, 2, \ldots k;\ m = 1, 2, \ldots$$

So, for each fixed i, $\left\{\beta_i^{(m)}\right\}_{m=1}^{\infty}$ is bounded and hence, in particular, $\left\{\beta_1^{(m)}\right\}_{m=1}^{\infty}$ is also so. Thus, $\left\{\beta_1^{(m)}\right\}_{m=1}^{\infty}$ has a convergent subsequence converging to β_1. Let $\{z_{1,m}\}$ denote the corresponding subsequence of $\{z_m\}$. By the same argument, as above $\{z_{1,m}\}$ has a subsequence $\{z_{2,m}\}$ for which the corresponding subsequence of scalars $\left\{\beta_2^{(m)}\right\}$ converges to β_2. Continuing in this way after k steps we obtain a subsequence $\{z_{k,m}\}_{m=1}^{\infty}$, where $z_{k,m} = \sum\limits_{i=1}^{k} \gamma_i^{(m)} x_i$ with $\sum\limits_{i=1}^{k} \mid \gamma_i^{(m)} \mid = 1$ and $\gamma_i^{(m)} \to \beta_i$ as $m \to \infty$ $\ i = 1, 2, .., k$. Let $y = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k$. Then we have, $\forall t \in \mathbb{R}^+$ and $\forall\, y_1,\, y_2,\, \ldots,\, y_{n-1} \in U$,

$$\lim_{m \to \infty} N\left(y_1,\, y_2,\, ..,\, y_{n-1}, \left(z_{k,m} - y\right), t\right)$$

$$= \lim_{m \to \infty} N\left(y_1,\, y_2,\, ..,\, y_{n-1}, \left(\gamma_1^{(m)} - \beta_1\right) x_1 + \left(\gamma_2^{(m)} - \beta_2\right) x_2 + \ldots + \left(\gamma_k^{(m)} - \beta_k\right) x_k,\ \frac{nt}{t}\right)$$

$$\begin{aligned} \geq \lim_{m \to \infty} & \left(N\left(y_1,\, y_2,\, \ldots,\, y_{n-1}, \left(\gamma_1^{(m)} - \beta_1\right) x_1,\ \frac{t}{n}\right)\right) \\ & * \left(N\left(y_1,\, y_2,\, \ldots,\, y_{n-1}, \left(\gamma_2^{(m)} - \beta_2\right) x_2,\ \frac{t}{n}\right)\right) \\ & . \\ & . \\ & . \\ & * \left(N\left(y_1,\, y_2,\, \ldots,\, y_{n-1}, \left(\gamma_k^{(m)} - \beta_k\right) x_k,\ \frac{t}{n}\right)\right) \end{aligned}$$

$$= \lim_{m\to\infty}\{N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x_1,\, \frac{t}{n \mid \gamma_1^{(m)} - \beta_1 \mid}\right)$$

$$N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x_2,\, \frac{t}{n \mid \gamma_2^{(m)} - \beta_2 \mid}\right)$$

$$.$$

$$.$$

$$.N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x_k,\, \frac{t}{n \mid \gamma_k^{(m)} - \beta_k \mid}\right)\}$$

$$= \lim_{m\to\infty} N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x_1,\, \frac{t}{n \mid \gamma_1^{(m)} - \beta_1 \mid}\right)$$

$$\lim_{m\to\infty} N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x_2,\, \frac{t}{n \mid \gamma_2^{(m)} - \beta_2 \mid}\right)$$

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$$\lim_{m\to\infty} N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x_k,\, \frac{t}{n \mid \gamma_k^{(m)} - \beta_k \mid}\right)$$

$$= 1 * 1 * \ldots. * 1 \text{ (By the continuity of } t - norm\, at\, (1,1)).$$

$$= 1.$$

Hence $\lim_{m\to\infty} N\left(y_1,\, y_2,\, \ldots, y_{n-1},\, (z_{k,m} - y)\,,\, t\right) = 1, \forall t \geq 0$. Now for $s > 0$, choose m such that $\frac{1}{m} < s$. Then we have,

$$N\left(y_1,\, y_2,\, \ldots, y_{n-1}, z_{km},\, s\right)$$

$$= N\left(y_1,\, y_2,\, \ldots, y_{n-1}, z_{km} + \underline{0},\, s + \frac{1}{m} - \frac{1}{m}\right)$$

$$\geq N\left(y_1,\, y_2,\, \ldots, y_{n-1}, z_{km},\, \frac{1}{m}\right)$$

$$*N\left(y_1,\, y_2,\, \ldots, y_{n-1}, \underline{0},\, s - \frac{1}{m}\right)$$

$$\geq \left(1 - \frac{1}{m}\right) * N\left(y_1,\, y_2,\, \ldots, y_{n-1}, \underline{0},\, s - \frac{1}{m}\right)$$

$$= \left(1 - \tfrac{1}{m}\right) * 1$$

{ since $y_1,\ y_2,\ \ldots,\ y_{n-1},\ \underline{0}$ are linearly dependent so that $\mathrm{N}(y_1,\ y_2,\ \ldots, y_{n-1},\ \underline{0},\ t) = 1, \forall t \in \mathbb{R}^+$.}

Thus $z_{k,m} \to 0$ as $m \to \infty$.

But we have seen that $z_{k,m} \to y$ as $m \to \infty$. Also, limit of a convergent sequence in (U, N) is unique. So $y = \underline{0} \Rightarrow \beta_1 = \beta_2 = \ \ldots = \beta_k = 0$.

Now $\gamma_i^{(m)} \to \beta_i$ as $m \to \infty$ for i = 1, 2, . . , ., k, where $\sum\limits_{i=1}^{k} \mid \gamma_i^{(m)} \mid = 1,\ m = 1,\ 2,\ \ldots\ldots$

Thus $\sum\limits_{i=1}^{k} \mid \beta_i \mid = 1$, which contradicts the fact that $\beta_1 = \beta_2 = \ \ldots = \beta_k = 0$.

So (2) holds. □

Theorem 3.8. *Let (U, N) be a finite dimensional fuzzy $n-$normed linear space with underlying $t-$norm being continuous at $(1, 1)$. Then (U, N) is complete.*

Proof. Let $dimU = k$ and $\{e_1, e_2, \ldots, e_k\}$ be a basis for U. Let $\{x_n\}$ be a Cauchy sequence in (U, N). Then $\exists$ scalars $\beta_i^{(n)},\ i = 1, 2, \ldots, k\,;\ \ n = 1, 2, \ldots,$ s.t.

$$x_n = \sum_{i=1}^{k} e_i \beta_i^{(n)}, n = 1, 2, \ldots$$

Since $\{x_n\}$ is Cauchy, so, for each $y_1,\ y_2,\ \ldots, y_{n-1} \in U$,

$$\lim_{r,s\to\infty} N\left(y_1,\ y_2,\ \ldots,\ y_{n-1}, x_r - x_s, t\right) = 1\ \forall\ t \in \mathbb{R}^+ \ldots\ldots(1)$$

Now by Lemma 3.7, $\exists c > 0$ and $\delta \in (0, 1)$ such that

$$N\left(y_1,\ y_2,\ \ldots, y_{n-1},\ \sum_{i=1}^{k} e_i \left(\beta_i^{(r)} - \beta_i^{(s)}\right),\ c\sum_{i=1}^{k} \mid \beta_i^{(r)} - \beta_i^{(s)} \mid\right) < 1 - \delta$$

(1) implies that for $t > 0$ and $s > 0$ $\exists p \in \mathbb{N}$ such that

$$N\left(y_1,\ y_2\,,\ \ldots, y_{n-1}, x_r - x_s, t\right) > 1 - \delta\, \forall\, r, s \in \mathbb{N},\ r \geq p, s \geq p\ and\ \forall t > 0.$$

Hence,

$$N\left(y_1,\ y_2,\ \ldots, y_{n-1},\ \sum_{i=1}^{k} e_i \left(\beta_i^{(r)} - \beta_i^{(s)}\right),\ c\sum_{i=1}^{k} \mid \beta_i^{(r)} - \beta_i^{(s)} \mid\right) < 1 - \delta < N\left(y_1,\ y_2,\ \ldots y_{n-1}, x_r - x_s, t\right),$$

$\forall r, s \in \mathbb{N}$ with $r \geq p, s \geq p$ and $\forall t > 0$. Hence,

$$c\sum_{i=1}^{k} \mid \beta_i^{(r)} - \beta_i^{(s)} \mid \leq t\, \forall r, s {\in}\ \mathbb{N}\ with\ r \geq p, s \geq p\, \&\, \forall t > 0.$$

i.e,

$$\sum_{i=1}^{k} \mid \beta_i^{(r)} - \beta_i^{(s)} \mid \leq \frac{t}{c} \forall\, r, s {\in}\ \mathbb{N}\ with\, r \geq p, s \geq p\ \&\ \forall t > 0.$$

i.e,

$$\Rightarrow \mid \beta_i^{(r)} - \beta_i^{(s)} \mid \leq \frac{t}{c}\ \forall r, s \in \mathbb{N}\, with\ r \geq p, s \geq p\ \&\ \forall t > 0,\ for\, i = 1, 2, \ldots, k.$$

Thus, for each $i = 1, 2, \ldots, k$, $\left\{\beta_i^{(n)}\right\}$is a Cauchy sequence in $\mathbb{C}$. As $\mathbb{C}$ is complete so, $\left\{\beta_i^{(n)}\right\}_{n=1}^{\infty}$is convergent,$\forall\, i = 1, 2, \ldots, k$. Let $y = \sum\limits_{i=1}^{k} \beta_i e_i$.

So for $y_1,\, y_2,\, \ldots,\, y_{n-1} \in U$,

$$N\left(y_1,\, y_2,\, \ldots, y_{n-1},\, x_p - x,\, t\right)$$

$$= N\left(y_1,\, y_2,\, \ldots, y_{n-1},\, \sum_{i=1}^{k} e_i \left(\beta_i^{(p)} - \beta_i\right),\, t\right)$$

$$= N\left(y_1,\, y_2,\, \ldots, y_{n-1},\, \sum_{i=1}^{k} e_i \left(\beta_i^{(p)} - \beta_i\right),\, \frac{kt}{k}\right)$$

$$\geq N\left(y_1,\, y_2,\, \ldots, y_{n-1},\, e_1,\, \frac{t}{k \mid \beta_1^{(p)} - \beta_1 \mid}\right)$$

$$* N\left(y_1,\, y_2,\, \ldots, y_{n-1},\, e_2,\, \frac{t}{k \mid \beta_2^{(p)} - \beta_2 \mid}\right)$$

$$\vdots$$

$$* N\left(y_1,\, y_2,\, \ldots, y_{n-1},\, e_k,\, \frac{t}{k \mid \beta_k^{(p)} - \beta_k \mid}\right)$$

Now as $p \to \infty$, $\frac{t}{k|\beta_i^{(p)} - \beta|} \to \infty$ for $i = 1,\, 2,\, \ldots,\, k$.

So,

$$\lim_{p\to\infty} N\left(y_1,\, y_2,\, \ldots, y_{n-1},\, e_i,\, \frac{t}{k \mid \beta_i^{(p)} - \beta_i \mid}\right) = 1 \, for \, i = 1,\, 2,\, \ldots,\, k.$$

Using continuity of $t-$ norm at $(1, 1)$, we get

$$\lim_{p\to\infty} N\left(y_1,\, y_2,\, \ldots, y_{n-1},\, x_p - x,\, t\right) = 1$$

Thus (U, N) is complete. □

Theorem 3.9. *Let (U, N) be a fuzzy $n-$normed linear space. Let M be a finite dimensional subspace of U of dimension at least n then M is closed.*

Proof. For simplicity of notation the restriction of N on $M^n \times [0, 1]$ is also denoted by N. By Theorem 3.8, M is complete. Let $\{x_n\}$ be a sequence in M converging to x in U. As$\{x_n\}$is convergent so is Cauchy. But(M, N) is complete, hence $\{x_n\}$converges to a point in M. But limit of a convergent sequence is unique so $x \in M$. Thus M is closed. □

Theorem 3.10. *Let* (U, N) *be a fuzzy* $n-$*normed linear space, with underlying* $t-$*norm is continuous at* $(1,1)$*. Then a subset* A *of* U *is compact implies* A *is closed and bounded.*

Proof. First we establish that$A$ is bounded. If possible, suppose that A is not bounded. Then $\exists y_1,\, y_2,\, \ldots,\, y_{n-1} \in U$ and $r \in (0,1)$ such that for each $m > 0$, $\exists x_m \in A$ such that

$$N\left(y_1,\, y_2,\, \ldots,\, x_m,\, m\right) \leq 1 - r \ldots . (1)$$

Now $\{x_m\}$ is a Cauchy sequence in A. As A is compact and$\{x_m\}$is a sequence in A, so there exists a convergent subsequence $\{x_{m_l}\}$ of $\{x_m\}$ converging to a point, "say" x in$A$. So for each $y_1,\, y_2,\, \ldots,\, y_{n-1} \in U$

$$\underset{l \to \infty}{lim} N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x_{m_l} - x,\, t\right) = 1, \forall t > 0 \ldots . (2)$$

From (1),

$$N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x_{m_l},\, m_l\right) \leq 1 - r.$$

So,

$$\begin{aligned} 1 - r &\geq N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x_{m_l} - x + x,\, m_l - t + t\right) \\ &\geq N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x_{m_l} - x,\, t\right) * N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x,\, m_l - t\right) \end{aligned}$$

$\Rightarrow 1 - r \geq 1 * 1 = 1 \,\text{as}\, l \to \infty$, using continuity of $t-$norm at $(1,1)$,
$\Rightarrow r \leq 0$, a contradiction.
So A is bounded.

Next, consider a sequence $\{x_n\}$ in A converging to a point x in U. As A is compact the sequence $\{x_n\}$ has a convergent subsequence$\{x_{n_l}\}$, "say", converging to a point x in A. As every subsequence of a convergent sequence converge to the same limit of the convergent sequence and the limit of a convergent sequence in (U,N) is unique, the subsequence $\{x_{n_l}\}$converge to x. Hence $x \in A$. Thus A is closed. □

Remark 3.11. If, however, X is finite-dimensional then the converse also holds, which is shown in the following theorem.

Theorem 3.12. *Let* (U, N)*be a finite-dimensional fuzzy* $n-$*normed linear space with underlying* $t-$*norm be continuous at* $(1,1)$ *then if* $A \subseteq U$ *be closed and bounded then* A *is compact.*

Proof. Let A be closed and bounded and let $dimU = k$. Let $B = \{e_1, e_2, \ldots, e_k\}$be a basis for U and let $\{x_m\}$ be a sequence in A. As B is a basis for U, $\exists$ scalars $\beta_i^{(m)}$, $i = 1, 2, \ldots, k$ such that

$$x_m = \sum_{i=1}^{k} \beta_i^{(m)} e_i,\; m = 1, 2, \ldots$$

By Lemma 3.7, $\exists c > 0 \,\text{and}\, \delta \in (0,1)$ such that for scalars $\beta_i^{(m)}, i = 1, 2, \ldots, k$, $\exists$ $y_1, y_2, \ldots, y_{n-1} \in U$, such that

$$N\left(y_1,\, y_2,\, \ldots, y_{n-1},\, \sum_{i=1}^{k} \beta_i^{(m)} e_i,\, c \sum_{i=1}^{k} \mid \beta_i^{(m)} \mid\right) < 1 - \delta.$$

Since $\{x_m\}$ is bounded, $\delta \in (0,1)$ and $y_1, y_2, \ldots, y_{n-1} \in U$, $\exists t > 0$ such that

$$N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, \sum_{i=1}^{k} \beta_i^{(m)} e_i, t\right) > 1 - \delta; m = 1, 2, \ldots$$

Hence,

$$N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, \sum_{i=1}^{k} \beta_i^{(m)} e_i,\, c\sum_{i=1}^{k} \mid \beta_i^{(m)} \mid\right) < 1 - \delta < N\left(y_1,\, y_2,\, \ldots y_{n-1},\, \sum_{i=1}^{k} \beta_i^{(m)} e_i, t\right), m = 1, 2, \ldots$$

$$\therefore c\sum_{i=1}^{k} \mid \beta_i^{(m)} \mid \leq t,\ m = 1,\, 2,\, ..$$

$$\Rightarrow \sum_{i=1}^{k} \mid \beta_i^{(m)} \mid \leq \frac{t}{c},\, m = 1\,, 2, \ldots$$

So

$$\mid \beta_i^{(m)} \mid \leq \frac{t}{c}, \forall 1 \leq i \leq k, \forall m = 1,\, 2,\, \ldots.$$

Thus for each $i = 1,\, 2,\, \ldots, k$, $\left\{\beta_i^{(m)}\right\}_{m=1}^{\infty}$ is bounded in $\mathbb{R}$ or$\mathbb{C}$ and hence, in particular, $\left\{\beta_i^{(m)}\right\}_{m=1}^{\infty}$ is bounded and so it has a convergent subsequence$\left\{\beta_1^{(m,1)}\right\}_m$(say) converging to a point β_1 (say) in $\mathbb{R}$or $\mathbb{C}$. Let $\{x_{m,1}\}$be the corresponding subsequence of $\{x_m\}$.Again considering the bounded sequence$\left\{\beta_2^{(m,2)}\right\}$(say) , and let $\{x_{m,2}\}$ be the corresponding subsequence of $\{x_{m,1}\}$. Let after $k-$steps $\{x_{m,k}\}$ be the corresponding subsequence of $\{x_m\}$, where $x_{m,k} = \sum\limits_{i=1}^{k} \beta_1^{(m,k)} e_i, i = 1,\, 2,\, \ldots.$ and $\left\{\beta_i^{(m,k)}\right\}_m$ are convergent, $\forall 1 \leq i \leq k$. Let $\underset{m\to\infty}{lim} \beta_i^{(m,k)} = \beta_i, 1 \leq i \leq k$. and $y = \sum\limits_{i=1}^{k} \beta_i e_i$

Now for $y_1,\, y_2 \ldots,\, y_{n-1} \in U$,

$$N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x_{k,m} - y,\, t\right)$$

$$\begin{aligned}
&= N\left(y_1,\, y_2,\, .,\, \ldots,\, y_{n-1},\, \sum_{i=1}^{k}\left(\beta_i^{(m,k)} - \beta_i\right) e_i,\, \tfrac{kt}{k}\right) \\
&\geq N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, \left(\beta_1^{(m,k)} - \beta_1\right) e_1,\, \tfrac{t}{k}\right) \\
&\quad * N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, \left(\beta_2^{(m,k)} - \beta_2\right) e_2,\, \tfrac{t}{k}\right) \\
&\quad . \\
&\quad . \\
&\quad . \\
&\quad * N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, \left(\beta_k^{(m,k)} - \beta_k\right) e_k,\, \tfrac{t}{k}\right)
\end{aligned}$$

$$
\begin{aligned}
= \quad & N\left(y_1,\, y_2,\, \ldots, y_{n-1},\, e_1,\, \tfrac{t}{k|\beta_1^{(m,k)}-\beta_1|}\right) \\
& *N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, e_2,\, \tfrac{t}{k|\beta_2^{(m,k)}-\beta_2|}\right) \\
& . \\
& . \\
& . \\
& *N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, e_k,\, \tfrac{t}{k|\beta_k^{(m,k)}-\beta_k|}\right)
\end{aligned}
$$

As $m \to \infty, \beta_i^{(m,k)} \to \beta_i$ and hence $\frac{t}{k|\beta_i^{(m,k)}-\beta_i|} \to \infty, \forall\ 1 \leq i \leq k.$

Hence

$$
\begin{aligned}
N\left(y_1,\, y_2,\, \ldots,\, y_{n-1},\, x_{k,m} - y,\, t\right) &\geq 1 * 1 *\ \ldots * 1\,(k\,\text{times}) \\
&= 1,
\end{aligned}
$$

(using continuity of $t-$norm at$(1,1)$).

$$\Rightarrow x_{k,m} \to y\, as\, m \to \infty.$$

So, the sequence $\{x_m\}$ has a convergent subsequence converging to a point in X. The set A is closed and $\{x_m\}$ is a sequence in A. Also $\{x_{k,m}\}$ is a convergent subsequence of $\{x_m\}$ converging to y, and hence $y \in A$. As $\{x_m\}$ is an arbitrary sequence in A, it follows that A is compact. □

4. Conclusions

Extending a recent approach of Bag & Samanta [1] towards the study of fuzzy normed linear spaces with general $t-$norm, we have been able to develop finite dimensional fuzzy $n-$normed linear spaces and have studied compactness and completeness in such spaces. As for fuzzy normed linear spaces with general t-norm, the Bag-Samanta decomposition theorem is not applicable, so that a different technique is required to handle such situations. There is a wide scope of research in studying fuzzy normed linear spaces as well as fuzzy $n-$normed linear space with underlying general $t-$norm setting in the triangle inequality of the fuzzy norm (fuzzy $n-$norm), because it is just a begining.

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