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A study on fuzzy soft set and its operations

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Abstract. In this paper, we study some operations of fuzzy soft sets and give fundamental properties of fuzzy soft sets. We discuss properties of fuzzy soft sets and their interrelation with respect to different operations such as union, intersection, restricted union and extended intersection. Then, we illustrate properties of OR, AND operations by giving counter examples. Also we prove that certain De Morgan's laws hold in fuzzy soft set theory with respect to different operations on fuzzy soft sets.

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1. Introduction

Modeling uncertain data has been an interesting subject in engineering, economics, environmental and social sciences. Crisp set theory helped a little for formal modeling, reasoning and computing uncertain data. However, there are many complex problems in various branches of science that involve data which is not always crisp. We can not successfully use classical methods because of various types of uncertainties present in these problems. Some mathematical theories such as theory of probability [24], theory of fuzzy sets [6, 9, 11, 27, 28], theory of intuitionistic fuzzy sets [3, 4], theory of rough sets [21], theory of vague sets [7], theory of interval mathematics [5, 8], are useful approaches to describe uncertainty. But, all these theories have their inherent difficulties as pointed out by Molodtsov in [16]. He pointed out the reasons for these difficulties are possibly, the inadequacy of the parameterization tool of these theories. In 1999 he introduced the Soft set theory (SST) as a new mathematical tool to deal with uncertain data which is free from such difficulties. This theory has proved to be useful in many different fields such as: decision making

[12, 13, 16, 23], data analysis [29], forecasting [25] and simulation [10]. Basic notion of soft sets was presented by [16]. Later on Maji et al. [15] introduced several operations of soft sets. Pei and Miao [20], M. Irfan. Ali et al. [2] introduced and studied several soft set operations. Sezgin and Atagun [23] studied several soft set operations and properties of restricted symmetric difference. Recently J. H. Park et al. [19] broaden the theoretical aspects of SST by defining the equivalence relation of soft sets and related results.

In recent times, researches are contributing towards fuzzification of Soft set theory due to fuzzy nature of parameters in complex real world problems. In 2001 Maji et al. [14] initiated the concept of Fuzzy Soft Sets and give some properties, followed and improved by Ahmad and Kharal [1]. Later on Neog and Sut [17, 18] studied and improved the notion of fuzzy soft complement and some properties of fuzzy soft sets along with examples and proofs of certain results.

This paper is about fuzzification of some results of SST. The paper is organized as follows: First we study in detail, operations of fuzzy soft sets. Then, we discuss some properties of fuzzy soft sets with respect to different fuzzy soft operations and interconnections between each other. Finally, we discuss certain De Morgan's laws in fuzzy soft set theory.

2. Preliminaries

In this section, we recall some basic notions in fuzzy soft set theory. Let $\mathbf{X}$ be initial universe set and $\mathbf{E_X}$ be the set of all possible parameters under consideration with respect to $\mathbf{X}$.

$A, B \subseteq \mathbf{E}_X$ and $\widetilde{P}(X)$ is the set of all fuzzy subsets of X. Usually, parameters are attributes, characteristics or properties of objects in X. In what follows, $\mathbf{E}_X$ (simply denoted by E) always stands for the universe set of parameters with respect to X, unless otherwise specified.

Maji et al.[25] defined a fuzzy soft set in the following manner.

Definition 2.1. ([25]) A pair (F, A) is called a fuzzy soft set over X,

$$\text{where } F\colon A \to \widetilde{P}(X) \text{ is a mapping from } A \text{ into } \widetilde{P}(X).$$

In other words, a fuzzy soft set over X is a parameterized family of fuzzy subsets of the universe X. For $e \in A$, $F(e)$ may be considered as the fuzzy subset of e-approximate elements of the fuzzy soft set (F, A).

Definition 2.2. ([26]) Let X be a universe and E a set of parameters. Then the pair $\widehat{(X, E)}$ denotes the collection of all fuzzy soft sets on X with parameters from E and is called fuzzy soft class.

Definition 2.3. ([25]) For two fuzzy soft sets (F, A) and (G, B) in a fuzzy soft class $\widehat{(X, E)}$, we say that (F, A) is a fuzzy soft subset of (G, B), if

(i) $A \subseteq B$

(ii) $F(e) \leq G(e)$ for all $e \in A$

It is denoted by $(F, A) \mathrel{\widetilde{\subseteq}} (G, B)$

Definition 2.4. For two fuzzy soft sets (F, A) and (G, B) in a fuzzy soft class $\widehat{(X, E)}$, we say that (G, B) is a fuzzy soft superset of (F, A), if $(F, A) \mathrel{\widetilde{\subset}} (G, B)$. i.e

(i) $A \subset B$
(ii) $F(e) < G(e)$ for all $e \in A$
It is denoted by $(G, B) \widetilde{\supseteq} (F, A)$

Definition 2.5. For two fuzzy soft sets (F, A) and (G, B) in a fuzzy soft class $\widehat{(X, E)}$, we say that (F, A) is equal to (G, B), if $(F, A) \widetilde{\subseteq} (G, B)$ and $(G, B) \widetilde{\subseteq} (F, A)$

Definition 2.6. ([25]) Union of two fuzzy soft sets (F, A) and (G, B) in a fuzzy soft class $\widehat{(X, E)}$ is a fuzzy soft set (H, C), where $C = A \cup B$ and for all $e \in C$.

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B \\ G(e) & \text{if } e \in B \setminus A \\ F(e) \vee G(e) & \text{if } e \in A \cap B \end{cases}$$

This relation is denoted by $(F, A) \widetilde{\bigvee} (G, B) = (H, C)$.

Definition 2.7. ([25]) Intersection of two fuzzy soft sets (F, A) and (G, B) in a fuzzy soft class $\widehat{(X, E)}$ is a fuzzy soft set (H, C), where $C = A \cap B$ and for all $e \in C$. $H(e) = F(e)$ or $G(e)$,(as both are the same set).

This relation is denoted by$(F, A) \widetilde{\bigwedge} G, B) = (H, C)$

Pei and Miao et al. [21] improved the definition for the intersection of soft sets and B.Ahmad et al. [26] use the same notion for the fuzzy soft set as follows:

Definition 2.8. ([26]) Let (F, A) and (G, B) be two fuzzy soft sets in a fuzzy soft class $\widehat{(X, E)}$. The intersection of two fuzzy soft sets (F, A) and (G, B) is a fuzzy soft set (H, C), where $H(e) = F(e) \wedge G(e)$ and for all $e \in C = A \cap B \neq \varnothing$.

Since the notation of Maji et al. [25], B.Ahmad and A.kharal et al. [26] are same, thus may mislead the readers. We denote "(F, A) intersection (G, B)" by "$(F, A) \Cap (G, B) = (H, C)$" as M.Irfan.Ali et al. [22] used for soft sets. We call it restricted intersection of fuzzy soft set.

Definition 2.9. Let (F, A) and (G, B) be two fuzzy soft sets in a fuzzy soft class $\widehat{(X, E)}$. The extended intersection of (F, A) and (G, B) is a fuzzy soft set (H, C),where $C = A \cup B$ and for all $e \in C$.

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B \\ G(e) & \text{if } e \in B \setminus A \\ F(e) \wedge G(e) & \text{if } e \in A \cap B \end{cases}$$

This relation is denoted by $(F, A) \sqcap_{\varepsilon} (G, B) = (H, C)$

Definition 2.10. Let (F, A) and (G, B) be two fuzzy soft sets in a fuzzy soft class $\widehat{(X, E)}$. The restricted union of (F, A) and (G, B) is a fuzzy soft set $(H, A \cap B)$,

where $H(e) = F(e) \vee G(e)$ for all $e \in A \cap B \neq \varnothing$.

This relation is denoted by $(F, A) \widetilde{\bigvee}_{\Re} (G, B) = (H, A \cap B)$

Definition 2.11. ([14]) Let $E = \{e_1, e_2, e_3, ..., e_n\}$ be a set of parameters of the universe X. The NOT set of E denoted by $\rceil E$ is defined as $\rceil E = \{\neg e_1, \neg e_2, \neg e_3, ...,$ $\neg e_n\}$, where $\neg e_i =$ not e_i $\forall i \in \{1, 2, 3, ..., n\}$.

Proposition 2.12. ([14]) *Let E be a set of parameters and $A, B \subseteq E$, then*
(a) $\rceil(\rceil A) = A$
(b) $\rceil(A \cup B) = \rceil A \cup \rceil B$
(c) $\rceil(A \cap B) = \rceil A \cap \rceil B$

Definition 2.13. ([25]) A fuzzy soft set (F, A) is said to be a null fuzzy soft set, denoted by Φ, if $F(e) = \overline{0}$ for all $e \in A$. Where, $\overline{0}$ is the null fuzzy subset of X.

Definition 2.14. ([25]) A fuzzy soft set (F, A) is said to be an absolute fuzzy soft set, denoted by $\widetilde{A}$, if $F(e) = \overline{1}$ for all $e \in A$. Where, $\overline{1}$ is the absolute fuzzy subset of X.

Definition 2.15. ([25]) The complement of a fuzzy soft set (F, A) is denoted by $(F, A)^c$ and is defined as $(F, A)^c = (F^c, \rceil A)$,

where $F^c : \rceil A \longrightarrow \widetilde{P}(X)$ is a mapping given by $F^c(e) = [F(\neg e)]^c = \widetilde{A} \setminus F(\neg e)$ for all $e \in \rceil A$.

Neog and Sut [27] defined an alternative definition for the complement of fuzzy soft set given as follows:

Definition 2.16. ([27]) The complement of fuzzy soft set (F, A) is denoted by $(F, A)^c$ and is defined as $(F, A)^c = (F^c, A)$,

where $F^c : A \rightarrow \widetilde{P}(X)$ is a mapping given by $F^c(e) = [F(e)]^c = \widetilde{A} \setminus F(e)$ for all $e \in A$.

Since the notation of Maji et al.[25] and Neog and Sut [27] are same, thus may mislead the readers. So we propose a different notation and name for the complement of fuzzy soft set defined by [27] with the same notion as proposed by M.Irfan.Ali et al. [22] for the soft sets as follows:

Definition 2.17. The relative complement of a fuzzy soft set (F, A) is denoted by $(F, A)^r$ and is defined as $(F, A)^r = (F^r, A)$,

where $F^r : A \rightarrow \widetilde{P}(X)$ is a mapping given by $F^r(e) = [F(e)]^r = \widetilde{A} \setminus F(e)$ for all $e \in A$.

Definition 2.18. A fuzzy soft set (F, A) is said to be a relative null fuzzy soft set (with respect to parameter set A), denoted by Φ_A , if $F(e) = \overline{0}_A$, for all $e \in A$. Where, $\overline{0}_A$ is the null fuzzy subset of X with respect to parameter set A.

Definition 2.19. A fuzzy soft set (F, A) is said to be a relative absolute fuzzy soft set (with respect to parameter set A), denoted by $\mathfrak{U}_A$, if $F(e) = \overline{1}_A$, for all $e \in A$. Where, $\overline{1}_A$ is the absolute fuzzy subset of X with respect to parameter set A. The relative absolute fuzzy soft set $\mathfrak{U}_E$ with respect to the universe set of parameters E is the absolute fuzzy soft set $\widetilde{A}$.

Definition 2.20. ([25]) If (F, A) and (G, B) are two fuzzy soft sets in a fuzzy soft class $\widehat{(X, E)}$, then " (F, A) AND (G, B)" denoted by $(F, A) \wedge (G, B)$ is defined by

$(F, A) \wedge (G, B) = (H, A \times B)$, where $H((e_1, e_2)) = F(e_1) \wedge G(e_2)$ for all$(e_1, e_2) \in A \times B$.

Definition 2.21. ([25]) If $(F, A)$and (G, B) are two fuzzy soft sets in a fuzzy soft class $\widehat{(X, E)}$, then "(F, A) OR (G, B) " denoted by $(F, A) \vee (G, B)$ is defined by

$(F, A) \vee (G, B) = (H, A \times B)$, where $H((e_1, e_2)) = F(e_1) \vee G(e_2)$ for all $(e_1, e_2) \in A \times B$.

Remark 2.22. From definition of equal fuzzy soft sets it is clear that two fuzzy soft sets are equal if and only if they have same set-valued mapping.

i.e $(F, A) = (G, B)$ if and only if $F(e) = G(e)$ for all $e \in A = B$

3. Major section

In this section we will illustrate commutative, identity, inclusion, associative and distributive properties of fuzzy soft sets by using different fuzzy soft operations like union, restricted union, extended intersection and restricted intersection. Some examples will illustrate the solution and applicability of these properties. We will also discuss the absorption and idempotent laws by using these operations. At the end of this section we will discuss some properties of OR, AND fuzzy soft operations.

Theorem 3.1. ***Properties of the unoin*** $(\widetilde{\bigvee})$ ***operation***

(a) $(F, A)\widetilde{\bigvee}(G, B) = (G, B)\widetilde{\bigvee}(F, A)$

(b) $(F, A)\widetilde{\bigvee}\Phi_A = (F, A)$, *and* $(F, A)\widetilde{\bigvee}\mathfrak{U}_A = \mathfrak{U}_A$ [29]

(c) $(F, A)\widetilde{\bigvee}\Phi_B = (F, A) \Leftrightarrow B \subseteq A$ [29]

(d) $(F, A)\widetilde{\bigvee}(G, A) = \Phi_A \Leftrightarrow (F, A) = \Phi_A$ *and* $(G, A) = \Phi_A$

(e) $(F, A)\widetilde{\bigvee}((G, B)\widetilde{\bigvee}(H, C)) = ((F, A)\widetilde{\bigvee}(G, B))\widetilde{\bigvee}(H, C)$

(f) $(F, A)\widetilde{\nsubseteq}((F, A)\widetilde{\bigvee}(G, B))$ *in-general. But, if* $(F, A)\widetilde{\subseteq}(G, B)$*, then* $(F, A)\widetilde{\subseteq}((F, A)\widetilde{\bigvee}(G, B))$*, moreover* $(F, A) = (F, A)\widetilde{\bigvee}(G, B)$.

(g) $(F, A) \Cap ((G, B))\widetilde{\bigvee}(H, C)) = ((F, A) \Cap (G, B))\widetilde{\bigvee}((F, A) \Cap (H, C))$

(h) $(F, A)\widetilde{\bigvee}((G, B) \Cap (H, C)) = ((F, A)\widetilde{\bigvee}(G, B)) \Cap ((F, A)\widetilde{\bigvee}(H, C))$

(i) $((F, A) \Cap (G, B))\widetilde{\bigvee}(H, C) = ((F, A)\widetilde{\bigvee}(H, C)) \Cap ((G, B)\widetilde{\bigvee}(H, C))$

Proof. **(d)**. Let $(F, A)\widetilde{\bigvee}(G, A) = (H, A)$ and for all $e \in A$, $H(e) = F(e) \vee G(e)$. Now, it is given that $(F, A)\widetilde{\bigvee}(G, B) = \Phi_A$, then $H(e) = \overline{0}_A$ for all $e \in A$. This implies $F(e) \vee G(e) = \overline{0}_A$. Thus, $F(e) = \overline{0}_A$ and $G(e) = \overline{0}_A$ for all $e \in A$. Hence $(F, A) = \Phi_A$ and $(G, A) = \Phi_A$.

Conversely, suppose that $(F, A) = \Phi_A$ and $(G, A) = \Phi_A$. By definition of Φ_A, $F(e) = \overline{0}_A$ and $G(e) = \overline{0}_A$ for all $e \in A$. Thus, $F(e)\vee G(e) = \overline{0}_A$. Hence $(F, A)\widetilde{\bigvee}(G, A) = \Phi_A$.

(f). Let $(F, A)\widetilde{\bigvee}(G, B) = (H, C)$, where $C = A \cup B$. For all $e \in C$, there are three possibilities, either $e \in A$ or $e \in B$ or $e \in A \cap B$. If $e \in A$, then $F(e) = H(e)$. If $e \in B$, then $H(e) = G(e)$. If $e \in A \cap B$, then $H(e) = F(e) \vee G(e)$ implies $F(e) \leq H(e)$ for all $e \in A \cap B$. But, for all the cases $F(e) \nleq H(e)$ for all $e \in C$. Therefore, $(F, A) \widetilde{\nsubseteq} ((F, A)\widetilde{\bigvee}(G, B))$ in-general.

Now, suppose that $(F, A)\widetilde{\subseteq}(G, B)$, then $A \subseteq B$ and $F(e) \leq G(e)$ for all $e \in A$, then $A \setminus B = \varnothing$. For $e \in A \cap B, H(e) = F(e) \vee G(e)$, since $A \subseteq B$ and $F(e) \leq G(e)$, then $H(e) = F(e) \vee F(e) = F(e)$ for all $e \in A \cap B = A$. This follows that F and H are same set-valued mapping, hence the result.

(g). First, we investigate left hand side of the equality. Suppose $(G, B)\widetilde{\bigvee}(H, C) = (I, D)$, where $D = B \cup C$ and for all $e \in D$

$$I(e) = \begin{cases} G(e) & \text{if } e \in B \setminus C \\ H(e) & \text{if } e \in C \setminus B \\ G(e) \vee H(e) & \text{if } e \in B \cap C \end{cases}$$

Also $(F, A) \Cap ((G, B))\widetilde{\bigvee}(H, C) = (F, A) \Cap (I, D) = (J, A \cap D)$ for all $e \in A \cap D \neq \varnothing$, $J(e) = F(e) \wedge I(e)$, where $A \cap D = A \cap (B \cup C)$. By considering I as piecewise defined function, we have

$$J(e) = \begin{cases} F(e) \wedge G(e) & \text{if } e \in A \cap (B \setminus C) \\ F(e) \wedge H(e) & \text{if } e \in A \cap (C \setminus B) \\ F(e) \wedge (G(e) \vee H(e)) & \text{if } e \in A \cap (B \cap C) \end{cases}$$

$$J(e) = \begin{cases} F(e) \wedge G(e) & \text{if } e \in (A \cap B) \setminus (A \cap C) \\ F(e) \wedge H(e) & \text{if } e \in (A \cap C) \setminus (A \cap B) \\ (F(e) \wedge G(e)) \vee (F(e) \wedge H(e)) & \text{if } e \in (A \cap B) \cap C \end{cases} \tag{3.1}$$

Consider the right hand side of the equality. Suppose that $(F, A) \Cap (G, B) = (K, A \cap B)$, where $K(e) = F(e) \wedge G(e)$ for all $e \in A \cap B \neq \varnothing$, also let $(F, A) \Cap (H, C) = (L, A \cap C)$, where $L(e) = F(e) \wedge H(e)$ for all $e \in A \cap C \neq \varnothing$.

Now, $((F, A) \Cap (G, B))\widetilde{\bigvee}((F, A) \Cap (H, C)) = (K, A \cap B)\widetilde{\bigvee}(L, A \cap C) = (P, R)$, where $R = (A \cap B) \cup (A \cap C) = A \cap (B \cup C)$ and for all $e \in R$

$$P(e) = \begin{cases} K(e) & \text{if } e \in (A \cap B) \setminus (A \cap C) \\ L(e) & \text{if } e \in (A \cap C) \setminus (A \cap B) \\ K(e) \vee L(e) & \text{if } e \in (A \cap B) \cap (A \cap C) \end{cases}$$

Considering K and L as piecewise functions, we have

$$P(e) = \begin{cases} F(e) \wedge G(e) & \text{if } e \in (A \cap B) \setminus (A \cap C) \\ F(e) \wedge H(e) & \text{if } e \in (A \cap C) \setminus (A \cap B) \\ (F(e) \wedge G(e) \vee (F(e) \wedge H(e)) & \text{if } e \in (A \cap B) \cap C \end{cases} \tag{3.2}$$

From (3.1) and (3.2) it is clear that J and P are same set-valued mapping. Hence $(F, A) \Cap ((G, B))\widetilde{\bigvee}(H, C)) = ((F, A) \Cap (G, B))\widetilde{\bigvee}((F, A) \Cap (H, C))$.

(h). By similar techniques used to prove (g), (h) can be illustrated, therefore, omitted.

(i). Let $(F, A) \Cap (G, B) = (I, A \cap B)$,where, $I(e) = F(e) \wedge G(e)$ for all $e \in A \cap B \neq \varnothing$.

Also, $((F, A) \Cap (G, B))\widetilde{\bigvee}(H, C) = (I, A \cap B)\widetilde{\bigvee}(H, C) = (J, D)$, where $D = (A \cap B) \cup C$ and for all $e \in D$, we have

$$J(e) = \begin{cases} I(e) & \text{if } e \in (A \cap B) \setminus C \\ H(e) & \text{if } e \in C \setminus (A \cap B) \\ I(e) \vee H(e) & \text{if } e \in (A \cap B) \cap C \end{cases}$$

By taking into account the definition of I along with J, we can write

$$J(e) = \begin{cases} F(e) \wedge G(e) & \text{if } e \in (A \cap B) \setminus C \\ H(e) & \text{if } e \in C \setminus (A \cap B) \\ (F(e) \wedge G(e)) \vee H(e) & \text{if } e \in (A \cap B) \cap C \end{cases}$$

By using properties of operations in set theory, we have

$$J(e) = \begin{cases} F(e) \wedge G(e) & \text{if } e \in (A \setminus C) \cap (B \setminus C) \\ H(e) & \text{if } e \in (C \setminus A) \cup (C \setminus B) \\ (F(e) \vee H(e)) \wedge (G(e) \vee H(e)) & \text{if } e \in A \cap (B \cap C) \end{cases} \tag{3.3}$$

Consider the right hand side of the equality. Let $(F, A)\widetilde{\bigvee}(H, C) = (K, M)$, where $M = A \cup C$ and for all $e \in M$

$$K(e) = \begin{cases} F(e) & \text{if } e \in A \setminus C \\ H(e) & \text{if } e \in C \setminus A \\ F(e) \vee H(e) & \text{if } e \in A \cap C \end{cases}$$

Also, suppose that $(G, B)\widetilde{\bigvee}(H, C) = (L, N)$, where $N = B \cup C$ and for all $e \in N$

$$L(e) = \begin{cases} G(e) & \text{if } e \in B \setminus C \\ H(e) & \text{if } e \in C \setminus B \\ G(e) \vee H(e) & \text{if } e \in B \cap C \end{cases}$$

Now,$((F, A)\widetilde{\bigvee}(H, C)) \Cap ((G, B)\widetilde{\bigvee}(H, C)) = (K, M) \Cap (L, N) = (P, M \cap N)$, where $P(e) = K(e) \wedge L(e)$ for all $e \in M \cap N = (A \cup C) \cap (B \cup C) = (A \cap B) \cup C \neq \varnothing$. By taking into account the properties of operations in set theory and considering K, L as piecewise defined functions, we have

$$P(e) = \begin{cases} F(e) \wedge G(e) & \text{if } e \in (A \setminus C) \cap (B \setminus C) \\ H(e) \wedge H(e) & \text{if } e \in (C \setminus A) \cup (C \setminus B) \\ (F(e) \vee H(e)) \vee (G(e) \vee H(e)) & \text{if } e \in A \cap (B \cap C) \end{cases}$$

$$P(e) = \begin{cases} F(e) \wedge G(e) & \text{if } e \in (A \setminus C) \cap (B \setminus C) \\ H(e) & \text{if } e \in (C \setminus A) \cup (C \setminus B) \\ (F(e) \vee H(e)) \vee (G(e) \vee H(e)) & \text{if } e \in A \cap (B \cap C) \end{cases} \tag{3.4}$$

From (3.3) and (3.4) it is clear that J and P are same set-valued mapping, which completes the proof. □

The following example illustrates theorem 1(g).

Example 3.2. Let $X = \{a, b, c, d\}$ be the set of four cars under consideration and $E = \{e_1 =$ expensive, $e_2 =$ beautiful, $e_3 =$ cheap, $e_4 =$ fuel efficient, $e_5 =$ latest model$\}$ be the set of parameters. Consider the fuzzy soft sets (F, A), (G, B) and (H, C) representing the attractiveness of the cars are defined as follows:

$$\begin{aligned} (F, A) &= \left\{ \begin{array}{c} e_1 = \{a_{0.1}, b_{0.3}, c_{0.8}, d_{0.1}\}, e_2 = \{a_{0.5}, b_{0.7}, c_{0.8}, d_1\}, \\ e_3 = \{a_{0.8}, b_{0.7}, c_{0.9}, d_{0.5}\} \end{array} \right\}. \\ (G, B) &= \{e_2 = \{a_{0.4}, b_{0.3}, c_0, d_{0.4}\}, e_4 = \{a_{0.4}, b_{0.9}, c_{0.3}, d_{0.7}\}\}. \\ (H, C) &= \{e_3 = \{a_{0.5}, b_{0.7}, c_{0.8}, d_1\}, e_4 = \{a_{0.8}, b_{0.7}, c_{0.9}, d_{0.5}\}\}. \end{aligned}$$

Now consider $(G,B)\widetilde{\bigvee}(H,C) = \left\{ \begin{array}{c} e_2 = \{a_{0.4}, b_{0.3}, c_0, d_{0.4}\}, e_3 = \{a_{0.5}, b_{0.7}, c_{0.8}, d_1\}, \\ e_4 = \{a_{0.8}, b_{0.9}, c_{0.9}, d_{0.7}\} \end{array} \right\}$.
This imply

$$(F,A) \Cap ((G,B))\widetilde{\bigvee}(H,C)) = \left\{ \begin{array}{c} e_2 = \{a_{0.4}, b_{0.3}, c_0, d_{0.4}\}, \\ e_3 = \{a_{0.5}, b_{0.7}, c_{0.8}, d_{0.5}\} \end{array} \right\}. \tag{3.5}$$

Also

$$(F,A) \Cap (G,B) = \{e_2 = \{a_{0.4}, b_{0.3}, c_0, d_{0.4}\},\}.$$

and

$$(F,A) \Cap (H,C) = \{e_3 = \{a_{0.5}, b_{0.7}, c_{0.8}, d_{0.5}\}\}.$$

This imply

$$((F,A) \Cap (G,B))\widetilde{\bigvee}((F,A) \Cap (H,C)) = \left\{ \begin{array}{c} e_2 = \{a_{0.4}, b_{0.3}, c_0, d_{0.4}\}, \\ e_3 = \{a_{0.5}, b_{0.7}, c_{0.8}, d_{0.5}\} \end{array} \right\}. \tag{3.6}$$

From (3.5) and (3.6) it is clear that

$$(F,A) \Cap ((G,B))\widetilde{\bigvee}(H,C)) = ((F,A) \Cap (G,B))\widetilde{\bigvee}((F,A) \Cap (H,C)).$$

Proposition 3.3. *For any fuzzy soft sets (F,A) and (G,B) in $\widehat{(X,E)}$, the following properties hold.*

(a) $(F,A)\widetilde{\bigvee}(F,A) = (F,A)$
(b) $(F,A) \Cap (F,A) = (F,A)$
(c) $(F,A)\widetilde{\bigvee}((F,A) \Cap (G,B)) = (F,A)$
(d) $(F,A) \Cap ((F,A)\widetilde{\bigvee}(G,B)) = (F,A)$

Proof. **(a)** and **(b)** are straightforward and can be proved easily.

(c).Suppose that $(F,A) \Cap (G,B) = (H, A \cap B)$, where $H(e) = F(e) \wedge G(e)$ for all $e \in A \cap B \neq \varnothing$. Now $(F,A)\widetilde{\bigvee}((F,A) \Cap (G,B)) = (I,D)$, where $D = A \cup (A \cap B) = A$, then for all $e \in A$ we have, $I(e) = F(e)$. Since I and F are same set-valued mapping. Hence $(F,A)\widetilde{\bigvee}((F,A) \Cap (G,B)) = (F,A)$

By using similar technique we can prove (d). □

Theorem 3.4. ***Properties of the Restricted union*** $(\widetilde{\bigvee}_{\Re})$ ***operation***

(a) $(F,A)\widetilde{\bigvee}_{\Re}(G,B) = (G,B)\widetilde{\bigvee}_{\Re}(F,A)$
(b) $(F,A)\widetilde{\bigvee}_{\Re}\Phi_A = (F,A)$ *and* $(F,A)\widetilde{\bigvee}_{\Re}\mathfrak{U}_A = \mathfrak{U}_A$
(c) $(F,A)\widetilde{\bigvee}_{\Re}\Phi_B = (F,A) \Leftrightarrow B \subseteq A$
(d) $(F,A)\widetilde{\bigvee}_{\Re}(G,A) = \Phi_A \Leftrightarrow (F,A) = \Phi_A$ *and* $(G,A) = \Phi_A$
(e) $(F,A)\widetilde{\bigvee}_{\Re}((G,B)\widetilde{\bigvee}_{\Re}(H,C)) = ((F,A)\widetilde{\bigvee}_{\Re}(G,B))\widetilde{\bigvee}_{\Re}(H,C)$
(f) $(F,A)\widetilde{\nsubseteq}((F,A)\widetilde{\bigvee}_{\Re}(G,B))$ *in-general. But, if* $(F,A)\widetilde{\subseteq}(G,B)$, *then* $(F,A)\widetilde{\subseteq}((F,A)\widetilde{\bigvee}_{\Re}(G,B))$ *moreover* $(F,A) = (F,A)\widetilde{\bigvee}_{\Re}(G,B)$
(g) $(F,A)\widetilde{\bigvee}_{\Re}((G,B) \Cap (H,C)) = ((F,A)\widetilde{\bigvee}_{\Re}(G,B)) \Cap ((F,A)\widetilde{\bigvee}_{\Re}(H,C))$
(h) $(F,A) \Cap ((G,B))\widetilde{\bigvee}_{\Re}(H,C)) = ((F,A) \Cap (G,B))\widetilde{\bigvee}_{\Re}((F,A) \Cap (H,C))$
(i) $((F,A) \Cap (G,B))\widetilde{\bigvee}_{\Re}(H,C) = ((F,A) \Cap (H,C))\widetilde{\bigvee}_{\Re}((G,B) \Cap (H,C))$

Proof. **(c)**. Suppose $(F,A)\widetilde{\bigvee}_{\Re}\Phi_B = (H, A\cap B)$, where $H(e) = F(e)\vee \overline{0}_B$ for all $e\in A\cap B\neq\varnothing$. Now, it is given that $B\subseteq A$ then $A\cap B = B$ this implies that $H(e) = F(e)\vee\overline{0}_B = F(e)$ for all $e\in A\cap B = B$. This shows that H and F are same set valued mapping, thus $(F,A)\widetilde{\bigvee}_{\Re}\Phi_B = (F,A)$. Conversely, let $(F,A)\widetilde{\bigvee}_{\Re}\Phi_B = (F,A)$. Then, $(H, A\cap B) = (F,A)$ which implies $A\cap B = A$. Thus, $B = A$ implies $B\subseteq A$ hence the required result.

(d). Let $(F,A)\widetilde{\bigvee}_{\Re}(G,A) = (H,A)$ where, $H(e) = F(e)\vee G(e)$ for all $e\in A$. Now, it is given that $(F,A)\widetilde{\bigvee}_{\Re}(G,A) = \Phi_A$ then $H(e) = \overline{0}_A$, implies $F(e)\vee G(e) = \overline{0}_A$ for all $e\in A$. Thus,$F(e) = \overline{0}_A$ and $G(e) = \overline{0}_A$, therefore, $(F,A) = \Phi_A$ and $(G,A) = \Phi_A$.Conversely, suppose that $(F,A) = \Phi_A$ and $(G,A) = \Phi_A$. Then, by definition of Φ_A, $F(e) = \overline{0}_A$ and $G(e) = \overline{0}_A$ for all $e\in A \Rightarrow F(e)\vee G(e) = \overline{0}_A$. Hence, $(F,A)\widetilde{\bigvee}_{\Re}(G,A) = \Phi_A$.

(e). First, we investigate the left hand side of the equality. Suppose $(G,B)\widetilde{\bigvee}_{\Re}(H,C)$ $= (I, B\cap C)$, where $I(e) = G(e)\vee H(e)$ for all $e\in B\cap C\neq\varnothing$. Also consider

$$(F,A)\widetilde{\bigvee}_{\Re}((G,B)\widetilde{\bigvee}_{\Re}(H,C) = (F,A)\widetilde{\bigvee}_{\Re}(I, B\cap C) = (J, A\cap(B\cap C)),$$

where $J(e) = F(e)\vee I(e)$ for all $e\in A\cap(B\cap C)\neq\varnothing$. By taking into account the definition of I, we have

$$J(e) = F(e)\vee(G(e)\vee H(e)) = (F(e)\vee G(e))\vee H(e)$$

Consider the right hand side of the equality. Let $(F,A)\widetilde{\bigvee}_{\Re}(G,B) = (K, A\cap B)$, where $K(e) = F(e)\vee G(e)$ for all $e\in A\cap B\neq\varnothing$.Now

$$((F,A)\widetilde{\bigvee}_{\Re}(G,B))\widetilde{\bigvee}_{\Re}(H,C) = (K, A\cap B)\widetilde{\bigvee}_{\Re}(H,C) = (L, (A\cap B)\cap C),$$

where $L(e) = K(e)\vee H(e)$ for all $e\in(A\cap B)\cap C\neq\varnothing$. By taking into account the definition of K, we have $L(e) = (F(e)\vee G(e))\vee H(e)$ for all $e\in(A\cap B)\cap C\neq\varnothing$. Since J and L are same set-valued mapping, hence the result.

(f). Let $(F,A)\widetilde{\bigvee}_{\Re}(G,B) = (H, A\cap B)$, where $H(e) = F(e)\vee G(e)$ for all $e\in A\cap B\neq\varnothing$. Then, $F(e)\leq H(e)$, but $A\nsubseteq A\cap B$. Hence, $(F,A)\widetilde{\nsubseteq}((F,A)\widetilde{\bigvee}(G,B))$ in-general.

Now, suppose that $(F,A)\widetilde{\subseteq}(G,B)$,then $A\subseteq B$ and $F(e)\leq G(e)$ for all $e\in A$. $\Rightarrow$ $A = A\cap B$ and $H(e) = F(e)\vee F(e) = F(e)$ for all $e\in A$, .

It conclude that F and H are same set-valued mapping, thus the desired result.

(g). Suppose $(G,B)\Cap(H,C) = (I, B\cap C)$, where $I(e) = G(e)\wedge H(e)$ for all $e\in B\cap C\neq\varnothing$. Also assume that $(F,A)\widetilde{\bigvee}_{\Re}((G,B)\Cap(H,C) = (F,A)\widetilde{\bigvee}_{\Re}(I, B\cap C) = (J, A\cap(B\cap C))$,where $J(e) = F(e)\vee I(e)$ for all $e\in A\cap(B\cap C)\neq\varnothing$. By considering the definition of I, we have

$$J(e) = F(e)\vee(G(e)\wedge H(e)) = (F(e)\vee G(e))\wedge(F(e)\vee H(e)) \text{ for all } e\in A\cap(B\cap C)\neq\varnothing.$$

Now, we investigate the hand side of the equality. Let$(F,A)\widetilde{\bigvee}_{\Re}(G,B) = (K, A\cap B)$, where $K(e) = F(e)\vee G(e)$ for all $e\in A\cap B\neq\varnothing$. Also assume that $(F,A)\widetilde{\bigvee}_{\Re}(H,C) = (L, A\cap C)$, where $L(e) = F(e)\vee H(e)$ for all $e\in A\cap C\neq\varnothing$. Now consider $((F,A)\widetilde{\bigvee}_{\Re}(G,B))\Cap((F,A)\widetilde{\bigvee}_{\Re}(H,C) = (K, A\cap B)\Cap(L, A\cap C) = (M, (A\cap B)\cap (A\cap C))$, where $M(e) = K(e)\wedge L(e)$ for all $e\in(A\cap B)\cap(A\cap C)\neq\varnothing$. By taking into account the definitions of K and L along with M, we have $M(e) = (F(e) \vee G(e)) \wedge (F(e) \vee H(e))$ for all $e \in (A \cap B) \cap (A \cap C)$. Since J and M are same set-valued mapping, thus the result.

(h). By similar techniques used to prove **(g)**, we can prove **(h)** and **(i)**, therefore, we omitted. □

Now, we will illustrate theorem 2(g) with an example.

Example 3.5. Let $X = \{a, b, c, d\}$ and $E = \{e_1, e_2, e_3, e_4\}$ be initial universe and parameter set. Consider the fuzzy soft sets $(F, A), (G, B)$ and (H, C) defined as follows:

$$\begin{aligned}(F, A) &= \{e_3 = \{a_{0.1}, b_{0.3}, c_{0.8}, d_{0.1}\}, e_4 = \{a_{0.5}, b_{0.7}, c_{0.8}, d_1\}\}.\\(G, B) &= \{e_2 = \{a_{0.4}, b_{0.3}, c_0, d_{0.4}\}, e_4 = \{a_{0.4}, b_{0.9}, c_{0.3}, d_{0.7}\}\}.\\(H, C) &= \{e_1 = \{a_{0.5}, b_{0.7}, c_{0.8}, d_1\}, e_4 = \{a_{0.8}, b_{0.7}, c_{0.9}, d_{0.5}\}\}.\end{aligned}$$

Now $(G, B)\widetilde{\bigvee}_{\Re}(H, C) = \{e_4 = \{a_{0.8}, b_{0.9}, c_{0.9}, d_{0.7}\}\}$. Then,

$$(F, A) \Cap ((G, B)\widetilde{\bigvee}_{\Re}(H, C)) = \{e_4 = \{a_{0.5}, b_{0.7}, c_{0.8}, d_{0.7}\}. \tag{3.7}$$

Also

$$(F, A) \Cap (G, B) = \{e_4 = \{a_{0.4}, b_{0.7}, c_{0.3}, d_{0.7}\}\}.$$

and

$$(F, A) \Cap (H, C) = \{e_4 = \{a_{0.5}, b_{0.7}, c_{0.8}, d_{0.5}\}\}.$$

$$((F, A) \Cap (G, B))\widetilde{\bigvee}_{\Re}((F, A) \Cap (H, C)) = \{e_4 = \{a_{0.5}, b_{0.7}, c_{0.8}, d_{0.7}\}\}. \tag{3.8}$$

From (3.7) and (3.8), we have

$$(F, A) \Cap ((G, B))\widetilde{\bigvee}_{\Re}(H, C)) = ((F, A) \Cap (G, B))\widetilde{\bigvee}_{\Re}((F, A) \Cap (H, C)).$$

Proposition 3.6. *For any fuzzy soft sets (F, A) and (G, A) in $\widehat{(X, E)}$, the following properties hold.*

(a) $(F, A)\widetilde{\bigvee}_{\Re}(F, A) = (F, A)$

(b) $(F, A)\widetilde{\bigvee}(G, A) = (F, A)\widetilde{\bigvee}_{\Re}(G, A)$

Proof. **(a)**. Let $(F, A)\widetilde{\bigvee}(G, A) = (H, C)$, where $C = A \cup A = A$ and for $e \in A$, $H(e) = F(e) \vee G(e)$.

Also assume that $(F, A)\widetilde{\bigvee}_{\Re}(G, A) = (J, A)$, where $J(e) = F(e) \vee G(e)$ for all $e \in A$. since H and J are same set-valued mapping, hence the result.

Similarly we can prove (b). □

Theorem 3.7. ***Properties of the Extended Intersection*** $(\sqcap_\varepsilon)$***operation***

(a) $(F, A) \sqcap_\varepsilon (G, B) = (G, B) \sqcap_\varepsilon (F, A)$

(b) $(F, A) \sqcap_\varepsilon \Phi_A = \Phi_A$ *and* $(F, A) \sqcap_\varepsilon \mathfrak{U}_A = (F, A)$

(c) $(F, A) \sqcap_\varepsilon \Phi_B = \Phi_B \Leftrightarrow A \subseteq B$

(d) $(F, A) \sqcap_\varepsilon (G, A) = \Phi_A \Leftrightarrow (F, A) = \Phi_A$ *and* $(G, A) = \Phi_A$

(e) $(F, A) \sqcap_\varepsilon ((G, B) \sqcap_\varepsilon (H, C)) = ((F, A) \sqcap_\varepsilon (G, B)) \sqcap_\varepsilon (H, C)$

(f) $((F,A) \sqcap_\varepsilon (G,B)) \widetilde{\nsubseteq} (F,A)$ *in-general. But, if* $(F,A) \widetilde{\subseteq} (G,B)$*, then* $((F,A) \sqcap_\varepsilon (G,B)) \widetilde{\subseteq} (F,A)$ *moreover* $(F,A) = (F,A) \sqcap_\varepsilon (G,B)$

(g) $(F,A) \sqcap_\varepsilon ((G,B) \widetilde{\bigvee}_{\Re} (H,C)) = ((F,A) \sqcap_\varepsilon (G,B)) \widetilde{\bigvee}_{\Re} ((F,A) \sqcap_\varepsilon (H,C))$

(h) $(F,A) \widetilde{\bigvee}_{\Re} ((G,B)) \sqcap_\varepsilon (H,C)) = ((F,A) \widetilde{\bigvee}_{\Re} (G,B)) \sqcap_\varepsilon ((F,A) \widetilde{\bigvee}_{\Re} (H,C))$

(i) $((F,A) \widetilde{\bigvee}_{\Re} (G,B)) \sqcap_\varepsilon H,C) = ((F,A) \sqcap_\varepsilon (H,C)) \widetilde{\bigvee}_{\Re} ((G,B) \sqcap_\varepsilon (H,C))$

Proof. **(c)**. Suppose $(F,A) \sqcap_\varepsilon \Phi_B = (H,C)$, where $C = A \cup B$ for all $e \in C$

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B \\ \overline{0}_B & \text{if } e \in B \setminus A \\ F(e) \wedge \overline{0}_B & \text{if } e \in A \cap B \end{cases}$$

Let $A \subseteq B$, then

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B = \varnothing \\ \overline{0}_B & \text{if } e \in B \setminus A \\ F(e) \wedge \overline{0}_B & \text{if } e \in A \cap B = A \end{cases}$$

$$H(e) = \begin{cases} \overline{0}_B & \text{if } e \in B \setminus A \\ \overline{0}_B & \text{if } e \in A \end{cases}$$

This shows that H and Φ_B are same set valued mapping. Thus, $(F,A) \sqcap_\varepsilon \Phi_B = \Phi_B$.

Conversely, let $(F,A) \sqcap_\varepsilon \Phi_B = \Phi_B$. Then, $(H,C) = \Phi_B$. $\Rightarrow C = A \cup B = B$, thus $A \subseteq A \cup B = B$, so the proof is completed.

(d). Let $(F,A) \sqcap_\varepsilon (G,A) = (H,A)$, where $H(e) = F(e) \wedge G(e)$ for all $e \in A$. Now it is given that $(F,A) \sqcap_\varepsilon (G,B) = \Phi_A$, then $H(e) = \overline{0}_A$ implies $F(e) \wedge G(e) = \overline{0}_A$ for all $e \in A$. Thus, $F(e) = \overline{0}_A$ and $G(e) = \overline{0}_A$. Therefore, $(F,A) = \Phi_A$ and $(G,A) = \Phi_A$.

Conversely, suppose that $(F,A) = \Phi_A$ and $(G,A) = \Phi_A$. Then by definition of Φ_A, $F(e) = \overline{0}_A$ and $G(e) = \overline{0}_A$ for all $e \in A$ implies $F(e) \wedge G(e) = \overline{0}_A$. Hence, $(F,A) \sqcap_\varepsilon (G,A) = \Phi_A$.

(f). Let $(F,A) \sqcap_\varepsilon (G,B) = (H,C)$, where $C = A \cup B$. For $e \in A \cap B$, $H(e) = F(e) \wedge G(e)$, implies $F(e) \nleq H(e)$, although $A \cap B \subseteq A$, Hence, $((F,A) \sqcap_\varepsilon (G,B)) \widetilde{\nsubseteq} (F,A)$ in-general. Now suppose that $(F,A) \widetilde{\subseteq} (G,B)$. Then, $A \subseteq B$ and for $e \in A$, this imply $F(e) \leq G(e)$, thus $H(e) = F(e)$ for all $e \in A$. Hence, H and F are same set-valued mapping, which completes the proof.

(g). Suppose $(G,B) \widetilde{\bigvee}_{\Re} (H,C) = (I, B \cap C)$, where $I(e) = G(e) \vee H(e)$ for all $e \in B \cap C \neq \varnothing$. Also $(F,A) \sqcap_\varepsilon ((G,B) \widetilde{\bigvee}_{\Re} (H,C) = (F,A) \sqcap_\varepsilon (I, B \cap C) = (J,D)$, where $D = A \cup (B \cap C)$ and for all $e \in D$.

$$J(e) = \begin{cases} F(e) & \text{if } e \in A \setminus (B \cap C) \\ I(e) & \text{if } e \in (B \cap C) \setminus A \\ F(e) \wedge I(e) & \text{if } e \in A \cap (B \cap C) \end{cases}$$

By taking into account the definition of I along with J as piecewise defined function, we have

$$J(e) = \begin{cases} F(e) & \text{if } e \in A \setminus (B \cap C) \\ G(e) \vee H(e) & \text{if } e \in (B \cap C) \setminus A \\ F(e) \wedge (G(e) \vee H(e)) & \text{if } e \in A \cap (B \cap C) \end{cases}$$

By using properties of operations in set theory, it follows that

$$(3.9)\qquad J(e)=\begin{cases} F(e) & \text{if } e\in (A\setminus B)\cup(A\setminus C)\\ G(e)\vee H(e) & \text{if } e\in (B\setminus A)\cap(C\setminus A)\\ (F(e)\wedge(G(e))\vee(F(e)\wedge H(e)) & \text{if } e\in A\cap(B\cap C)\end{cases}$$

Consider the right hand side of the equality. Let $(F,A)\sqcap_{\varepsilon}(G,B)=(K,M)$, where $M=A\cup B$ for all $e\in M$

$$K(e)=\begin{cases} F(e) & \text{if } e\in A\setminus B\\ G(e) & \text{if } e\in B\setminus A\\ F(e)\wedge G(e) & \text{if } e\in A\cap B\end{cases}$$

Also$(F,A)\sqcap_{\varepsilon}(H,C)=(L,N)$, where $N=A\cup C$ for all $e\in N$

$$L(e)=\begin{cases} F(e) & \text{if } e\in A\setminus C\\ H(e) & \text{if } e\in C\setminus A\\ F(e)\wedge H(e) & \text{if } e\in A\cap C.\end{cases}$$

Now $((F,A)\sqcap_{\varepsilon}(G,B))\widetilde{\bigvee}_{\Re}((F,A)\sqcap_{\varepsilon}(H,C)=(K,M)\widetilde{\bigvee}_{\Re}(L,N)=(P,M\cap N)$, where $P(e)=K(e)\vee L(e)$ for all $e\in M\cap N=(A\cup B)\cap(A\cup C)=A\cup(B\cap C)\neq\varnothing$. By assuming K and L as piecewise defined functions, we have

$$(3.10)\qquad P(e)=\begin{cases} F(e) & \text{if } e\in (A\setminus B)\cup(A\setminus C)\\ G(e)\vee H(e) & \text{if } e\in (B\setminus A)\cap(C\setminus A)\\ (F(e)\wedge(G(e))\vee(F(e)\wedge H(e)) & \text{if } e\in A\cap(B\cap C)\end{cases}$$

From (3.9) and (3.10) it is clear that J and M are same set-valued mapping, so the proof is completed.

(h). First we investigate the left hand side of the equality. Let $(G,B)\sqcap_{\varepsilon}(H,C)=(I,D)$, where $D=B\cup C$ for all $e\in D$.

$$I(e)=\begin{cases} G(e) & \text{if } e\in B\setminus C\\ H(e) & \text{if } e\in C\setminus B\\ G(e)\wedge H(e) & \text{if } e\in B\cap C\end{cases}$$

Also, $(F,A)\widetilde{\bigvee}_{\Re}((G,B))\sqcap_{\varepsilon}(H,C))=(F,A)\widetilde{\bigvee}_{\Re}(I,D)=(J,A\cap(B\cup C))$, where $J(e)=F(e)\vee I(e)$ for all $e\in A\cap(B\cup C)\neq\varnothing$. By taking into account the definition of I along with J as piecewise defined function, we have

$$J(e)=\begin{cases} F(e)\vee G(e) & \text{if } e\in A\cap(B\setminus C)\\ F(e)\vee H(e) & \text{if } e\in A\cap(C\setminus B))\\ F(e)\vee(G(e)\wedge H(e)) & \text{if } e\in A\cap(B\cap C)\end{cases}$$

By using properties of operations in set theory, it follows that

$$(3.11)\qquad J(e)=\begin{cases} F(e)\vee G(e) & \text{if } e\in (A\setminus B)\cap C\\ H(e) & \text{if } e\in (B\setminus A)\cap C\\ ((F(e)\wedge G(e))\vee(G(e)\wedge H(e)) & \text{if } e\in A\cap B\cap C\end{cases}$$

Now, we consider the right hand side of the equality. Suppose that $(F,A)\widetilde{\bigvee}_{\Re}(G,B)=(K,A\cap B)$, where $K(e)=F(e)\vee G(e)$ for all $e\in A\cap B\neq\varnothing$. Also, assume that $(F,A)\widetilde{\bigvee}_{\Re}(H,C)=(L,A\cap C)$, where $L(e)=F(e)\wedge H(e)$ for all $e\in A\cap C\neq\varnothing$.

Now, $((F,A)\widetilde{\bigvee}_{\Re}(G,B)) \sqcap_{\varepsilon} ((F,A)\widetilde{\bigvee}_{\Re}(H,C)) = (K, A\cap B) \sqcap_{\varepsilon} (L, A\cap C) = (P,R)$, where $R = (A\cap B)\cup(A\cap C)$.

$$P(e) = \begin{cases} K(e) & \text{if } e \in (A\cap B)\setminus(A\cap C) \\ L(e) & \text{if } e \in (A\cap C)\setminus(A\cap B) \\ K(e)\wedge L(e) & \text{if } e \in (A\cap B)\cap(A\cap C) \end{cases}$$

By taking into account the definitions of K and L along with P as piecewise defined function, we have

$$P(e) = \begin{cases} F(e)\vee G(e) & \text{if } e \in (A\cap B)\setminus(A\cap C) \\ F(e)\wedge H(e) & \text{if } e \in (A\cap C)\setminus(A\cap B) \\ (F(e)\vee G(e))\wedge(F(e)\wedge H(e)) & \text{if } e \in (A\cap B)\cap(A\cap C) \end{cases}$$

By using properties of operations in set theory for all $e \in R = (A\cap B)\cup(A\cap C) = A\cap(B\cup C)$, we have

$$P(e) = \begin{cases} F(e)\vee G(e) & \text{if } e \in A\cap(B\setminus C) \\ F(e)\wedge H(e) & \text{if } e \in A\cap(C\setminus B)) \\ (F(e)\vee G(e))\wedge(F(e)\wedge H(e)) & \text{if } e \in A\cap(B\cap C) \end{cases} \tag{3.12}$$

From (13) and (14) it is clear that J and P are same set-valued mapping, hence the result.

(i). By similar techniques used to prove **(g)** and **(h)**, we can prove **(i)**, therefore, omitted. □

Now, we give a corresponding example of theorem 3(h).

Example 3.8. Let $X = \{a,b,c\}$ and $E = \{e_1,e_2,e_3,e_4\}$ be initial universe and parameter set, respectively. Consider the fuzzy soft sets $(F,A),(G,B)$ and (H,C) are given by

$$\begin{aligned} (F,A) &= \{e_2 = \{a_{0.9}, b_{0.3}, c_{0.8}\}, e_3 = \{a_{0.5}, b_{0.7}, c_{0.8}\}\}. \\ (G,B) &= \{e_1 = \{a_{0.5}, b_{0.3}, c_0\}, e_3 = \{a_{0.8}, b_{0.9}, c_{0.3}\}\}. \\ (H,C) &= \{e_1 = \{a_0, b_{0.7}, c_{0.8}\}, e_2 = \{a_{0.8}, b_{0.7}, c_{0.9}\}\}. \end{aligned}$$

Now $(G,B) \sqcap_{\varepsilon} (H,C) = \{e_1 = \{a_0, b_{0.3}, c_0\}, e_2 = \{a_{0.8}, b_{0.7}, c_{0.9}\}, e_3 = \{a_{0.8}, b_{0.9}, c_{0.3}\}\}$. Then,

$$(F,A)\widetilde{\bigvee}_{\Re}((G,B)) \sqcap_{\varepsilon} (H,C)) = \left\{ \begin{array}{c} e_2 = \{a_{0.9}, b_{0.7}, c_{0.9}\}, \\ e_3 = \{a_{0.8}, b_{0.9}, c_{0.8}\} \end{array} \right\}. \tag{3.13}$$

Also,

$$(F,A)\widetilde{\bigvee}_{\Re}(G,B) = \{e_3 = \{a_{0.8}, b_{0.9}, c_{0.8}\}\}.$$

and

$$(F,A)\widetilde{\bigvee}_{\Re}(H,C) = \{e_2 = \{a_{0.9}, b_{0.7}, c_{0.9}\}\}.$$

So

$$((F,A)\widetilde{\bigvee}_{\Re}(G,B)) \sqcap_{\varepsilon} ((F,A)\widetilde{\bigvee}_{\Re}(H,C)) = \left\{ \begin{array}{c} e_2 = \{a_{0.9}, b_{0.7}, c_{0.9}\}, \\ e_3 = \{a_{0.8}, b_{0.9}, c_{0.8}\} \end{array} \right\}. \tag{3.14}$$

From (3.13) and (3.14), we have

$$(F,A)\widetilde{\bigvee}_{\Re}((G,B))\sqcap_{\varepsilon}(H,C)) = ((F,A)\widetilde{\bigvee}_{\Re}(G,B))\sqcap_{\varepsilon}((F,A)\widetilde{\bigvee}_{\Re}(H,C)$$

Proposition 3.9. *For any fuzzy soft sets (F,A) and (G,B) in $\widehat{(X,E)}$, the following properties hold.*

(a) $(F,A)\sqcap_{\varepsilon}(F,A) = (F,A)$
(b) $(F,A)\widetilde{\bigvee}_{\Re}((F,A)\sqcap_{\varepsilon}(G,B)) = (F,A)$
(c) $(F,A)\sqcap_{\varepsilon}((F,A)\widetilde{\bigvee}_{\Re}(G,B)) = (F,A)$
(d) $(F,A)\sqcap_{\varepsilon}(G,A) = (F,A)\Cap(G,A)$
$A\cup(A\cap B) = (A\cup A)\cap(A\cup B) = A, H(e) = F(e)\vee G(e)$

Proof. **(b).** Consider $(F,A)\sqcap_{\varepsilon}(G,B) = (H,C)$, where $C = A\cup B$ and for all $e\in C$

$$H(e) = \begin{cases} F(e) & \text{if } e\in A\setminus B \\ H(e) & \text{if } e\in B\setminus A \\ F(e)\wedge H(e) & \text{if } e\in A\cap B \end{cases}$$

Also, $(F,A)\widetilde{\bigvee}_{\Re}((F,A)\sqcap_{\varepsilon}(G,B)) = (J,A\cap C)$, where $J(e) = F(e)\vee H(e)$ for all $e\in A\cap C = A\cap(A\cup B) = A\neq\varnothing$. Then, for all $e\in A$, we have $H(e) = F(e)\vee F(e) = F(e)$. Since H and F are same set-valued mapping, so the proof is completed.

(c). By using similar techniques used to prove (b), we can prove (c).

(d). First, we investigate the left hand side of the equality. Suppose $(F,A)\sqcap_{\varepsilon}(G,A) = (H,A\cup A)$, where $H(e) = F(e)\wedge G(e)$ for all $e\in A$. Now we consider the right hand side of the equality. Assume $(F,A)\Cap(G,A) = (I,A\cap A)$, where $I(e) = F(e)\wedge G(e)$ for all $e\in A$. Since H and I are same set-valued mapping, which completes the proof. □

Theorem 3.10. ***Properties of the Restricted Intersection*** ($\Cap$) ***operation***

(a) $(F,A)\Cap(G,B) = (G,B)\Cap(F,A)$
(b) $(F,A)\Cap\Phi_A = \Phi_A$ *and* $(F,A)\Cap\mathfrak{U}_A = (F,A)$
(c) $(F,A)\Cap\Phi_B = \Phi_B \Leftrightarrow B\subseteq A$
(d) $(F,A)\Cap(G,A) = \Phi_A \Leftrightarrow (F,A) = \Phi_A$ *and* $(G,A) = \Phi_A$
(e) $(F,A)\Cap((G,B)\Cap(H,C)) = ((F,A)\Cap(G,B))\Cap(H,C)$
(f) $((F,A)\Cap(G,B))\widetilde{\nsubseteq}(F,A)$ *in-general. But, if* $(F,A)\widetilde{\subseteq}(G,B)$ *then* $((F,A)\Cap(G,B))\widetilde{\subseteq}(F,A)$ *moreover* $(F,A) = (F,A)\Cap(G,B)$
(g) $((F,A)\widetilde{\bigvee}(G,B))\Cap(H,C) = ((F,A)\Cap(H,C))\widetilde{\bigvee}((G,B)\Cap(H,C))$
(h) $(F,A)\Cap((G,B)\widetilde{\bigvee}_{\Re}(H,C)) = ((F,A)\Cap(G,B))\widetilde{\bigvee}_{\Re}((F,A)\Cap(H,C))$
(i) $((F,A)\widetilde{\bigvee}_{\Re}(G,B))\Cap(H,C)) = ((F,A)\Cap(G,B))\widetilde{\bigvee}_{\Re}((F,A)\Cap(H,C))$

Proof. **(c)**. Consider $\Phi_B = (G,B)$ and $(F,A)\Cap\Phi_B = (F,A)\Cap(G,B) = (H,A\cap B)$, where $H(e) = F(e)\wedge G(e)$ for all $e\in A\cap B\neq\varnothing$. Now, it is given that $B\subseteq A$. Then, $H(e) = F(e)\wedge\overline{0}_B = \overline{0}_B$ for all $e\in A\cap B = B$, hence $(F,A)\Cap\Phi_B = \Phi_B$. Conversely, suppose that $(F,A)\Cap\Phi_B = \Phi_B$. Then, $(H,A\cap B) = (F,A)$, it follows that $A\cap B = B$, therefore, $B\subseteq A$, which completes the proof.

(d). Let $(F,A)\Cap(G,A) = (H,A)$, where $H(e) = F(e)\wedge G(e)$ for all $e\in A$.It is given that $(F,A)\Cap(G,B) = \Phi_A$, then $H(e) = F(e)\wedge G(e) = \overline{0}_A$ for all $e\in A$. Thus, $F(e) = \overline{0}_A$ and $G(e) = \overline{0}_A$, therefore $(F,A) = \Phi_A$ and $(G,A) = \Phi_A$.

Conversely, assume that $(F, A) = \Phi_A$ and $(G, A) = \Phi_A$ then, by definition of Φ_A, we have $F(e) = \overline{0}_A$ and $G(e) = \overline{0}_A$ for all $e \in A$, this implies $F(e) \wedge G(e) = \overline{0}_A$.Hence $(F, A) \Cap (G, A) = \Phi_A$.

(e). It is straightforward, therefore, omitted.

(f). Let $(F, A) \Cap (G, B) = (H, A \cap B)$,where $H(e) = F(e) \wedge G(e)$ for all $e \in A \cap B \neq \varnothing$. Since $F(e) \nleq H(e)$ and $A \nsubseteq A \cap B$. Hence $(F, A)\widetilde{\nsubseteq}((F, A)\Cap (G, B))$ in-general.

Now, suppose that $(F, A)\widetilde{\subseteq}(G, B)$. Then, $A \subseteq B$ and $F(e) \leq G(e)$ for all $e \in A$, this implies $H(e) = F(e)$. Since, F and H are same set-valued mapping. This completes the proof.

(g). Suppose that $(F, A)\widetilde{\bigvee}(G, B) = (I, D)$, where $D = A \cup B$ and for all $e \in D$.

$$I(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B \\ G(e) & \text{if } e \in B \setminus A \\ F(e) \vee G(e) & \text{if } e \in A \cap B \end{cases}$$

Also, $((F, A)\widetilde{\bigvee}(G, B)) \Cap (H, C) = (I, D) \Cap (H, C) = (J, D \cap C)$, where $J(e) = I(e) \wedge H(e)$ for all $e \in D \cap C = (A \cup B) \cap C \neq \varnothing$. By taking into account the definition of I along with H, we have

$$J(e) = \begin{cases} F(e) \wedge H(e) & \text{if } e \in (A \setminus B) \cap C \\ G(e) \wedge H(e) & \text{if } e \in (B \setminus A) \cap C \\ (F(e) \vee G(e)) \wedge H(e) & \text{if } e \in (A \cap B) \cap C \end{cases} \tag{3.15}$$

Now, consider the right hand side of the equality. Let $(F, A) \Cap (H, C) = (K, A \cap C)$, where $K(e) = F(e) \wedge H(e)$ for all $e \in A \cap C \neq \varnothing$. Also, $(G, B) \Cap (H, C) = (L, B \cap C)$, where $L(e) = G(e) \wedge H(e)$ for all $e \in B \cap C \neq \varnothing$. Now, $((F, A) \Cap (H, C))\widetilde{\bigvee}((G, B) \Cap (H, C)) = (K, A \cap C) \Cap (L, B \cap C) = (P, R)$, where $R = (A \cap C) \cup (B \cap C)$ for all $e \in (A \cap C) \cup (B \cap C)$.

$$P(e) = \begin{cases} K(e) & \text{if } e \in (A \cap C) \setminus (B \cap C) \\ L(e) & \text{if } e \in (B \cap C) \setminus (A \cap C) \\ K(e) \vee L(e) & \text{if } e \in (A \cap C) \cap (B \cap C) \end{cases}$$

By assuming K and L as piecewise defined functions, we have for all $e \in (A \cap C) \cup (B \cap C) = (A \cap B) \cup C$

$$P(e) = \begin{cases} F(e) \wedge H(e) & \text{if } e \in (A \cap C) \setminus (B \cap C) \\ G(e) \wedge H(e) & \text{if } e \in (B \cap C) \setminus (A \cap C) \\ (F(e) \vee G(e)) \wedge H(e) & \text{if } e \in (A \cap C) \cap (B \cap C) \end{cases}$$

$$P(e) = \begin{cases} F(e) \wedge H(e) & \text{if } e \in (A \setminus B) \cap C \\ G(e) \wedge H(e) & \text{if } e \in (B \setminus A) \cap C \\ (F(e) \vee G(e)) \wedge H(e) & \text{if } e \in (A \cap B) \cap C \end{cases} \tag{3.16}$$

From (3.15) and (3.16) it is clear that J and P are same set-valued mapping. Therefore, the proof is completed.

(h). Suppose that $(G, B)\widetilde{\bigvee}_{\mathfrak{R}}(H, C) = (I, B \cap C)$, where $I(e) = G(e) \vee H(e)$ for all $e \in B \cap C \neq \varnothing$. Also, $(F, A) \Cap ((G, B)\widetilde{\bigvee}_{\mathfrak{R}}(H, C)) = (F, A) \Cap (I, B \cap C) =$

$(J, A\cap(B\cap C))$, where $J(e) = F(e)\wedge I(e)$ for all $e \in A\cap(B\cap C) \neq \varnothing$. It follows that $J(e) = F(e)\wedge(G(e)\vee H(e)) = (F(e)\wedge G(e))\vee(F(e)\wedge H(e))$ for all $A\cap(B\cap C) \neq \varnothing$.

Consider the hand side of the equality. Let $(F, A) \Cap (G, B) = (K, A \cap B)$, where $K(e) = F(e) \wedge G(e)$ for all $e \in A \cap B \neq \varnothing$. Also $(F, A) \Cap (H, C) = (L, A \cap C)$, where $L(e) = F(e)\wedge H(e)$ for all $e \in A\cap C \neq \varnothing$. Now, $((F, A)\Cap(G, B))\widetilde{\bigvee}_{\Re}((F, A)\Cap(H, C)) = (K, A \cap B)\widetilde{\bigvee}_{\Re}(L, A \cap C) = (M,(A \cap B) \cap (A \cap C))$, where $M(e) = K(e) \wedge L(e)$. for all $e \in (A \cap B) \cap (A \cap C) \neq \varnothing$.

By taking into account the definition of K and L along with M, we can rewrite M as $M(e) = (F(e) \wedge G(e)) \vee (F(e) \wedge H(e))$ for all $e \in A \cap (B \cap C)$. Since J and M are same set-valued mapping, the proof is completed.

(i). By similar techniques used to prove **(h)**, **(i)** can be shown, too, therefore we skip the proof. □

Now, we give a corresponding example of theorem 4(h).

Example 3.11. Let $X = \{a, b, c\}$ and $E = \{e_1, e_2, e_3, e_4\}$ be initial universe and parameter set, respectively. Consider the fuzzy soft sets $(F, A), (G, B)$ and (H, C) are given by

$$\begin{aligned}(F, A) &= \{e_1 = \{a_{0.9}, b_{0.3}, c_{0.8}\}, e_4 = \{a_{0.5}, b_{0.7}, c_{0.8}\}\}.\\ (G, B) &= \{e_1 = \{a_{0.5}, b_{0.3}, c_0\}, e_3 = \{a_{0.8}, b_{0.9}, c_{0.3}\}\}.\\ (H, C) &= \{e_1 = \{a_0, b_{0.7}, c_{0.8}\}, e_2 = \{a_{0.8}, b_{0.7}, c_{0.9}\}\}.\end{aligned}$$

Now $(F, A)\widetilde{\bigvee}(G, B) = \{e_1 = \{a_{0.9}, b_{0.3}, c_{0.8}\}, e_3 = \{a_{0.8}, b_{0.9}, c_{0.3}\}, e_4 = \{a_{0.5}, b_{0.7}, c_{0.8}\}\}$. Then,

$$((F, A)\widetilde{\bigvee}(G, B)) \Cap (H, C) = \{e_1 = \{a_0, b_{0.3}, c_{0.8}\}\}. \tag{3.17}$$

Also consider, $(F, A) \Cap (H, C) = \{e_1 = \{a_0, b_{0.3}, c_{0.8}\}\}$, and $(G, B) \Cap (H, C) = \{e_1 = \{a_0, b_{0.3}, c_0\}\}$.

$$((F, A) \Cap (G, B))\widetilde{\bigvee}((F, A) \Cap (H, C)) = \{e_1 = \{a_0, b_{0.3}, c_{0.8}\}\}. \tag{3.18}$$

From (3.17) and (3.18), we have

$$(F, A)\widetilde{\bigvee}(G, B)) \Cap (H, C) = ((F, A) \Cap (H, C))\widetilde{\bigvee}((G, B) \Cap (H, C))$$

Proposition 3.12. *For any fuzzy soft sets (F, A) and (G, B) in $\widehat{(X, E)}$, the following properties hold.*

(a) $(F, A) \wedge (G, B) \neq (G, B) \wedge (F, A)$
(b) $(F, A) \vee (G, B) \neq (G, B) \vee (F, A)$
(c) $((F, A) \wedge (G, B)) \wedge (H, C) \neq (F, A) \wedge ((G, B) \wedge (H, C))$
(d) $((F, A) \vee (G, B)) \vee (H, C) \neq (F, A) \vee ((G, B) \vee (H, C))$

Proof. We shall show by counter example that the equality in above proposition does not hold. □

3.1. **Counter Example.** Let $X = \{a, b, c\}$ and $E = \{e_1, e_2, e_3, e_4\}$ be initial universe and parameter sets respectively. Consider the fuzzy soft sets (F, A), (G, B) and (H, C) are given by

$$\begin{aligned}(F, A) &= \left\{\begin{array}{c} e_1 = \{a_{0.1}, b_{0.3}, c_{0.8}\}, \\ e_3 = \{a_{0.8}, b_{0.7}, c_{0.9}\} \end{array}\right\}. \\ (G, B) &= \left\{\begin{array}{c} e_2 = \{a_{0.4}, b_{0.3}, c_0\}, \\ e_4 = \{a_{0.4}, b_{0.9}, c_{0.3}\} \end{array}\right\}. \\ (H, C) &= \{e_1 = \{a_{0.8}, b_{0.5}, c_0\}\}.\end{aligned}$$

$$\begin{aligned}(F, A) \vee (G, B) &= (I, A \times B) \\ &= \left\{\begin{array}{c} (e_1, e_2) = \{a_{0.4}, b_{0.3}, c_{0.8}\}, (e_1, e_4) = \{a_{0.4}, b_{0.9}, c_{0.8}\} \\ (e_3, e_2) = \{a_{0.8}, b_{0.7}, c_{0.9}\}, (e_3, e_4) = \{a_{0.8}, b_{0.9}, c_{0.9}\} \end{array}\right\}.\end{aligned}$$

Also

$$((F, A) \vee (G, B)) \vee (H, C) = (I, A \times B) \vee (H, C) = (J, (A \times B) \times C)$$

$$((F, A) \vee (G, B)) \vee (H, C) = \left\{\begin{array}{c} ((e_1, e_2), e_1) = \{a_{0.8}, b_{0.5}, c_{0.8}\}, \\ ((e_1, e_4), e_1) = \{a_{0.8}, b_{0.9}, c_{0.8}\}, \\ ((e_3, e_2), e_1) = \{a_{0.8}, b_{0.7}, c_{0.9}\}, \\ ((e_3, e_4), e_1) = \{a_{0.8}, b_{0.9}, c_{0.9}\} \end{array}\right\}. \tag{3.19}$$

Now, considering right hand side

$$(G, B) \vee (H, C) = (K, B \times C) = \left\{\begin{array}{c} (e_2, e_1) = \{a_{0.8}, b_{0.5}, c_0\}, \\ (e_4, e_1) = \{a_{0.8}, b_{0.9}, c_{0.3}\} \end{array}\right\}.$$

further

$$(F, A) \vee ((G, B) \vee (H, C)) = (L, A \times (B \times C))$$

$$(F, A) \vee ((G, B) \vee (H, C)) = \left\{\begin{array}{c} (e_1, (e_2, e_1)) = \{a_{0.8}, b_{0.5}, c_{0.8}\}, \\ (e_1, (e_4, e_1)) = \{a_{0.8}, b_{0.9}, c_{0.8}\}, \\ (e_3, (e_2, e_1)) = \{a_{0.8}, b_{0.7}, c_{0.9}\}, \\ (e_3, (e_4, e_1)) = \{a_{0.8}, b_{0.9}, c_{0.9}\} \end{array}\right\}. \tag{3.20}$$

From (3.19) and (3.20), we have

$$((F, A) \wedge (G, B)) \wedge (H, C) \neq (F, A) \wedge ((G, B) \wedge (H, C).$$

Remark 3.13. In set theory Cartesian product is neither commutative nor associative, i.e $A \times B \neq B \times A$ and $A \times (B \times C) \neq (A \times B) \times C$, therefore, above proposition is true in-general. But, the equality in the above proposition can hold iff $A = B = C$. We illustrate it in following proposition.

Proposition 3.14. *For any fuzzy soft sets (F, A) and (G, B) in $\widehat{(X, E)}$, the following properties hold.*

(a) $(F, A) \wedge (G, A) = (G, A) \wedge (F, A)$
(b) $(F, A) \vee (G, A) = (G, A) \vee (F, A)$
(c) $((F, A) \wedge (G, A)) \wedge (H, A) = (F, A) \wedge ((G, A) \wedge (H, A))$
(d) $((F, A) \vee (G, A)) \vee (H, A) = (F, A) \vee ((G, A) \vee (H, A))$

Proof. **(a)**. First we consider the left hand side of the equality. Suppose that $(F,A)\wedge(G,A)=(H,A\times A)$, for $e\in A$ we have $H((e,e))=F(e)\wedge G(e)$ for all $(e,e)\in A\times A$. Now, we consider the right hand side of the equality, let $(G,A)\wedge(F,A)=(I,A\times A)$, where $I((e,e))=G(e)\wedge F(e)=F(e)\wedge G(e)$ for all $(e,e)\in A\times A$. Since H and I are same set-valued mapping, the proof is completed.

(b). By using similar techniques used to prove (a), we can prove (b).

(c). Suppose that $(F,A)\wedge(G,A)=(I,A\times A)$, where $I((e,e))=F(e)\wedge G(e)$ for all $(e,e)\in A\times A$. Also, consider that $((F,A)\wedge(G,A))\wedge(H,A)=(I,A\times A)\wedge(H,A)=(J,(A\times A)\times A)$, where $J((e,e),e))=I((e,e))\wedge H(e)$ for all $((e,e),e)\in(A\times A)\times A$. By taking into account the definition of I, we have $J((e,e),e))=(F(e)\wedge G(e))\wedge H(e)$. Now, we investigate the right hand side of the equality, let $(G,A)\wedge(H,A)=(K,A\times A)$, where $K((e,e))=G(e)\wedge H(e)$ for all $(e,e)\in A\times A$. Further assume that $(F,A)\wedge((G,A)\wedge(H,A))=(F,A)\wedge(K,A\times A)=(L,A\times(A\times A))$, where $L((e,(e,e))=F(e)\wedge K(e,e)$ for all $(e,(e,e))\in A\times(A\times A)$. By using definition of K along L, it follows that $L((e,(e,e))=F(e)\wedge(G(e)\wedge H(e))=(F(e)\wedge G(e))\wedge H(e)$ for all $(e,(e,e))\in A\times(A\times A)=(A\times A)\times A$. This shows that J and L are same set-valued mapping, which completes the proof.

(d). The proof can be illustrated similar to (c), therefore omitted. □

4. De Morgan's laws in fuzzy soft set theory

In this section first we will show by counter example that instead of De Morgan's inclusion laws as proved by [5] , how De Morgan laws hold by using same complementation operation . Later on we will further prove De Morgan laws by using relative complementation operation over fuzzy union and extended intersection, restricted union and restricted intersection and OR, AND operations.

4.1. **Counter Example.** Let $X=\{a,b,c\}$ and $E=\{e_1,e_2,e_3,e_4\}$ be initial universe and parameter sets respectively. Consider the fuzzy soft sets (F,A) and (G,B) are given by

$$(F,A)=\left\{\begin{array}{c} e_1=\{a_{0.1},b_{0.3},c_{0.8}\},\\ e_2=\{a_{0.5},b_{0.7},c_{0.8}\},\\ e_3=\{a_{0.8},b_{0.7},c_{0.9}\}\end{array}\right\}.$$

$$(G,B)=\left\{\begin{array}{c} e_1=\{a_{0},b_{0.8},c_{0.4}\},\\ e_2=\{a_{0.4},b_{0.3},c_{0}\},\\ e_4=\{a_{0.4},b_{0.9},c_{0.3}\}\end{array}\right\}.$$

Now

$$(F,A)\widetilde{\bigvee}(G,B)=\left\{\begin{array}{c} e_1=\{a_{0.1},b_{0.8},c_{0.8}\},e_2=\{a_{0.5},b_{0.7},c_{0.8}\},\\ e_3=\{a_{0.8},b_{0.7},c_{0.9}\},e_4=\{a_{0.4},b_{0.9},c_{0.3}\}\end{array}\right\}.$$

and

$$((F,A)\widetilde{\bigvee}(G,B))^c=\left\{\begin{array}{c} \neg e_1=\{a_{0.9},b_{0.2},c_{0.2}\},\neg e_2=\{a_{0.5},b_{0.3},c_{0.2}\},\\ \neg e_3=\{a_{0.2},b_{0.3},c_{0.1}\},\neg e_4=\{a_{0.6},b_{0.1},c_{0.7}\}\end{array}\right\}. \quad (4.1)$$

Now consider

$$(F,A)^c=(F^c,\rceil A)=\left\{\begin{array}{c} \neg e_1=\{a_{0.9},b_{0.7},c_{0.2}\},\\ \neg e_2=\{a_{0.5},b_{0.3},c_{0.2}\},\\ \neg e_3=\{a_{0.2},b_{0.3},c_{0.1}\}\end{array}\right\}.$$

and

$$(G,B)^c = (G^c, \rceil B) = \left\{ \begin{array}{l} \neg e_1 = \{a_1, b_{0.2}, c_{0.6}\}, \\ \neg e_2 = \{a_{0.6}, b_{0.7}, c_1\}, \\ \neg e_4 = \{a_{0.6}, b_{0.1}, c_{0.7}\} \end{array} \right\}.$$

$$(F,A)^c \Cap (G,B)^c = \left\{ \begin{array}{l} \neg e_1 = \{a_{0.9}, b_{0.2}, c_{0.2}\}, \\ \neg e_2 = \{a_{0.5}, b_{0.3}, c_{0.2}\} \end{array} \right\}. \tag{4.2}$$

From (4.1) and (4.2), we have

$$((F,A)\widetilde{\bigvee}(G,B))^c \neq (F,A)^c \Cap (G,B)^c$$

Thus De Morgan's laws do not hold in fuzzy soft sets, but instead of restricted intersection if we use extended intersection, then these laws hold under same complementation operation.

$$(F,A)^c \sqcap_\varepsilon (G,B)^c = \left\{ \begin{array}{l} \neg e_1 = \{a_{0.9}, b_{0.7}, c_{0.2}\}, \neg e_2 = \{a_{0.5}, b_{0.3}, c_{0.2}\}, \\ \neg e_3 = \{a_{0.2}, b_{0.3}, c_{0.1}\}, \neg e_4 = \{a_{0.6}, b_{0.1}, c_{0.7}\} \end{array} \right\}. \tag{4.3}$$

From (4.1) and (4.3), we have

$$((F,A)\widetilde{\bigvee}(G,B))^c = (F,A)^c \sqcap_\varepsilon (G,B)^c$$

This led us to the following proposition.

Proposition 4.1. *For any fuzzy soft sets (F,A) and (G,B) in $\widehat{(X,E)}$, the following axioms hold*

(a) $((F,A)\widetilde{\bigvee}(G,B))^c = (F,A)^c \sqcap_\varepsilon (G,B)^c$

(b) $((F,A) \sqcap_\varepsilon (G,B))^c = (F,A)^c \widetilde{\bigvee}(G,B)^c$

Proof. **(a).** Suppose (F,A) and (G,B) are fuzzy soft sets in a fuzzy soft class $\widehat{(X,E)}$ and $(F,A)\widetilde{\bigvee}(G,B) = (H,C)$, where$C = A \cup B$ and for all $e \in C$

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B \\ G(e) & \text{if } e \in B \setminus A \\ F(e) \vee G(e) & \text{if } e \in A \cap B \end{cases}$$

Now $((F,A)\widetilde{\bigvee}(G,B))^c = (H,C)^c = (H^c, \rceil C)$ for all $\neg e \in \rceil C = \rceil A \cup \rceil B$

$$H^c(\neg e) = \begin{cases} [F(e)]^c & \text{if } \neg e \in \ \rceil A \ \setminus \ \rceil B \\ [G(e)]^c & \text{if } \neg e \in \ \rceil B \ \setminus \ \rceil A \\ [F(e) \vee G(e)]^c & \text{if } \neg e \in \ \rceil A \ \cap \ \rceil B \end{cases}$$

$$H^c(\neg e) = \begin{cases} F^c(\neg e) & \text{if } \neg e \in \ \rceil A \setminus \rceil B \\ G^c(\neg e) & \text{if } \neg e \in \ \rceil B \ \setminus \ \rceil A \\ F^c(\neg e) \wedge G^c(\neg e) & \text{if } \neg e \in \ \rceil A \ \cap \ \rceil B \end{cases}$$

Consider the right hand side of the equality.$(F,A) \sqcap_\varepsilon (G,B)^c = (F^c, \rceil A) \sqcap_\varepsilon (G^c, \rceil B)$ $= (J,D)$, where $D = \rceil A \cup \rceil B$ and for all $\neg e \in D$.

$$J(\neg e) = \begin{cases} F^c(\neg e) & \text{if } \neg e \in \ \rceil A \ \setminus \ \rceil B \\ G^c(\neg e) & \text{if } \neg e \in \ \rceil B \ \setminus \ \rceil A \\ F^c(\neg e) \wedge G^c(\neg e) & \text{if } \neg e \in \ \rceil A \ \setminus \ \rceil B \end{cases}$$

Since H^c and J are same set-valued mapping. We conclude that $((F,A)\widetilde{\bigvee}(G,B))^c = (F,A)^c \sqcap_\varepsilon (G,B)^c$

(b). Straightforward, therefore, omitted. □

Proposition 4.2. *For any fuzzy soft sets* (F,A) *and* (G,B) *in* $\widehat{(X,E)}$*, the following axioms hold.*

(a) $((F,A)\widetilde{\bigvee}(G,B))^r = (F,A)^r \sqcap_\varepsilon (G,B)^r$

(b) $((F,A) \sqcap_\varepsilon (G,B))^r = (F,A)^r \widetilde{\bigvee}(G,B)^r$

Proof. **(a)**. Suppose (F,A) and (G,B) are fuzzy soft sets in a fuzzy soft class $\widehat{(X,E)}$ and $(F,A)\widetilde{\bigvee}(G,B) = (H,C)$, where $C = A \cup B$ and for all $e \in C$

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B \\ G(e) & \text{if } e \in B \setminus A \\ F(e) \vee G(e) & \text{if } e \in A \cap B \end{cases}$$

Now $((F,A)\widetilde{\bigvee}(G,B))^r = (H,C)^r = (H^r,C)$ for all $e \in C$

$$H^r(e) = \begin{cases} [F(e)]^r & \text{if } e \in A \setminus B \\ [G(e)]^r & \text{if } e \in B \setminus A \\ [F(e) \vee G(e)]^r & \text{if } e \in A \cap B \end{cases}$$

$$H^r(e) = \begin{cases} F^r(e) & \text{if } e \in A \setminus B \\ G^r(e) & \text{if } e \in B \setminus A \\ F^r(e) \vee G^r(e) & \text{if } e \in A \cap B \end{cases}$$

Consider the right hand side of the equality. $(F,A)^r \sqcap_\varepsilon (G,B)^r = (F^r,A) \sqcap_\varepsilon (G^r,B) = (J,D)$, where $D = A \cup B$ and for all $e \in D$

$$J(e) = \begin{cases} F^r(e) & \text{if } e \in A \setminus B \\ G^r(e) & \text{if } e \in B \setminus A \\ F^r(e) \vee G^r(e) & \text{if } e \in A \cap B \end{cases}$$

Since H^r and J are same set-valued mapping. We conclude that $((F,A)\widetilde{\bigvee}(G,B))^r = (F,A)^r \sqcap_\varepsilon (G,B)^r$

(b). Proof is similar as in (a). □

In the following example we will illustrate (b) of the above proposition.

Example 4.3. Let $X = \{a,b,c,d\}$ and $E = \{e_1,e_2,e_3,e_4\}$ be initial universe and parameter set. Consider the fuzzy soft sets (F,A) and (G,B) defined as follows:

$$\begin{aligned} (F,A) &= \{e_1 = \{a_{0.1}, b_{0.3}, c_{0.8}\}, e_3 = \{a_{0.8}, b_{0.7}, c_{0.9}\}\}. \\ (G,B) &= \{e_3 = \{a_{0.4}, b_{0.3}, c_0\}, e_4 = \{a_{0.4}, b_{0.9}, c_{0.3}\}\}. \end{aligned}$$

Then,

$$(F,A)\widetilde{\bigvee}(G,B) = \{e_1 = \{a_{0.1}, b_{0.3}, c_{0.8}\}, e_3 = \{a_{0.8}, b_{0.7}, c_{0.9}\}, e_4 = \{a_{0.4}, b_{0.9}, c_{0.3}\}\}.$$

Relative complement of $(F,A)\widetilde{\bigvee}(G,B)$ is

$$((F,A)\widetilde{\bigvee}(G,B))^r = \left\{ \begin{array}{c} e_1 = \{a_{0.9}, b_{0.7}, c_{0.2}\}, e_3 = \{a_{0.2}, b_{0.3}, c_{0.1}\}, \\ e_4 = \{a_{0.6}, b_{0.1}, c_{0.7}\} \end{array} \right\}. \tag{4.4}$$

Now consider

$$(F,A)^r = (F^r, A) = \{e_1 = \{a_{0.9}, b_{0.7}, c_{0.2}\}, e_3 = \{a_{0.2}, b_{0.3}, c_{0.1}\}\}.$$

and

$$(G,B)^r = (G^r, B) = \{e_3 = \{a_{0.6}, b_{0.7}, c_1\}, e_4 = \{a_{0.6}, b_{0.1}, c_{0.7}\}\}.$$

Then, we have

$$(F,A)^r \sqcap_\varepsilon (G,B)^r = \left\{ \begin{array}{c} e_1 = \{a_{0.9}, b_{0.7}, c_{0.2}\}, e_3 = \{a_{0.2}, b_{0.3}, c_{0.1}\}, \\ e_4 = \{a_{0.6}, b_{0.1}, c_{0.7}\} \end{array} \right\}. \tag{4.5}$$

Hence,

$$\begin{aligned} ((F,A)\widetilde{\bigvee}(G,B))^r &= (F,A)^r \sqcap_\varepsilon (G,B)^r \\ &= \left\{ \begin{array}{c} e_1 = \{a_{0.9}, b_{0.7}, c_{0.2}\}, e_3 = \{a_{0.2}, b_{0.3}, c_{0.1}\}, \\ e_4 = \{a_{0.6}, b_{0.1}, c_{0.7}\} \end{array} \right\}. \end{aligned}$$

Proposition 4.4. *For any fuzzy soft set (F,A) in $\widehat{(X,E)}$, we have the following properties.*

(a) $(\Phi_A)^r = \mathfrak{U}_A$ *and* $(\mathfrak{U}_A)^r = \Phi_A$
(b) $((F,A)^r)^r = (F,A)$
(c) $(F,A)\widetilde{\bigvee}(F,A)^r = (F,A)\widetilde{\bigvee}_{\Re}(F,A)^r = \mathfrak{U}_A$
(d) $(F,A) \sqcap_\varepsilon (F,A)^r = (F,A) \Cap (F,A)^r = \Phi_A$

Proof. Straightforward. □

Proposition 4.5. *For any fuzzy soft sets (F,A) and (G,B) in $\widehat{(X,E)}$, the following axioms hold*

(a) $((F,A)\widetilde{\bigvee}_{\Re}(G,B))^r = (F,A)^r \Cap (G,B)^r$
(b) $((F,A) \Cap (G,B))^r = (F,A)^r\widetilde{\bigvee}_{\Re}(G,B)^r$
(c) $((F,A) \vee (G,B))^r = (F,A)^r \wedge (G,B)^r$
(d) $((F,A) \wedge (G,B))^r = (F,A)^r \vee (G,B)^r$

Proof. **(a)**. Assume that (F,A) and (G,B) are fuzzy soft sets in a fuzzy soft class $\widehat{(X,E)}$ and $(F,A)\widetilde{\bigvee}_{\Re}(G,B) = (H, A\cap B)$, where $H(e) = F(e) \vee G(e)$ for all $e \in A\cap B \neq \varnothing$. Now $((F,A)\widetilde{\bigvee}_{\Re}(G,B))^r = (H,C)^r = (H^r, C)$, where $H^r(e) = (F(e) \vee G(e))^r = F^r(e) \wedge G^r(e))$ for all $e \in A\cap B \neq \varnothing$. Now consider the right hand side of the equality. Suppose that $(F,A)^r \Cap (G,B)^r = (F^r,A) \Cap (G^r,B) = (J,D)$, where $J(e) = F^r(e) \wedge G^r(e)$ for all $e \in A\cap B \neq \varnothing$. Then, since H^r and J are same set-valued mapping,.we conclude that $((F,A)\widetilde{\bigvee}_{\Re}(G,B))^r = (F,A)^r \Cap (G,B)^r$

(b). By similar technique, we can prove (b).

(c). Let (F,A) and (G,B) are fuzzy soft sets in a fuzzy soft class $\widehat{(X,E)}$ and $(F,A)\vee(G,B) = (H, A\times B)$, where $H(e_1,e_2) = F(e_1)\vee G(e_2)$ for all $(e_1,e_2) \in A\times B$. The relative complement is given by $((F,A)\vee(G,B))^r = (H, A\times B)^r = (H^r, A\times B)$, where $H^r(e_1,e_2) = (F(e_1) \vee G(e_2))^r = F^r(e_1) \wedge G^r(e_2)$ for all $(e_1,e_2) \in A\times B$.

Now consider the right hand side of the equality $(F,A)^r \wedge (G,B)^r = (F^r,A) \wedge (G^r,B) = (J, A\times B)$,where $J(e_1,e_2) = F^r(e_1) \wedge G^r(y)$for all $(e_1,e_2) \in A\times B$.

Since H^r and J are same set-valued mapping. We conclude that $((F,A) \vee (G,B))^r = (F,A)^r \wedge (G,B)^r$.

(d) By similar technique, we can prove (d). □

In the following example we will illustrate (d) of the above proposition.

Example 4.6. Let $X = \{a, b, c\}, E = \{e_1, e_2, e_3, e_4\}$. Consider the fuzzy soft sets (F, A) and (G, B) defined as follows:

$$\begin{aligned} (F,A) &= \{e_1 = \{a_{0.1}, b_{0.3}, c_{0.8}\}, e_3 = \{a_{0.8}, b_{0.7}, c_{0.9}\}\}. \\ (G,B) &= \{e_3 = \{a_{0.4}, b_{0.3}, c_0\}, e_4 = \{a_{0.4}, b_{0.9}, c_{0.3}\}\}. \end{aligned}$$

Then, we have

$$(F,A) \vee (G,B) = \left\{ \begin{array}{l} (e_1, e_3) = \{a_{0.4}, b_{0.3}, c_{0.8}\}, (e_1, e_4) = \{a_{0.4}, b_{0.9}, c_{0.8}\}, \\ (e_3, e_3) = \{a_{0.8}, b_{0.7}, c_{0.9}\}, (e_3, e_4) = \{a_{0.8}, b_{0.9}, c_{0.9}\}. \end{array} \right\}.$$

This imply

$$((F,A) \vee (G,B))^r = \left\{ \begin{array}{l} (e_1, e_3) = \{a_{0.6}, b_{0.7}, c_{0.2}\}, (e_1, e_4) = \{a_{0.6}, b_{0.1}, c_{0.2}\}, \\ (e_3, e_3) = \{a_{0.2}, b_{0.3}, c_{0.1}\}, (e_3, e_4) = \{a_{0.2}, b_{0.1}, c_{0.1}\}. \end{array} \right\}.$$

Now consider

$$(F,A)^r = (F^r, A) = \{e_1 = \{a_{0.9}, b_{0.7}, c_{0.2}\}, e_3 = \{a_{0.2}, b_{0.3}, c_{0.1}\}\}.$$

and

$$(G,B)^r = (G^r, B) = \{e_3 = \{a_{0.6}, b_{0.7}, c_1\}, e_4 = \{a_{0.6}, b_{0.1}, c_{0.7}\}\}.$$

This imply

$$(F,A)^r \wedge (G,B)^r = \left\{ \begin{array}{l} (e_1, e_3) = \{a_{0.6}, b_{0.7}, c_{0.2}\}, (e_1, e_4) = \{a_{0.6}, b_{0.1}, c_{0.2}\}, \\ (e_3, e_3) = \{a_{0.2}, b_{0.3}, c_{0.1}\}, (e_3, e_4) = \{a_{0.2}, b_{0.1}, c_{0.1}\}. \end{array} \right\}.$$

Hence

$$((F,A) \vee (G,B))^r = (F,A)^r \wedge (G,B)^r$$

5. Conclusions

In this paper, we have presented a detailed theoretical study of fuzzy soft sets. We successfully made fuzzification of some algebraic properties of Soft set theory and proved that they also hold in Fuzzy soft set theory under same kind of operations. We have investigated the interrelation of different fuzzy soft operations with each other. We have proved certain De Morgan's laws hold in fuzzy soft sets with respect to different operations.

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References

[1] S. R. S. Varadhan, Probability Theory, American Mathematical Society. 2001.
[2] L. A Zadeh, Fuzzy Sets, Information and Control. 8 (1965) 338–353.
[3] L. A Zadeh, Is there a need for fuzzy logic, Information Sciences. 178 (13) (2008) 2751–2779.
[4] E. E. Kerre, A first view on the alternatives of fuzzy set theory, in: B. Reusch, K. H. Temme (Eds), Computational intelligence inTheory and Pratice, physica-Verlag, Heidelberg. (2001) 55–72.
[5] D. Dubois, W. Ostasiewicz and H. Prade, Fuzzy sets: history and basic notions, in: D. Dubois, H. Prade (Eds), Fundamentals of Fuzzy Sets, Kluwer Academic Publishers, Dordrecht. (2000) 80–93.
[6] J. Goguen, L-fuzzy sets, Journal of Mathematical Analysis and Applications. 18 (1967) 145–147.
[7] K. Atanassov, Intuitionistic fuzzy Sets, Fuzzy Sets and systems. 20 (1) (1986) 87–89.

[8] K. Atanassov, Intuitionistic fuzzy Sets, Physica-Verlag, Heidelberg, New York. 1999.
[9] Z. Pawlak, Rough Sets: Theoratical Aspects of Reasoning about Data, Kluwer Academic publishers. 1991.
[10] W. L. Gau and D.J.Buehrer, Vague sets, IEEE Transactions on Systems, Man and Cybernetics 23 (2) (1993) 610–614.
[11] K. Atanassov, Operators over interval valued intuitionistic fuzzy sets, Fuzzy Sets and Systems. 64 (1994) 159–174.
[12] M. B. Gorzalczany, A method of inference in approximate reasoning based on interval valued fuzzy sets, Fuzzy Sets and Systems. 21 (1) (1987) 1–17.
[13] D. Molodtsov, Soft set theory-first results, Comput. Math. Appl. 37 (4-5) (1999) 19–31.
[14] P. K. Maji, A. R. Roy, and R. Biswas, An application of soft sets in decision making problem, Comput. Math. Appl. 44 (8-9) (2002).
[15] A. R. Roy and P. K. Maji, A fuzzy soft set theoretic approach to decision making problems, Journal of Computational and Applied Mathematics. 203 (2) (2007) 412–418.
[16] Z. Kong, L. Gao and L. Wang, Comment on ”A fuzzy soft set theoretic approach to decision making problems”, Journal of Computational and Applied Mathematics. 223 (2) (2009) 540–542.
[17] Y. Zou and Xiao, Data analysis approaches of soft sets under incomplete information, Knowledge-Based Systems. 21 (8) (2008) 941–945.
[18] Z. Xaio, K. Gong and Y. Zou, A combined forecasting approach based on fuzzy soft sets, Journal of Computational and Applied Mathematics. 228 (1) (2009) 326–333.
[19] S. J. Kalayathankal and G. S. Singh, A fuzzy soft flood alarm model, Mathematics and Computers in Simulation. 80 (5) (2010) 887–893.
[20] P. K. Maji, A. R. Roy, and R. Biswas, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.
[21] D. Pei and D. Miao, From soft sets to information systems, in: X. Hu, Q. Liu, A. Skowron, T. Y Lin, R. R Yager, B. Zhang (Eds), Proceedings of Granular Computing, IEEE. (2) 2005 617–621.
[22] M. I. Ali, F. Feng, X. Liu and W. K. Min, On some new operations in soft set theory, comput. Math. Appl. 57 (9) (2009) 1547–1553.
[23] A. Sezgin and A. O. Atagun, On operations of soft sets, Comput. Math. Appl. 61 (2011) 1475–1467.
[24] J. H. Park, O. H. Kim and Y. C. Kwun, Some properties of equivalence soft set relations, Comput. Math. Appl. 63 (2012) 1079–1088.
[25] P. K. Maji, A. R. Roy and R. Biswas, Fuzzy soft sets, Journal of Fuzzy Mathematics. 9 (3) 589–602 (2001).
[26] B. Ahmad, and A. Kharal, On Fuzzy Soft Sets, Hindawi Publishing Corporation, Advances in Fuzzy Systems, volume 2009, Article ID 586507, 6pages doi: 10.1155/2009/586507.
[27] T. J. Neog and D. K. Sut, “On Union and Intersection of FuzzySoft Sets”, Int.J. Comp. Tec. Appl. Vol 2 (5) 1160–1176.
[28] T. J. Neog and D. K. Sut, On Fuzzy Soft Complement and Related Properties, Accepted for publication in International, Journal of Energy, Information and communications (IJEIC).
[29] M. Borah, T. J. Neog and D. K. Sut, A study on some operations of fuzzy soft sets, International Journal of Modern Engineering Research (IJMER), Vol.2, Issue. 2 (2012) 219–225.

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