Interval-valued fuzzy quasi-ideals and bi-ideals of semirings

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Received 01 July 2013; Revised 17 July 2013; Accepted 26 July 2013

ABSTRACT. The interval-valued prime fuzzy ideals (in brevity, the i-v. prime fuzzy ideals) of a semigroup have been recently studied by Kar, Sarkar and Shum [18]. As a continued study of i-v fuzzy ideals, we are going to investigate the properties of i-v fuzzy quasi-ideal and i-v fuzzy bi-ideal of a semiring and then we characterize the regularity and intra-regularity of a semiring in terms of the above i-v fuzzy ideals.

2010 AMS Classification: 08A72

Keywords: Semirings, Interval Numbers, i-v Fuzzy Quasi-ideals, i-v Fuzzy Bi-ideals, Regular Semirings, Intra-regular Semirings.

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1. Introduction

Quasi-ideals of rings and semigroups were introduced and investigated by O. Steinfeld [24], [25], [26]. In 1975, R. A. Good and D. R. Hughes [11] also considered the bi-ideals in semirings. The quasi-ideals are generalization of left ideals and right ideals whereas the bi-ideals are the generalization of quasi-ideals. The properties of fuzzy subquasi semigroup of a quasigroup were investigated by W. A. Dudek in [7]. S. Kar and P. Sarkar considered the fuzzy quasi-ideals and fuzzy bi-ideals of a ternary semigroup in [16].

The notion of i-v fuzzy set was introduced by L. A. Zadeh in 1975 [29]. Later on, I. Grattan-Guinness [12], K. U. Jahn [14] and R. Sambuc [23] studied the i-v fuzzy sets and they regarded this kind of fuzzy sets as a generalization of the ordinary fuzzy set. In fact, i-v fuzzy sets (in short, IVFS) are defined in terms of i-v membership functions.

After the i-v fuzzy sets have been introduced (see [3], [4], [5], [6], [13], [15], [17], [20], [21], [27]), some theories related with i-v fuzzy sets have been developed. There are natural ways to fuzzify various algebraic structures and the approaches
have already been extensively studied in the literature. In particular, A. Rosenfeld [22] studied the fuzzy subgroups in 1971. Also, N. Kuroki in 1979 [19] further mentioned the fuzzy semigroups. In 1993, J. Ahsan, K. Saifullah and M. Farid Khan [1] introduced the fuzzy semirings. Recently, many interesting results of semirings have been obtained and given by using the context of fuzzy sets.

In this paper, we first introduce i-v fuzzy quasi-ideals and i-v fuzzy bi-ideals of a semiring. Then, we proceed to characterize the regular and intra-regular semirings by using the i-v fuzzy quasi-ideals and i-v fuzzy bi-ideals of the semirings. Our study of i-v fuzzy quasi-ideals in this paper is a continued study of our recent work on i-v fuzzy ideals of a semigroup [18].

2. Preliminaries

**Definition 2.1.** [10] A non-empty set $S$ together with two binary operations ‘$+$’ and ‘$\cdot$’ is said to be a semiring if (i) $(S, +)$ is an abelian semigroup; (ii) $(S, \cdot)$ is a semigroup and (iii) $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$ for all $a, b, c \in S$.

Let $(S, +, \cdot)$ be a semiring. If there exists an element ‘$0_S$’ in $S$ such that $a + 0_S = a$ and $a \cdot 0_S = 0_S = 0_S \cdot a$ for all $a \in S$; then ‘$0_S$’ is called the zero element of $S$.

Throughout this paper, we consider a semiring $(S, +, \cdot)$ with a zero element ‘$0_S$’. Unless otherwise stated, a semiring $(S, +, \cdot)$ will be simply denoted by $S$ and the multiplication ‘$\cdot$’ will be denoted by juxtaposition. In this paper, by the product $AB$ of two subsets $A$ and $B$ of a semiring $S$, we mean the finite sum $\sum_{i=1}^{n} a_i b_i$, for some $a_i \in A$, $b_i \in B$ and $n \in \mathbb{Z}^+$. 

**Definition 2.2.** [15] An interval number on $[0, 1]$, denoted by $\tilde{a}$, is defined as the closed subinterval of $[0, 1]$, where $\tilde{a} = [a^-, a^+]$ satisfying $0 \leq a^- \leq a^+ \leq 1$.

For any two interval numbers $\tilde{a} = [a^-, a^+]$ and $\tilde{b} = [b^-, b^+]$, we define the followings:

(i) $\tilde{a} \leq \tilde{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$.
(ii) $\tilde{a} = \tilde{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$.
(iii) $\tilde{a} < \tilde{b}$ if and only if $\tilde{a} \neq \tilde{b}$ and $\tilde{a} \leq \tilde{b}$.

**Note 2.3.** We write $\tilde{a} \geq \tilde{b}$ whenever $\tilde{b} \leq \tilde{a}$ and $\tilde{a} > \tilde{b}$ whenever $\tilde{b} < \tilde{a}$.

We denote the interval number $[0, 0]$ by $\tilde{0}$ and $[1, 1]$ by $\tilde{1}$.

**Definition 2.4.** [6] Let $\{\tilde{a}_i : i \in \Lambda\}$ be a family of interval numbers, where $\tilde{a}_i = [a_i^-, a_i^+]$. Then we define $\sup\{\tilde{a}_i\} = [\sup_{i \in \Lambda} a_i^-, \sup_{i \in \Lambda} a_i^+]$ and $\inf\{\tilde{a}_i\} = [\inf_{i \in \Lambda} a_i^-, \inf_{i \in \Lambda} a_i^+]$.

We denote the set of all interval numbers on $[0, 1]$ by $D[0, 1]$.

Let us recall the following known definitions.

**Definition 2.5.** [28] Let $S$ be a non-empty set. Then a mapping $\mu : S \rightarrow [0, 1]$ is called a fuzzy subset of $S$. 

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Lemma 2.14. Then we define their intersection and union by
\[ \mu^{-}(x), \mu^{+}(x) \] for all \( x \in S \), for any i-v fuzzy subset \( \mu \) of a non-empty set \( S \), where \( \mu^{-} \) and \( \mu^{+} \) are some fuzzy subsets of \( S \).

We state below several definitions which will be useful in further study of this paper.

Definition 2.8. Let \( \mu_{1} \) and \( \mu_{2} \) be two i-v fuzzy subsets of a set \( X \neq \emptyset \). Then \( \mu_{1} \) is said to be subset of \( \mu_{2} \), denoted by \( \mu_{1} \subseteq \mu_{2} \) if \( \mu_{1}(x) \leq \mu_{2}(x) \) i.e. \( \mu_{1}(x) \leq \mu_{2}(x) \) and \( \mu_{1}^{+}(x) \leq \mu_{2}^{+}(x) \), for all \( x \in X \) where \( \mu_{1}(x) = [\mu_{1}^{-}(x), \mu_{1}^{+}(x)] \) and \( \mu_{2}(x) = [\mu_{2}^{-}(x), \mu_{2}^{+}(x)] \).

Definition 2.9. The interval Min-norm is a function \( \text{Min}^{i} : D(0, 1] \times D(0, 1] \rightarrow D(0, 1] \), defined by:
\[
\text{Min}^{i}(\tilde{a}, \tilde{b}) = [\text{min}(a^{-}, b^{-}), \text{min}(a^{+}, b^{+})] \quad \text{for all} \quad \tilde{a}, \tilde{b} \in D(0, 1], \quad \text{where} \quad \tilde{a} = [a^{-}, a^{+}] \quad \text{and} \quad \tilde{b} = [b^{-}, b^{+}].
\]

Definition 2.10. The interval Max-norm is a function \( \text{Max}^{i} : D(0, 1] \times D(0, 1] \rightarrow D(0, 1] \), defined by:
\[
\text{Max}^{i}(\tilde{a}, \tilde{b}) = [\text{max}(a^{-}, b^{-}), \text{max}(a^{+}, b^{+})] \quad \text{for all} \quad \tilde{a}, \tilde{b} \in D(0, 1], \quad \text{where} \quad \tilde{a} = [a^{-}, a^{+}] \quad \text{and} \quad \tilde{b} = [b^{-}, b^{+}].
\]

Definition 2.11. Let \( X \neq \emptyset \) be a set and \( A \subseteq X \). Then the i-v characteristic function \( \tilde{\chi}_{A} \) of \( A \) is an i-v fuzzy subset of \( S \) which is defined as follows:
\[
\tilde{\chi}_{A}(x) = \begin{cases} 
1 & \text{when} \quad x \in A, \\
0 & \text{when} \quad x \notin A.
\end{cases}
\]

Definition 2.12. Let \( \tilde{\mu}_{1} \) and \( \tilde{\mu}_{2} \) be two i-v fuzzy subsets of a non-empty set \( X \). Then we define their intersection and union by \( (\tilde{\mu}_{1} \cap \tilde{\mu}_{2})(x) = \text{Min}^{i}(\tilde{\mu}_{1}(x), \tilde{\mu}_{2}(x)) \) and \( (\tilde{\mu}_{1} \cup \tilde{\mu}_{2})(x) = \text{Max}^{i}(\tilde{\mu}_{1}(x), \tilde{\mu}_{2}(x)) \) for all \( x \in X \).

The following results can be easily observed.

Lemma 2.13. Let \( S \) be a non-empty set and \( A, B \) be two subsets of \( S \). Then \( \tilde{\chi}_{A \cup B} = \tilde{\chi}_{A} \vee \tilde{\chi}_{B} \) and \( \tilde{\chi}_{A \cap B} = \tilde{\chi}_{A} \wedge \tilde{\chi}_{B} \).

Lemma 2.14. Let \( A \) and \( B \) be two non-empty subsets of a semiring \( S \). Then \( \tilde{\chi}_{A} \tilde{\chi}_{B} = \tilde{\chi}_{A \wedge B} \).

We first state the definition of a fuzzy point in a semiring.

Definition 2.15. Let \( S \) be a semiring and \( x \in S \). Let \( \tilde{a} \in D(0, 1] \setminus \{1\} \). Then an i-v fuzzy subset \( x\tilde{a} \) of \( S \) is called an i-v fuzzy point of \( S \) if
\[
x\tilde{a}(y) = \begin{cases} 
\tilde{a} & \text{if} \quad x = y, \\
0 & \text{otherwise}.
\end{cases}
\]

We now state the definitions of i-v fuzzy left(right) ideals of a semiring.

Definition 2.16. Let \( \tilde{\mu} \) be a non-empty i-v fuzzy subset of a semiring \( S \) (i.e. \( \tilde{\mu}(x) \neq 0 \) for some \( x \in S \)). Then \( \tilde{\mu} \) is called an i-v fuzzy left (resp. i-v fuzzy right) ideal of \( S \) if the following conditions hold.
Deﬁnition 3.1. Their product, denoted by \( i-v \) fuzzy quasi-ideal of \( S \), is a non-empty \( i-v \) fuzzy subset of \( S \) which is an \( i-v \) fuzzy left ideal as well as an \( i-v \) fuzzy right ideal of \( S \).

Deﬁnition 2.17. \([9]\) Let \( \tilde{\mu}_1 \) and \( \tilde{\mu}_2 \) be two \( i-v \) fuzzy subsets of a semiring \( S \). Then their product, denoted by \( \tilde{\mu}_1 \tilde{\mu}_2 \), is deﬁned by :

\[
(\tilde{\mu}_1 \tilde{\mu}_2)(x) = \left\{ \begin{aligned}
\sup \left\{ \inf_{1 \leq i \leq n} \left\{ \text{Min}^i(\tilde{\mu}_1(u_i), \tilde{\mu}_2(v_i)) \right\} : x = \sum_{i=1}^{n} u_i v_i; \\
\text{for any } u_i, v_i \in S, n \in \mathbb{Z}^+; \\
\hat{0} \quad \text{if } x \text{ cannot be expressed as } x = \sum_{i=1}^{n} u_i v_i;
\end{aligned} \right.
\]

Throughout this paper, we assume that any two interval numbers in \( D[0, 1] \) are comparable, i.e. for any two interval numbers \( \tilde{a} \) and \( \tilde{b} \) in \( D[0, 1] \), we have either \( \tilde{a} \leq \tilde{b} \) or \( \tilde{a} > \tilde{b} \).

3. \( i-v \) Fuzzy Quasi-ideals of a Semiring

We begin with the following deﬁnition of \( i-v \) fuzzy quasi-ideal of a semiring. Some properties of the quasi subsemigroups of a quasigroup have already been studied in \([7]\).

Deﬁnition 3.1. A non-empty \( i-v \) fuzzy subset \( \tilde{\mu} \) of a semiring \( S \) is said to be an \( i-v \) fuzzy quasi-ideal of \( S \) if for any \( x, y \in S \), \( \tilde{\mu}(x + y) \geq \text{Min}^i(\tilde{\mu}(x), \tilde{\mu}(y)) \) and \( \tilde{\mu}X \subseteq \tilde{\mu} \).

Lemma 3.2. For any three \( i-v \) fuzzy subsets \( \tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3 \) of a semiring \( S \), we have the following properties :

(i) \( \tilde{\mu}_1 (\tilde{\mu}_2 \cup \tilde{\mu}_3) = (\tilde{\mu}_1 \tilde{\mu}_2) \cup (\tilde{\mu}_1 \tilde{\mu}_3); \quad (\tilde{\mu}_2 \cup \tilde{\mu}_3) \tilde{\mu}_1 = (\tilde{\mu}_2 \tilde{\mu}_1) \cup (\tilde{\mu}_3 \tilde{\mu}_1) \)

(ii) \( \tilde{\mu}_1 (\tilde{\mu}_2 \cap \tilde{\mu}_3) \subseteq (\tilde{\mu}_1 \tilde{\mu}_2) \cap (\tilde{\mu}_1 \tilde{\mu}_3); \quad (\tilde{\mu}_2 \cap \tilde{\mu}_3) \tilde{\mu}_1 \subseteq (\tilde{\mu}_2 \tilde{\mu}_1) \cap (\tilde{\mu}_3 \tilde{\mu}_1). \)

Lemma 3.3. A non-empty subset \( A \) of a semiring \( S \) is a quasi-ideal of \( S \) if and only if \( \tilde{\chi}_A \) is an \( i-v \) fuzzy quasi-ideal of \( S \).

Lemma 3.4. Let \( S \) be a semiring. Then the following statements hold.

(i) Every \( i-v \) fuzzy left (or right) ideal of \( S \) is an \( i-v \) fuzzy quasi-ideal of \( S \).

(ii) The intersection of an \( i-v \) fuzzy left ideal and an \( i-v \) fuzzy right ideal of \( S \) is an \( i-v \) fuzzy quasi-ideal of \( S \).

(iii) If \( \tilde{\mu} \) be a non-empty \( i-v \) fuzzy subset of \( S \), then \( \tilde{\chi}_S \tilde{\mu} \) is an \( i-v \) fuzzy left ideal, \( \tilde{\mu}X \) is an \( i-v \) fuzzy right ideal, \( \tilde{\mu}X \tilde{\chi}_S \) is an \( i-v \) fuzzy ideal and \( \tilde{\chi}_S \tilde{\mu} \cap \tilde{\mu}X \) is an \( i-v \) fuzzy quasi-ideal of \( S \).
Note 3.5. It can be easily seen that each i-v fuzzy left ideal or an i-v fuzzy right ideal of a semiring \( S \) is an i-v fuzzy quasi-ideal of \( S \). But the converse is in general not true. We have the following example.

Example 3.6. We consider the semiring \( S = M_2(\mathbb{N}_0) \) with respect to the usual addition and multiplication of matrices. Suppose that \( P \) is the set
\[
P = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{N}_0) \right\}.
\]
Now we define an i-v fuzzy subset
\[
\tilde{\mu} : M_2(\mathbb{N}_0) \to [0, 1]
\]
by
\[
\tilde{\mu}(A) = \begin{cases} [0.8, 0.9] & \text{when } A \in P; \\ [0.3, 0.4] & \text{otherwise}. \end{cases}
\]
We can easily check that \( \tilde{\mu} \) is an i-v fuzzy quasi-ideal of \( S \) but is not an i-v fuzzy left ideal and an i-v fuzzy right ideal of \( S \) either.

We state the following proposition concerning the i-v fuzzy quasi-ideals of a semiring.

Proposition 3.7. Let \( x \tilde{\mu} \) and \( y \tilde{\theta} \) be two idempotent i-v fuzzy points of a semiring \( S \). Also let \( \tilde{\mu} \) and \( \tilde{\theta} \) be an i-v fuzzy left ideal and i-v fuzzy right ideal of \( S \), respectively. Then, we deduce the following equalities:
\[
x \tilde{\mu} = x \tilde{\tilde{\mu}} \tilde{\tilde{\theta}} = \tilde{\tilde{\mu}} \tilde{\tilde{\theta}} x \tilde{\tilde{\mu}} = x \tilde{\tilde{\mu}} \tilde{\tilde{\theta}} x \tilde{\tilde{\mu}} \tilde{\tilde{\theta}} = x \tilde{\tilde{\mu}} \tilde{\tilde{\theta}} x \tilde{\tilde{\mu}} \tilde{\tilde{\theta}} \]
and so each of these i-v fuzzy subsets is an i-v fuzzy quasi-ideal of \( S \).

Proof. We first prove that \( x \tilde{\mu} = x \tilde{\tilde{\mu}} \tilde{\tilde{\theta}} \). Clearly, \( x \tilde{\tilde{\mu}} \subseteq x \tilde{\tilde{\mu}} \). Also since \( \tilde{\mu} \) is an i-v fuzzy left ideal of \( S \), we have \( x \tilde{\tilde{\mu}} \subseteq x \tilde{\tilde{\mu}} \). It hence follows that \( x \tilde{\mu} = x \tilde{\tilde{\mu}} \).

For the reverse inclusion, we let \( y \in S \). Then, we have
\[
(x \tilde{\mu} x \tilde{\mu})(y) = \sup_{y = \sum_{i=1}^{m} p_i q_i} \inf_{1 \leq i \leq m} \{ \text{Min}^i(x \tilde{\mu} (p_i), x \tilde{\mu} (q_i)) \}.
\]
If \( (x \tilde{\mu} x \tilde{\mu})(y) = \tilde{a} \), then
\[
\inf_{1 \leq i \leq m} \{ \text{Min}^i(x \tilde{\mu} (p_i), x \tilde{\mu} (q_i)) \} = \tilde{a}.
\]

\[
\implies \text{Min}^i(x \tilde{\mu} (p_i), x \tilde{\mu} (q_i)) = \tilde{a} \quad \text{for each } 1 \leq i \leq m, \text{ where } y = \sum_{i=1}^{m} p_i q_i.
\]
\[
\implies p_i = x = q_i \quad \text{for each } 1 \leq i \leq m, \text{ where } y = \sum_{i=1}^{m} p_i q_i.
\]

Now \( x \tilde{\mu} (y) = (x \tilde{\mu} x \tilde{\mu})(y) \) implies that \( y = x \). Hence, we have \( x = \sum_{i=1}^{m} p_i q_i \). Thus we get \( x = \sum_{i=1}^{m} x^2 \). Let \( z \in S \). Then \( (x \tilde{\mu} x \tilde{\mu})(z) = \text{Min}^i \left( (x \tilde{\mu} x \tilde{\mu})(z), \tilde{\mu}(z) \right) \). Let \( T \) be
the set
\[ T = \left\{ \sum_{i=1}^{n} a_i b_i : a_i, b_i \in S; n \in \mathbb{N} \right\} \]

**Case I**: If \( z \notin T \), then \( (x_{\bar{a}S})_\cap \mu)(z) = 0 \). Therefore, \( (x_{\bar{a}S} \cap \mu)(z) = 0 \) and also \( (x_{\bar{a}S})(z) = 0 \).

**Case II**: Let \( z \in T \). Then, we have
\[
(x_{\bar{a}S} \cap \mu)(z) = \text{Min}^\dagger \left( \sup_{z = \sum_{i=1}^{n} a_i b_i} \left\{ \inf_{1 \leq i \leq n} \text{Min}^\dagger (x_{\bar{a}S}(a_i), \bar{S}(b_i)) \right\}, \mu(z) \right)
\]
\[
= \text{Min}^\dagger \left( \sup_{z = \sum_{i=1}^{n} a_i b_i} \left\{ \inf_{1 \leq i \leq n} \{x_{\bar{a}}(a_i)\} \right\}, \mu(z) \right).
\]

Now, let \( T_1 \) be the set \( T_1 = \left\{ \sum_{i=1}^{n} a_i b_i \in T : a_i = x \text{ for all } 1 \leq i \leq n \right\} \). If \( z \in T_1 \), then \( (x_{\bar{a}S} \cap \mu)(z) = \text{Min}^\dagger((\bar{a}, \mu(z))). \) Again, \( z \in T_1 \Rightarrow z = \sum_{i=1}^{n} x b_i = x \sum_{i=1}^{n} b_i = (\sum_{i=1}^{m} x^2) \sum_{i=1}^{n} b_i = \sum_{i=1}^{m} x (\sum_{i=1}^{n} b_i). \) Therefore,
\[
(x_{\bar{a}S})(z) = \inf_{1 \leq i \leq m} \left\{ \text{Min}^\dagger(x_{\bar{a}}(x), \mu(\sum_{i=1}^{n} x b_i)) \right\}
\]
\[
= \inf_{1 \leq i \leq m} \left\{ \text{Min}^\dagger((\bar{a}, \mu(z))) \right\} = \text{Min}^\dagger((\bar{a}, \mu(z)) = (x_{\bar{a}S} \cap \mu)(z).
\]

If \( z \in T \setminus T_1 \), then \( (x_{\bar{a}S} \cap \mu)(z) = 0 = (x_{\bar{a}S})(z) \). Thus we get \( x_{\bar{a}S} \cap \mu \subseteq x_{\bar{a}S} \mu \). Consequently, \( x_{\bar{a}S} = x_{\bar{a}S} \cap \mu \).

Again, we see that \( x_{\bar{a}S} \) is an intersection of an i-v fuzzy left ideal and an i-v fuzzy right ideal of \( S \). Hence \( x_{\bar{a}S} \mu \) is an i-v fuzzy quasi-ideal of \( S \), by Lemma 3.4 (ii).

Similarly, we can prove that \( \tilde{\theta} y_{\tilde{b}} = \tilde{\chi}_S y_{\tilde{b}} \cap \tilde{\theta} \) and \( \tilde{\theta} y_{\tilde{b}} \) is an i-v fuzzy quasi-ideal of \( S \), where \( \tilde{\theta} \) is an i-v fuzzy right ideal of \( S \) and \( y_{\tilde{b}} \) is an idempotent i-v fuzzy point of \( S \).

Now, \( x_{\bar{a}S} \chi_S y_{\tilde{b}} \subseteq x_{\bar{a}S} \tilde{\chi}_S \chi_S \chi_S \subseteq x_{\bar{a}S} \chi_S \). Also, \( x_{\bar{a}S} \chi_S y_{\tilde{b}} \subseteq \tilde{\chi}_S \chi_S y_{\tilde{b}} \subseteq \chi_S y_{\tilde{b}} \). This implies that \( x_{\bar{a}S} \chi_S y_{\tilde{b}} \subseteq x_{\bar{a}S} \chi_S \cap \tilde{\chi}_S \chi_S y_{\tilde{b}} \). To prove the reverse inclusion, let \( t \in S \). Since, \( x_{\bar{a}} \) and \( y_{\tilde{b}} \) are both idempotent, we have \( x = \sum_{i=1}^{m_1} x^2 \) and \( y = \sum_{j=1}^{m_2} y^2 \) for some \( m_1, m_2 \in \mathbb{N} \). If \( t \) can not be expressed as \( t = \sum_{i=1}^{m} a_i b_i \), for any \( a_i, b_i \in S \), then
\[(x_{\overline{a}}\overline{\chi}_S \cap \overline{\chi}y_g)(t) = \tilde{0} = (x_{\overline{a}}\overline{\chi}y_g)(t).\]

Now, suppose that \( t = \sum_{i=1}^{n_1} a_i b_i \), for some \( a_i, b_i \in S \). Then

\[
(x_{\overline{a}}\overline{\chi}_S \cap \overline{\chi}y_g)(t) = \text{Min}^t \left( (x_{\overline{a}}\overline{\chi}_S)(t), (\overline{\chi}y_g)(t) \right) = \text{Min}^t \left( \sup_{n_1} \left\{ \inf_{1 \leq i \leq n_1} \left\{ \text{Min}^t(x_{\overline{a}}(a_i), \overline{\chi}_S(b_i)) \right\} \right\}, \sup_{t = \sum_{i=1}^{n_1} a_i b_i} \left\{ \inf_{1 \leq i \leq n_1} \left\{ \text{Min}^t(\overline{\chi}_S(a_i), y_g(b_i)) \right\} \right\} \right),
\]

Now if \( a_i = x \) and \( b_i = y \) for each \( 1 \leq i \leq n_1 \), then

\[
(x_{\overline{a}}\overline{\chi}_S \cap \overline{\chi}y_g)(t) = \text{Min}^t \left( \inf_{1 \leq i \leq n_1} \{ x_{\overline{a}}(x) \}, \inf_{1 \leq i \leq n_2} \{ y_g(y) \} \right) = \text{Min}^t(\overline{a}, \overline{b}). \]

Again, \( a_i = x \) and \( b_i = y \) for each \( 1 \leq i \leq n_1 \) implies that \( t = \sum_{i=1}^{n_1} xy = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} x^2 y^2 \). Then, we deduce that

\[
(x_{\overline{a}}\overline{\chi}_S y_g)(t) = \sup_{k \leq i \leq k} \left\{ \inf_{1 \leq i \leq k} \left\{ \text{Min}^t(x_{\overline{a}}\overline{\chi}_S)(a_i), y_g(b_i) \right\} \right\},
\]

\[
\geq \inf_{1 \leq i \leq n_1} \left\{ \text{Min}^t((x_{\overline{a}}\overline{\chi}_S)(\sum_{i=1}^{n_1} x^2), y_g(\sum_{j=1}^{n_2} y^2)) \right\},
\]

\[
\geq \inf_{1 \leq i \leq n_1} \left\{ \text{Min}^t(\inf_{1 \leq i \leq n_1} \text{Min}^t(x_{\overline{a}}(x), \overline{\chi}_S(x)), y_g(y)) \right\}
\]

\[
= \text{Min}^t(\overline{a}, \overline{b}) = (x_{\overline{a}}\overline{\chi}_S \cap \overline{\chi}y_g)(t).
\]

Now let us consider the case where \( a_i \neq x \) or \( b_i \neq y \) for some \( 1 \leq i \leq n_1 \). Then

\[
(x_{\overline{a}}\overline{\chi}_S \cap \overline{\chi}y_g)(t) = \text{Min}^t \left( \sup_{n_1} \left\{ \inf_{1 \leq i \leq n_1} \{ x_{\overline{a}}(a_i) \} \right\}, \sup_{t = \sum_{i=1}^{n_1} a_i b_i} \left\{ \inf_{1 \leq i \leq n_1} \{ y_g(b_i) \} \right\} \right),
\]

\[
= \tilde{0} = (x_{\overline{a}}\overline{\chi}y_g)(t).
\]
This implies that \((x_\tilde{n} \tilde{\chi}_S \cap  \tilde{\chi}_S y_\tilde{r}) \subseteq (x_\tilde{n} \tilde{\chi}_S y_\tilde{r})\). Consequently, we get that \((x_\tilde{n} \tilde{\chi}_S \cap  \tilde{\chi}_S y_\tilde{r}) = x_\tilde{n} \tilde{\chi}_S y_\tilde{r}\). Being an intersection of i-v fuzzy left ideal and i-v fuzzy right ideal, \(x_\tilde{n} \tilde{\chi}_S y_\tilde{r}\) is an i-v fuzzy quasi-ideal of \(S\).

**Definition 3.8.** Let \(\tilde{\mu}\) be a non-empty i-v fuzzy subset of a semiring \(S\). The intersection of all i-v fuzzy left ideals of \(S\) containing \(\tilde{\mu}\) is said to be the i-v fuzzy left ideal of \(S\) generated by \(\tilde{\mu}\) and it is denoted by \((\tilde{\mu})_l\). The i-v fuzzy right ideal \((\tilde{\mu})_r\), and i-v fuzzy quasi-ideal \((\tilde{\mu})_q\) of \(S\), generated by \(\tilde{\mu}\) can be defined similarly.

**Definition 3.9.** Let \(\tilde{\mu}\) be an i-v fuzzy subset of a semiring \(S\). We define an i-v fuzzy subset \(< \tilde{\mu}> \) of \(S\) by \(< \tilde{\mu}> (x) = \sup \left\{ \inf_{1 \leq i \leq n} \tilde{\mu}(a_i) : x = \sum_{i=1}^{n} a_i, a_i \in S; n \in \mathbb{N} \right\}\), for all \(x \in S\).

**Lemma 3.10.** Let \(\tilde{\mu}\) be a non-empty i-v fuzzy subset of a semiring \(S\). Then

(i) \((\tilde{\mu})_l = < \tilde{\mu} \cup \tilde{\chi}_S \tilde{\mu} >\), (ii) \((\tilde{\mu})_r = < \tilde{\mu} \cup \tilde{\mu} \tilde{\chi}_S > \) and (iii) \((\tilde{\mu})_q = < \tilde{\mu} \cup (\tilde{\mu} \tilde{\chi}_S \cap \tilde{\chi}_S \tilde{\mu}) >\).

**Proof.** (i) We first prove that \(< \tilde{\mu} \cup \tilde{\chi}_S \tilde{\mu} >\) is an i-v fuzzy left ideal of \(S\) containing \(\tilde{\mu}\). Let \(x = \sum_{i=1}^{m} a_i\) and \(y = \sum_{j=1}^{n} b_j\) for some \(a_i, b_j \in S\), where \(1 \leq i \leq m\) and \(1 \leq j \leq n\). Then

\[
< \tilde{\mu} \cup \tilde{\chi}_S \tilde{\mu} > (x + y) = \sup \left\{ \inf_{1 \leq i \leq m} \left( \tilde{\mu} \cup \tilde{\chi}_S \tilde{\mu} \right)(c_i) : x + y = \sum_{i=1}^{m} c_i \right\}
\]

\[
\geq \sup \left\{ Min \left\{ \inf_{1 \leq i \leq m} \left( \tilde{\mu} \cup \tilde{\chi}_S \tilde{\mu} \right)(a_i), \inf_{1 \leq j \leq n} \left( \tilde{\mu} \cup \tilde{\chi}_S \tilde{\mu} \right)(b_j) \right\} : x = \sum_{i=1}^{m} a_i, y = \sum_{j=1}^{n} b_j \right\}
\]

\[
\geq Min \left\{ < \tilde{\mu} \cup \tilde{\chi}_S \tilde{\mu} > (x), < \tilde{\mu} \cup \tilde{\chi}_S \tilde{\mu} > (y) \right\}.
\]

Now \(\tilde{\chi}_S \tilde{\mu} \cup \tilde{\chi}_S \tilde{\mu} = \tilde{\chi}_S \tilde{\mu} \cup \tilde{\chi}_S \tilde{\mu} \subseteq \tilde{\chi}_S \tilde{\mu} \cup \tilde{\chi}_S \tilde{\mu} = \tilde{\chi}_S \tilde{\mu} \subseteq \tilde{\chi}_S \tilde{\mu} \cup \tilde{\mu}\). This implies that \(< \tilde{\chi}_S \tilde{\mu} \cup \tilde{\chi}_S \tilde{\mu} > \subseteq < \tilde{\chi}_S \tilde{\mu} \cup \tilde{\mu}\). Therefore, \(< \tilde{\chi}_S \tilde{\mu} \cup \tilde{\mu}\) is an i-v fuzzy left ideal of \(S\) and clearly, it contains \(\tilde{\mu}\). Let \(z \in S\) and IFL(S) be the set of all i-v fuzzy left ideals of \(S\). Then \((\tilde{\mu})_l(z) = \left( \cap_{\tilde{\mu} \in IFL(S)} \tilde{\mu} \right)(z) = \inf_{\tilde{\mu} \in IFL(S)} \tilde{\mu}(z) \leq \tilde{\chi}_S \tilde{\mu} \cup \tilde{\mu} > (z),\)

since \(< \tilde{\chi}_S \tilde{\mu} \cup \tilde{\mu}\) is an i-v fuzzy left ideal of \(S\) containing \(\tilde{\mu}\). Thus we obtain that \((\tilde{\mu})_l(z) \subseteq < \tilde{\chi}_S \tilde{\mu} \cup \tilde{\mu}\). Again \((\tilde{\mu})_r(z) = \left( \cap_{\tilde{\nu} \in IFL(S)} \tilde{\nu} \right)(z) = \inf_{\tilde{\nu} \in IFL(S)} \tilde{\nu}(z) \geq \inf_{\tilde{\mu} \in IFL(S)} \tilde{\mu}(z) = \tilde{\mu}(z),\)

Also, \((\tilde{\nu})_l(z) = \left( \cap_{\tilde{\mu} \in IFL(S)} \tilde{\mu} \right)(z) = \inf_{\tilde{\mu} \in IFL(S)} \tilde{\mu}(z) \geq \inf_{\tilde{\nu} \in IFL(S)} \tilde{\nu}(z) = \tilde{\nu}(z).\)

This shows that \((\tilde{\mu})_l(z) \geq Max \left\{ (\tilde{\mu})_l(z), (\tilde{\nu})_r(z) \right\} = (\tilde{\mu} \cup \tilde{\chi}_S \tilde{\mu})(z)\.\) Therefore \((\tilde{\mu} \cup \tilde{\chi}_S \tilde{\mu}\) \subseteq \((\tilde{\mu})_l\) which implies that \(< \tilde{\mu} \cup \tilde{\chi}_S \tilde{\mu} > \subseteq < \tilde{\mu} > = (\tilde{\mu})_l.\) Hence, we get that \((\tilde{\mu})_l = < \tilde{\mu} \cup \tilde{\chi}_S \tilde{\mu} > .\)
Theorem 3.14. \( S \)

Theorem 3.13. \( S \)

\( (ii) \) Proof of this part is similar to \((i)\).

\( (iii) \) We first prove \( \tilde{\mu} \cap (\tilde{\mu} \cap \tilde{\chi} S \tilde{\mu}) \) is an i-v fuzzy quasi-ideal of \( S \) containing \( \tilde{\mu} \). We have \( \tilde{\mu} \cap (\tilde{\mu} \cap \tilde{\chi} S \tilde{\mu}) = (\tilde{\mu} \cap \tilde{\chi} S \tilde{\mu}) \cap (\tilde{\mu} \cap \tilde{\chi} S \tilde{\mu}) \). Now, \( \tilde{\mu} \) and \( \tilde{\chi} S \tilde{\mu} \) are i-v fuzzy left ideal and i-v fuzzy right ideal of \( S \) containing \( \tilde{\mu} \) respectively. Therefore, \( \tilde{\mu} \) is an intersection of i-v fuzzy left ideal and i-v fuzzy right ideal of \( S \) respectively and clearly, it contains \( \tilde{\mu} \). Thus, \( \tilde{\mu} \cup (\tilde{\mu} \cap \tilde{\chi} S \tilde{\mu}) \) is an i-v fuzzy quasi-ideal of \( S \) containing \( \tilde{\mu} \).

The following theorem is known in regular semirings.

For each i-v fuzzy right ideal \( \tilde{\mu} \) and i-v fuzzy quasi-ideal \( \tilde{\mu} \), there exists an element \( x \in S \) such that \( a = axa \). A semiring \( S \) is said to be regular if its every element is regular.

The following theorem is known in regular semirings.

**Theorem 3.12.** \( S \) is regular if and only if \( \tilde{\mu} \cap \tilde{\chi} S \tilde{\mu} \) and \( \tilde{\mu} \) are i-v fuzzy quasi-ideals of \( S \) containing \( \tilde{\mu} \).

**Theorem 3.13.** The following statements are equivalent in a semiring \( S \).

1. \( S \) is regular.
2. For each i-v fuzzy right ideal \( \tilde{\mu} \) and i-v fuzzy left ideal \( \tilde{\chi} S \tilde{\mu} \) of \( S \), \( \tilde{\mu} = \tilde{\mu} \cap \tilde{\chi} S \tilde{\mu} \).
3. For each i-v fuzzy right ideal \( \tilde{\mu} \) and each i-v fuzzy left ideal \( \tilde{\chi} S \tilde{\mu} \) of \( S \), \( a \) is the i-v fuzzy quasi-ideal of \( S \).
4. The set \( IFQ(S) \) of all i-v fuzzy quasi-ideals of \( S \) forms a regular semigroup with respect to the usual product of i-v fuzzy subsets of \( S \).
5. Each i-v fuzzy quasi-ideal \( \tilde{\mu} \) of \( S \) satisfies \( \tilde{\mu} = \tilde{\mu} \cap \tilde{\chi} S \tilde{\mu} \).

The proof of this theorem is straightforward. We hence omit the proof.

In the following theorem, we study the type of i-v fuzzy quasi-ideals in a regular semiring \( S \).

**Theorem 3.14.** The following statements are equivalent in a semiring \( S \).

1. \( \tilde{\mu} \cap \tilde{\chi} S \tilde{\mu} \) for any i-v fuzzy right ideal \( \tilde{\mu} \) and i-v fuzzy quasi-ideal \( \tilde{\chi} S \tilde{\mu} \) of \( S \).
2. \( IFQ(S) \) forms an idempotent semigroup with respect to the usual product of i-v fuzzy subsets of \( S \).
3. \( \tilde{\mu} = \tilde{\mu} \cap \tilde{\chi} S \tilde{\mu} \) for any i-v fuzzy quasi-ideal \( \tilde{\mu} \) of \( S \).
Proof. (i) $\implies$ (ii) : Suppose that (i) hold. Then it follows from Theorem 3.13 that $IFQ(S)$ forms a regular semigroup with respect to the usual product of the i-v fuzzy subsets of $S$. It remains to prove that $IFQ(S)$ is idempotent. Let $\tilde{\eta} \in IFQ(S)$. Then, by Theorem 3.13, we get that $\tilde{\eta} = \tilde{\eta} \tilde{\eta} \tilde{\eta}$. Thus, we obtain that :

$$
\tilde{\eta} = \tilde{\eta} \tilde{x} s \tilde{\eta} = (\tilde{\eta} \tilde{x} s \tilde{\eta}) \tilde{x} s (\tilde{\eta} \tilde{x} s \tilde{\eta}) = \tilde{\eta} \tilde{x} s (\tilde{\eta} \tilde{x} s \tilde{\eta}) \tilde{x} s \tilde{\eta} \tilde{x} s \tilde{\eta} = \tilde{\eta} \tilde{x} s \tilde{\eta} \tilde{x} s \tilde{\eta} = \tilde{\eta} \tilde{x} s \tilde{\eta} \tilde{x} s \tilde{\eta} \tilde{x} s \tilde{\eta} = \tilde{\eta}^2.
$$

This shows that $\tilde{\eta} \subseteq \tilde{\eta}^2$. Now $\tilde{\eta}^2 \subseteq \tilde{x} s \tilde{\eta}$ and as well as $\tilde{\eta}^2 \subseteq \tilde{x} s \tilde{\eta}$ imply that $\tilde{\eta}^2 \subseteq \tilde{x} s \tilde{\eta} \cap \tilde{x} s \tilde{\eta} \subseteq \tilde{\eta}$, since $\tilde{\eta}$ is an i-v fuzzy quasi-ideal of $S$. Hence $\tilde{\eta} = \tilde{\eta}^2$. Thus $IFQ(S)$ forms an idempotent semigroup with respect to the usual product of i-v fuzzy subsets of $S$.

(ii) $\implies$ (iii) : This is just a restriction.

(iii) $\implies$ (i) : Let $\tilde{\eta} = \tilde{\eta}^2$ for any i-v fuzzy quasi-ideal $\tilde{\eta}$ of $S$. Let $\tilde{\mu}$ and $\tilde{\theta}$ be an i-v fuzzy right ideal and an i-v fuzzy left ideal of $S$ respectively. Then $\tilde{\mu} \tilde{\theta} \subseteq \tilde{\mu} \tilde{x} s \tilde{\theta} \subseteq \tilde{\mu}$ as well as, $\tilde{\mu} \tilde{\theta} \subseteq \tilde{x} s \tilde{\theta} \subseteq \tilde{\theta}$. This implies that $\tilde{\mu} \tilde{\theta} \subseteq \tilde{\mu} \tilde{\theta} \subseteq \tilde{\theta}$. Now being an intersection of an i-v fuzzy right ideal and an i-v fuzzy left ideal of $S$, $\tilde{\mu} \cap \tilde{\theta}$ is an i-v fuzzy quasi-ideal of $S$. Hence, we have $\tilde{\mu} \cap \tilde{\theta} = (\tilde{\mu} \cap \tilde{\theta})^2 = (\tilde{\mu} \cap \tilde{\theta})(\tilde{\mu} \cap \tilde{\theta}) \subseteq \tilde{\mu} \tilde{\theta}$. Similarly, $(\tilde{\mu} \cap \tilde{\theta}) \subseteq \tilde{\theta} \tilde{\mu}$. Thus we have proved that $\tilde{\mu} \tilde{\theta} = \tilde{\mu} \cap \tilde{\theta} \subseteq \tilde{\theta} \tilde{\mu}$. $\square$

4. INTERVAL-VALUED FUZZY BI-IDEALS OF A SEMIRING:

**Definition 4.1.** A non-empty i-v fuzzy subset $\tilde{\mu}$ of a semiring $S$ is said to be an i-v fuzzy bi-ideal of $S$ if for any $x, y, z \in S$, $\tilde{\mu}(x + y) \geq \text{Min}^i(\tilde{\mu}(x), \tilde{\mu}(y))$ and $\tilde{\mu}(xyz) \geq \text{Min}^i(\tilde{\mu}(x), \tilde{\mu}(y))$.

We characterize the i-v fuzzy bi-ideals of a semiring in the following lemma.

**Lemma 4.2.** A non-empty i-v fuzzy subset $\tilde{\mu}$ of a semiring $S$ is an i-v fuzzy bi-ideal of $S$ if and only if $\tilde{\mu}(x + y) \geq \text{Min}^i(\tilde{\mu}(x), \tilde{\mu}(y))$ for any $x, y \in S$ and $\tilde{\mu} \chi s \tilde{\mu} \subseteq \tilde{\mu}$.

In the following proposition, we state the relation between i-v fuzzy quasi-ideal and i-v fuzzy bi-ideal of a semiring.

**Proposition 4.3.** Every i-v fuzzy quasi-ideal of a semiring $S$ is also an i-v fuzzy bi-ideal of $S$.

**Proof.** Let $\tilde{\mu}$ be an i-v fuzzy quasi-ideal of a semiring $S$. Then $\tilde{\mu}(x + y) \geq \text{Min}^i(\tilde{\mu}(x), \tilde{\mu}(y))$, for any $x, y \in S$. Now $\tilde{\mu} \chi s \tilde{\mu} \subseteq \tilde{\mu} \chi s \tilde{x} s \tilde{\mu} \subseteq \tilde{\mu} \chi s \tilde{s} s \tilde{\mu} \subseteq \tilde{\mu} \chi s \tilde{\mu}$. Also, $\tilde{\mu} \chi s \tilde{\mu} \subseteq \tilde{x} s \tilde{\mu} \chi s \tilde{s} s \tilde{\mu} \subseteq \tilde{x} s \tilde{\mu} \chi s \tilde{\mu} \subseteq \tilde{x} s \tilde{\mu} \cap \tilde{x} s \tilde{\mu} \subseteq \tilde{\mu} \chi s \tilde{s} s \tilde{\mu}$. Hence, we get $\tilde{\mu} \chi s \tilde{\mu} \subseteq \tilde{x} s \tilde{\mu} \cap \tilde{x} s \tilde{\mu}$. Since, $\tilde{\mu}$ is an i-v fuzzy quasi-ideal of $S$, it follows that $\tilde{\mu} \chi s \tilde{\mu} \subseteq \tilde{x} s \tilde{\mu} \cap \tilde{x} s \tilde{\mu} \subseteq \tilde{\mu}$. Consequently, $\tilde{\mu}$ is an i-v fuzzy bi-ideal of $S$. $\square$

We note that the converse of the above Proposition does not hold in general.
Definition 4.4. Let $\tilde{\mu}$ be a non-empty i-v fuzzy subset of a semiring $S$. Then the i-v fuzzy bi-ideal of $S$ generated by $\tilde{\mu}$ is denoted by $(\tilde{\mu})_b$ and is defined as the intersection of all i-v fuzzy bi-ideals of $S$ containing $\tilde{\mu}$.

Lemma 4.5. Let $\tilde{\mu}$ be a non-empty i-v fuzzy subset of a semiring $S$. Then, we have the following equality.

$$(\tilde{\mu})_b = \inf_{x \in IFB(S)} (\tilde{\mu} \cup \tilde{\mu}^2 \cup \tilde{\mu} \tilde{x} \tilde{\mu})$$

Proof. We first prove that $<\tilde{\mu} \cup \tilde{\mu}^2 \cup \tilde{\mu} \tilde{x} \tilde{\mu}>$ is an i-v fuzzy bi-ideal of $S$, containing $\tilde{\mu}$. Similar to the proof given in Lemma 3.10 (i), we can show that $<\tilde{\mu} \cup \tilde{\mu}^2 \cup \tilde{\mu} \tilde{x} \tilde{\mu}> \subseteq (x + y) \geq \min^1 \left( <\tilde{\mu} \cup \tilde{\mu}^2 \cup \tilde{\mu} \tilde{x} \tilde{\mu}> (x), <\tilde{\mu} \cup \tilde{\mu}^2 \cup \tilde{\mu} \tilde{x} \tilde{\mu}> (y) \right)$, for any $x, y \in S$. Now, we easily deduce that

$$(\tilde{\mu} \cup \tilde{\mu}^2 \cup \tilde{\mu} \tilde{x} \tilde{\mu}) \tilde{x} (\tilde{\mu} \cup \tilde{\mu}^2 \cup \tilde{\mu} \tilde{x} \tilde{\mu}) = (\tilde{\mu} \cup \tilde{\mu}^2 \cup \tilde{\mu} \tilde{x} \tilde{\mu}) (\tilde{\mu} \tilde{x} \tilde{\mu} \cup \tilde{\mu} \tilde{x} \tilde{\mu} \cup \tilde{\mu} \tilde{x} \tilde{\mu} \tilde{x} \tilde{\mu})$$

$$(\tilde{\mu} \cup \tilde{\mu}^2 \cup \tilde{\mu} \tilde{x} \tilde{\mu}) \tilde{x} (\tilde{\mu} \cup \tilde{\mu}^2 \cup \tilde{\mu} \tilde{x} \tilde{\mu}) \subseteq (\tilde{\mu} \cup \tilde{\mu}^2 \cup \tilde{\mu} \tilde{x} \tilde{\mu}) (\tilde{\mu} \tilde{x} \tilde{\mu} \cup \tilde{\mu} \tilde{x} \tilde{\mu} \cup \tilde{\mu} \tilde{x} \tilde{\mu} \tilde{x} \tilde{\mu})$$

Consequently, $<\tilde{\mu} \cup \tilde{\mu}^2 \cup \tilde{\mu} \tilde{x} \tilde{\mu}>$ is an i-v fuzzy bi-ideal of $S$ and clearly, it contains $\tilde{\mu}$. Suppose that $IFB(S)$ denote the set of all i-v fuzzy bi-ideals of $S$. Let $x \in S$. Then, we have $(\tilde{\mu})_b = \inf_{x \in IFB(S)} (\tilde{\mu} \cup \tilde{\mu}^2 \cup \tilde{\mu} \tilde{x} \tilde{\mu}) (x)$

$$\inf_{x \in IFB(S)} (\tilde{\mu} \cup \tilde{\mu}^2 \cup \tilde{\mu} \tilde{x} \tilde{\mu}) (x) = \inf_{x \in IFB(S)} (\tilde{\mu} \cup \tilde{\mu}^2 \cup \tilde{\mu} \tilde{x} \tilde{\mu}) (x)$$

For i-v fuzzy bi-ideals of a semiring, we have the following Proposition.

Proposition 4.6. The product of an i-v fuzzy bi-ideal and an i-v fuzzy sub-semiring of a semiring $S$ is still an i-v fuzzy bi-ideal of $S$.

The following corollaries are easy consequence of the above Proposition.

Corollary 4.7. The product of two i-v fuzzy bi-ideals of a semiring is again an i-v fuzzy bi-ideal of $S$. 

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Corollary 4.8. The product of two i-v fuzzy quasi-ideals of a semiring is an i-v fuzzy bi-ideal of $S$.

In the following theorem, we state some properties of fuzzy quasi-ideals of a regular semiring.

Theorem 4.9. Let $S$ be a regular semiring. The following properties of a quasi-ideal of $S$ hold.

(i) Each i-v fuzzy quasi-ideal $\tilde{\mu}$ of $S$ satisfies $\tilde{\mu} = \tilde{\theta} \cap \tilde{\eta} = \tilde{\eta} \tilde{\eta}$, where $\tilde{\theta} = (\tilde{\mu})_r$, and $\tilde{\eta} = (\tilde{\mu})_l$.
(ii) Each i-v fuzzy quasi-ideal $\tilde{\mu}$ of $S$ satisfies $\tilde{\mu}^2 = \tilde{\mu}^3$.
(iii) Each i-v fuzzy bi-ideal of $S$ is an i-v fuzzy quasi-ideal of $S$.
(iv) Each i-v fuzzy bi-ideal of a two-sided ideal $T$ of $S$ is an i-v fuzzy quasi-ideal of $S$.

Proof. (i) In a regular semiring $S$, each i-v fuzzy quasi-ideal $\tilde{\mu}$ of $S$ satisfies $\tilde{\mu} = \tilde{\chi}_{S\tilde{\mu}}\cap \tilde{\chi}_{S\tilde{\mu}}$, by Theorem 3.13. Hence, it suffices to prove that $(\tilde{\mu})_l = \tilde{\chi}_{S\tilde{\mu}}$ and $(\tilde{\mu})_r = \tilde{\mu}\tilde{\chi}_{S}$. Now, we deduce the followings:

$$\tilde{\chi}_{S\tilde{\mu}} \subseteq < \tilde{\mu} \cup \tilde{\chi}_{S\tilde{\mu}} > = < \tilde{\mu} \cup \tilde{\chi}_{S\tilde{\mu}} > < \tilde{\mu} \cup \tilde{\chi}_{S\tilde{\mu}} >$$ (since, in a regular semiring $S$,

$\tilde{\mu}^2 = \tilde{\mu}_l^2$, where, $\tilde{\mu}_l$ is an i-v fuzzy left ideal of $S$, by Theorem 3.13)

$\subseteq < \tilde{\mu}^2 \cup \tilde{\mu}\tilde{\chi}_{S\tilde{\mu}} \cup \tilde{\chi}_{S\tilde{\mu}} > < \tilde{\mu}\tilde{\chi}_{S\tilde{\mu}} > < \tilde{\mu}\tilde{\chi}_{S\tilde{\mu}} >$ (since, by Theorem 3.14, $\tilde{\mu} = \tilde{\mu}^2$)

$\subseteq < \tilde{\chi}_{S\tilde{\mu}} \cup \tilde{\chi}_{S\tilde{\mu}} \cup \tilde{\chi}_{S\tilde{\mu}} \cup \tilde{\chi}_{S\tilde{\mu}} >$ (since, by Theorem 3.14, $\tilde{\mu} = \tilde{\mu}^2$)

Thus, we obtain $\tilde{\chi}_{S\tilde{\mu}} \subseteq < \tilde{\mu} \cup \tilde{\chi}_{S\tilde{\mu}} > \subseteq \tilde{\chi}_{S\tilde{\mu}}$. So, $\tilde{\chi}_{S\tilde{\mu}} = < \tilde{\mu} \cup \tilde{\chi}_{S\tilde{\mu}} > = (\tilde{\mu})_l$.

Similarly, we can get that $\tilde{\mu}_r = \tilde{\chi}_{S\tilde{\mu}} = (\tilde{\mu})_r$. Therefore, $\tilde{\mu} = \tilde{\chi}_{S\tilde{\mu}} \cap \tilde{\mu}\tilde{\chi}_{S} = < \tilde{\mu} \cup \tilde{\chi}_{S\tilde{\mu}} > \cap < \tilde{\mu} \cup \tilde{\mu}\tilde{\chi}_{S} > = (\tilde{\mu})_l \cap (\tilde{\mu})_r = (\tilde{\mu})_r$, by Theorem 3.12.

(ii) Let $\tilde{\mu}$ be an i-v fuzzy quasi-ideal of $S$. Then by Theorem 3.13, it follows that $\tilde{\mu}^2$ is a i-v fuzzy quasi-ideal of $S$, since $S$ is regular. Then by Theorem 3.13, we have $\tilde{\mu}^2 = \tilde{\mu}^2 \tilde{\chi}_S \tilde{\mu}^2 = \tilde{\mu}(\tilde{\mu}\tilde{\chi}_{S\tilde{\mu}}) \tilde{\mu} = \tilde{\mu}\tilde{\mu}\tilde{\mu} = \tilde{\mu}^3$.

(iii) Let $\tilde{\mu}$ be an i-v fuzzy bi-ideal of $S$. Then $\tilde{\mu}\tilde{\chi}_{S\tilde{\mu}} = \tilde{\mu}\tilde{\chi}_{S\tilde{\mu}} \subseteq \tilde{\mu}\tilde{\chi}_{S\tilde{\mu}} \subseteq \tilde{\mu}$ (since $\tilde{\mu}$ is an i-v fuzzy bi-ideal of $S$), thus $\tilde{\mu}$ is an i-v fuzzy quasi-ideal of $S$.

(iv) Suppose that $\tilde{\mu}_l$ be an i-v fuzzy bi-ideal of a two sided ideal $T$ of $S$. Let $t \in T \subseteq S$. Since $S$ is regular, there exist $u \in S$ such that $t = utu$. This implies that $t = (tut)$. Since $T$ is a two-sided ideal of $S$, $atu \in T$ and
Theorem 4.11. (follows from the regularity of $T$ and Theorem 3.13)

Then, we deduce that in intra-regular, there exist $a\in S$ such that $x = \sum_{i=1}^{m} a_i x^2 b_i$. A semiring $S$ is said to be intra-regular if its every element is intra-regular.

Proof. Let $S$ be an intra-regular semiring. Let $\mu\in L$ and $\theta$ be an i.v. fuzzy left ideal and an i-v fuzzy right ideal of $S$ respectively. Suppose that $x \in S$. Since, $S$ is intra-regular, there exist $a_i, b_i \in S$ such that $x = \sum_{i=1}^{n} a_i x^2 b_i$. So, $x = \sum_{i=1}^{n} (a_i x)(x b_i)$.

Then, we deduce that

$$
(\tilde{\mu} \tilde{\theta})(x) = \sup \left\{ \inf_{1 \leq i \leq k} \left\{ \min^{i}(\tilde{\mu}(p_i), \tilde{\theta}(q_i)) : x = \sum_{i=1}^{k} p_i q_i \right\} : p_i, q_i \in S \right\}
$$

$$
\geq \inf_{1 \leq i \leq n} \{ \max^{i}(\tilde{\mu}(a_i x), \tilde{\theta}(x b_i)) \}
$$

$$
\geq \inf_{1 \leq i \leq n} \{ \min^{i}(\tilde{\mu}(x), \tilde{\theta}(x)) \}
$$

(since, $\tilde{\mu}$ is an i-v fuzzy left ideal and $\tilde{\theta}$ is an i-v fuzzy right ideal of $S$)

$$
= \min^{i}(\tilde{\mu}(x), \tilde{\theta}(x))
$$

$$
= (\tilde{\mu} \cap \tilde{\theta})(x).
$$

Thus, we obtain $\tilde{\mu} \cap \tilde{\theta} \subseteq \tilde{\mu} \tilde{\theta}$.

Conversely, suppose that $\tilde{\mu} \cap \tilde{\theta} \subseteq \tilde{\mu} \tilde{\theta}$, for any i-v fuzzy left ideal $\tilde{\mu}$ and i-v fuzzy right ideal $\tilde{\theta}$ of $S$. Let $L$ and $R$ be a left ideal and a right ideal of $S$ respectively. Then,
by our assumption, we have \( \overline{x_L} \cap \overline{x_R} \subseteq \overline{x_Lx_R} \). This implies that \( \overline{x_{L\cap R}} \subseteq \overline{x_{LR}} \), by Lemma 2.13 and Lemma 2.14. Thus, we have shown that \( L \cap R \subseteq LR \). Hence, \( S \) is an intra-regular semiring, by Theorem 4.11.

Now we state the main theorem. This theorem is a characterization theorem of a regular and intra-regular semiring \( S \) in terms of their \( i\)-\( v \) fuzzy quasi-ideal and \( i\)-\( v \) fuzzy bi-ideal of \( S \).

**Theorem 4.13.** Let \( S \) be a semiring. Then the following statements are equivalent.

(i) \( S \) is regular and intra-regular.

(ii) Every \( i\)-\( v \) fuzzy quasi-ideal of \( S \) is idempotent.

(iii) Every \( i\)-\( v \) fuzzy bi-ideal of \( S \) is idempotent.

(iv) \( \tilde{\mu} \cap \tilde{\theta} \subseteq \tilde{\mu} \tilde{\theta} \) for all \( i\)-\( v \) fuzzy quasi-ideals \( \tilde{\mu} \) and \( \tilde{\theta} \) of \( S \).

(v) \( \mu \cap \theta \subseteq \mu \theta \) for every \( i\)-\( v \) fuzzy quasi-ideal \( \mu \) and \( i\)-\( v \) fuzzy bi-ideal \( \theta \) of \( S \).

(vi) \( \tilde{\mu} \cap \tilde{\theta} \subseteq \tilde{\mu} \tilde{\theta} \) for every \( i\)-\( v \) fuzzy bi-ideal \( \tilde{\mu} \) and \( i\)-\( v \) fuzzy quasi-ideal \( \tilde{\theta} \) of \( S \).

(vii) \( \tilde{\mu} \cap \tilde{\theta} \subseteq \tilde{\mu} \tilde{\theta} \) for all \( i\)-\( v \) fuzzy bi-ideals \( \tilde{\mu} \) and \( \tilde{\theta} \) of \( S \).

**Proof.** (i) \( \Rightarrow \) (vii): Let (i) hold and \( x \in S \). Since \( S \) is regular, there exists \( a \in S \) such that \( x = xax \). So we can write \( x = xax \). Again since \( S \) is intra-regular, there exist \( a_i, b_i \in S \) such that \( x = \sum_{i=1}^{m} a_ix^2b_i \), where \( m \in \mathbb{N} \). Then from (1), we have \( x = xax + \sum_{i=1}^{m} (xaax)(xbax) \). Now let \( \tilde{\mu} \) and \( \tilde{\theta} \) be two \( i\)-\( v \) fuzzy bi-ideals of \( S \). Then, the following conditions hold:

\[
(\tilde{\mu} \tilde{\theta})(x) = \sup \left\{ \inf_{1 \leq i \leq m} \{\text{Min}^1(\tilde{\mu}(p_i), \tilde{\theta}(q_i)) : x = \sum_{i=1}^{n} p_iq_i, p_i, q_i \in S\} \right\}
\geq \inf_{1 \leq i \leq m} \{\text{Min}^1(\tilde{\mu}(xaax), \tilde{\theta}(xbax))\}
\geq \inf_{1 \leq i \leq m} \{\text{Min}^1\left(\text{Min}^1(\tilde{\mu}(x), \tilde{\mu}(x)), \text{Min}^1(\tilde{\theta}(x), \tilde{\theta}(x))\right)\}
\]

(since \( \tilde{\mu} \) and \( \tilde{\theta} \) are \( i\)-\( v \) fuzzy bi-ideals of \( S \))

\[
= \text{Min}^1(\tilde{\mu}(x), \tilde{\theta}(x))
= (\tilde{\mu} \cap \tilde{\theta})(x).
\]

Consequently, \( \tilde{\mu} \cap \tilde{\theta} \subseteq \tilde{\mu} \tilde{\theta} \).

(vii) \( \Rightarrow \) (vi): This implication is clear since each \( i\)-\( v \) fuzzy quasi-ideal of \( S \) is also an \( i\)-\( v \) fuzzy bi-ideal of \( S \).

(vi) \( \Rightarrow \) (v): Suppose that (vi) holds. Let \( \tilde{\mu} \) be an \( i\)-\( v \) fuzzy quasi-ideal and \( \tilde{\theta} \) be an \( i\)-\( v \) fuzzy bi-ideal of \( S \). Then \( \tilde{\mu} \) is also an \( i\)-\( v \) fuzzy bi-ideal of \( S \). Now, by our assumption, we have \( \tilde{\mu} \cap (\tilde{\theta})_0 \subseteq \tilde{\mu} \tilde{\theta} = \tilde{\mu} \cup (\tilde{\theta} \tilde{x}_S \cap \tilde{x}_S \tilde{\theta}) \). As \( \tilde{x}_S \) is an \( i\)-\( v \) fuzzy right ideal of \( S \), it is an \( i\)-\( v \) fuzzy quasi-ideal as well as an \( i\)-\( v \) fuzzy bi-ideal of \( S \). Again \( \tilde{x}_S \tilde{\theta} \) is an \( i\)-\( v \) fuzzy left ideal and hence an \( i\)-\( v \) fuzzy quasi-ideal of \( S \). Thus, by our assumption, we conclude that \( \tilde{\theta} \tilde{x}_S \cap \tilde{x}_S \tilde{\theta} \subseteq \tilde{\theta} \tilde{x}_S \tilde{\theta} \subseteq \tilde{\theta} \), since \( \tilde{\theta} \) is an \( i\)-\( v \) fuzzy bi-ideal of \( S \). Then by (2), we have \( \tilde{\mu} \cap (\tilde{\theta})_0 \subseteq \tilde{\mu} \cup (\tilde{\theta} \cup \tilde{\theta}) \subseteq
< \mu \tilde{\theta} > = \tilde{\mu} \tilde{\theta}. \text{ Thus, } \tilde{\mu} \cap \tilde{\theta} \subseteq \tilde{\mu} \cap (\tilde{\theta})_q \subseteq \tilde{\mu} \tilde{\theta}.

(v) \implies (iv) : \text{ It is clear since each i-v fuzzy quasi-ideal of } S \text{ is also an i-v fuzzy bi-ideal of } S.

(iv) \implies (iii) : \text{ Suppose that (iv) holds. Let } \tilde{\mu} \text{ be an i-v fuzzy bi-ideal of } S. \text{ Now, by our assumption, we have } \tilde{\mu} \subseteq (\tilde{\mu})_q = (\tilde{\mu} \cap (\tilde{\mu})_q)_q = < \tilde{\mu} \cup (\tilde{\mu} \tilde{\chi} \tilde{s} \cap \tilde{s} \tilde{\mu}) > < \tilde{\mu} \cup (\tilde{\mu} \tilde{x} \tilde{s} \cap \chi \tilde{s} \tilde{\mu}) > \ldots \ldots (3).

\text{ Finally, by our assumption, we have } \tilde{\mu} \tilde{x} \tilde{s} \cap \tilde{x} \tilde{s} \tilde{\mu} \subseteq \tilde{\mu} \tilde{x} \tilde{e} \tilde{s} \tilde{\mu} \subseteq \tilde{\mu} \tilde{x} \tilde{s} \tilde{\mu} \subseteq \tilde{\mu} \text{ since } \tilde{\mu} \text{ is an i-v fuzzy bi-ideal of } S. \text{ Hence, from (iii), it follows that } \tilde{\mu} \subseteq < \tilde{\mu} > < \tilde{\mu} > \subseteq < \tilde{\mu}^2 > = \tilde{\mu}^2. \text{ Again, since } \tilde{\mu} \text{ is an i-v quasi-ideal of } S, \text{ it follows that } \tilde{\mu}^2 \subseteq \tilde{\mu}. \text{ Consequently, we have } \tilde{\mu} = \tilde{\mu}^2.

(iii) \implies (ii) : \text{ This part is clear since each i-v fuzzy quasi-ideal of } S \text{ is also an i-v fuzzy bi-ideal of } S.

(ii) \implies (i) : \text{ This implication follows from Theorem in 3.14, Theorem 3.12 and Theorem 4.12.} \quad \Box

5. Conclusions

We have characterized regular and intra-regular semiring in terms of i-v fuzzy quasi-ideals and i-v fuzzy bi-ideals of a semiring. So this paper helps us to realize that we can study different properties of semirings and even some other algebraic structures from the view of i-v fuzzy set theory. For example, as a continuation of this paper we shall study the $k$-regularity and $k$-intra-regularity of a semiring in terms of i-v fuzzy $k$-quasi ideal and i-v fuzzy $k$-bi-ideal of semirings.

Acknowledgements. The second author is grateful to CSIR-India for providing financial assistance. We are very much thankful to referees for their valuable comments which help a lot to enrich this paper.

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