Annals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (Month 201y), pp. 1–xx
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Somewhat fuzzy α-$\mathcal{I}$-continuous functions

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Received 26 June 2012; Accepted 03 January 2013

Abstract. Recently, El-Naschie has shown that the notion of fuzzy topology may be relevant to quantum particle physics in connection with string theory and E-infinite space time theory. In this paper, the concepts of somewhat fuzzy α-$\mathcal{I}$-continuous functions and somewhat fuzzy α-$\mathcal{I}$-open functions are introduced and studied. Somewhat fuzzy α-$\mathcal{I}$-open functions may be important in the study of quantum particle physics and related theory. Some characterizations and interesting properties of these functions are also discussed.

2010 AMS Classification: 54A40

Keywords: fuzzy α-$\mathcal{I}$-open sets, somewhat fuzzy α-$\mathcal{I}$-continuous functions, somewhat fuzzy α-$\mathcal{I}$-open functions.

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1. Introduction

The concept of fuzzy set was introduced by Zadeh [?]. Based on the concept of fuzzy sets, Chang [?] introduced and developed the concept of fuzzy topological spaces. In [?] Mohmoud and in [?], Sarkar indepentently presented some of the ideal concepts in the fuzzy trend and studied many of their fundamental properties. The concept of fuzzy topology may be relevent to quantum particle physics purticularly in connection with string theory and E-infinite theory [?, ?, ?, ?]. Recently, Yuksel et. al. [?] introduced the concepts of fuzzy α-$\mathcal{I}$-open sets and fuzzy α-$\mathcal{I}$-continuity in fuzzy ideal topological spaces. In the present paper, to introduce and study the concept of somewhat fuzzy α-$\mathcal{I}$-continuous functions and somewhat fuzzy α-$\mathcal{I}$-open functions in fuzzy ideal topological spaces.

2. Preliminaries

Throughtout this paper, (X, τ) always means fuzzy topolocial space in the sense of Chang [?]. For a fuzzy subset A of X, the fuzzy closure of A is denoted by $Cl(A)$ and is defined as $Cl(A) = \wedge\{B|B \geq A,\ B$ is a fuzzy closed subset of $X\}$. Also, the fuzzy interior of A is denoted by $Int(A)$ and is defined as $Int(A) = \vee\{B|B \leq A,\ B$ is a fuzzy open subset of $X\}$. A nonempty collection of fuzzy sets $\mathcal{I}$ of a set X is called a fuzzy ideal [?, ?] if and only if (i) $A \in \mathcal{I}$ and $A \leq B$, then $B \in \mathcal{I}$, (ii) if $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \vee B \in \mathcal{I}$. The triple $(X, \tau, \mathcal{I})$ means a fuzzy ideal topological space with a fuzzy ideal $\mathcal{I}$ and fuzzy topology τ. For $(X, \tau, \mathcal{I})$, the fuzzy local function of a fuzzy subset A of X with respect to τ and $\mathcal{I}$ denoted by $A^*(\tau, \mathcal{I})$ (briefly, A^*) and is defined $A^*(\tau, \mathcal{I}) = \vee\{x \in X : A \wedge U \notin \mathcal{I}$ for every U $\in \tau\}$. While A^* is the union of the fuzzy points such that if $U \in \tau$ and $E \in \mathcal{I}$, then there is atleast one $y \in X$ for which $U(y) + A(y) - 1 > E(y)$, where the fuzzy point x_p in X is a fuzzy set in X defined by

$$x_p(y) = \begin{cases} p, \quad p \in (0,1], & for\ y = x, y \in X \\ 0, & for\ y = x, y \in X \end{cases}$$

and x and p are respectively, called the support and the value of x. Also, for a fuzzy set A of $(X, \tau, \mathcal{I})$, $Cl^*(A) = A \vee A^*$. A fuzzy subset A of $(X, \tau, \mathcal{I})$ is said to be fuzzy α-$\mathcal{I}$-open [?] if $A \leq Int(Cl^*(Int(A)))$. The complement of fuzzy α-$\mathcal{I}$-open set is called fuzzy α-$\mathcal{I}$-closed. The family of all fuzzy α-$\mathcal{I}$-open (resp. fuzzy α-$\mathcal{I}$-closed) subsets of $(X, \tau, \mathcal{I})$ is denoted by $F\alpha\mathcal{I}O(X)$ (resp. $F\alpha\mathcal{I}C(X)$). The fuzzy α-$\mathcal{I}$-closure and fuzzy α-$\mathcal{I}$-interior of a fuzzy set A are respectively, denoted by $\alpha\mathcal{I}Cl(A)$ and $\alpha\mathcal{I}Int(A)$ and is defined as $\alpha\mathcal{I}Cl(A) = \wedge\{B|A \leq B, B \leq F\alpha\mathcal{I}C(X)\}$ and $\alpha\mathcal{I}Int(A) = \vee\{B|B \leq A, B \in F\alpha\mathcal{I}O(X)\}$. A fuzzy set A is said to be fuzzy α-$\mathcal{I}$-closed (resp. fuzzy α-$\mathcal{I}$-open) if and only if $\alpha\mathcal{I}Cl(A) = A$ (resp. $\alpha\mathcal{I}Int(A) = A$). Clearly, $\alpha\mathcal{I}Cl(1 - A) = 1 - \alpha\mathcal{I}Int(A)$ and $\alpha\mathcal{I}Int(1 - A) = 1 - \alpha\mathcal{I}Cl(A)$.

3. Somewhat fuzzy α-$\mathcal{I}$-continuous functions

Definition 3.1. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is called somewhat fuzzy α-$\mathcal{I}$-continuous if for every fuzzy open set A in Y such that $f^{-1}(A) \neq 0$, there exists a fuzzy α-$\mathcal{I}$-open set $B \neq 0$ in (X, τ) such that $B \leq f^{-1}(A)$.

Definition 3.2. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is called fuzzy continuous [?] if for $f^{-1}(\mu)$ is fuzzy open in X for every fuzzy open set μ in Y.

It is clear that every fuzzy continuous function is somewhat fuzzy α-$\mathcal{I}$-continuous but the converse is not true as the following example shows.

Example 3.3. Let μ_1, μ_2, μ_3 be fuzzy sets on $I = [0, 1]$ defined as follows:

$$\mu_1(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2} \\ 2x - 1, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$\mu_2(x) = \begin{cases} 1, & 0 \leq x \leq \frac{1}{4} \\ -4x + 2, & \frac{1}{4} \leq x \leq \frac{1}{2} \\ 0, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$\mu_3(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{4} \\ \frac{2}{3}(4x - 1), & \frac{1}{4} \leq x \leq 1 \end{cases}$$

. Clearly $\tau_1 = \{0, \mu_3, 1\}$, $\tau_2 = \{0, \mu_1, \mu_2, \mu_1 \vee \mu_2, 1\}$ are two topologies on I and $\mathcal{I}$ $= \{\varnothing\}$. Let $f : (I, \tau_1, \mathcal{I}) \to (I, \tau_2)$ be defined as $f(x) = \frac{x}{2}$ for each x in I. Then f is somewhat fuzzy α-$\mathcal{I}$-continuous but not fuzzy continuous.

Definition 3.4. A fuzzy set A in a fuzzy ideal topological space $(X, \tau, \mathcal{I})$ is called fuzzy α-$\mathcal{I}$-dense if there exists no fuzzy α-$\mathcal{I}$-closed set B such that $A < B < 1$ or equivalently $\alpha\mathcal{I}Cl(A) = 1$.

Proposition 3.5. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a somewhat fuzzy α-$\mathcal{I}$-continuous function and $g : (Y, \sigma) \to (Z, \eta)$ is fuzzy continuous function, then $g \circ f : (X, \tau, \mathcal{I}) \to (Z, \eta)$ is somewhat fuzzy α-$\mathcal{I}$-continuous function.*

Proof. Clear. □

Theorem 3.6. *For a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following statements are equivalent:*
(i) *f is somewhat fuzzy α-$\mathcal{I}$-continuous.*
(ii) *If A is a fuzzy closed set of Y such that $f^{-1}(A) \neq 1$, then there exists a proper fuzzy α-$\mathcal{I}$-closed set B of X such that $B \geq f^{-1}(A)$.*
(iii) *If A is a fuzzy α-$\mathcal{I}$-dense set, then $f(A)$ is a fuzzy dense set in Y.*

Proof. (i)⇒(ii): Suppose f is somewhat fuzzy α-$\mathcal{I}$-continuous and A is any fuzzy closed set in Y such that $f^{-1}(A) \neq 1$. Therefore, clearly $1 - A$ is a fuzzy open set and $f^{-1}(1 - A) = 1 - f^{-1}(A) \neq 0$. But by (i), there exists a fuzzy α-$\mathcal{I}$-open set B in $(X, \tau, \mathcal{I})$ such that $B \neq 0$ and $B \leq f^{-1}(1 - A)$. Therefore, $1 - B \geq 1 - f^{-1}(1 - A) = 1 - (1 - f^{-1}(A)) = f^{-1}(A)$. Put $1 - B = C$. Clearly, C is a proper fuzzy α-$\mathcal{I}$-closed set such that $C \geq f^{-1}(A)$.
(ii)⇒(iii): Let A be a fuzzy α-$\mathcal{I}$-dense set in X and suppose $f(A)$ is not fuzzy dense in Y. Then there exists a fuzzy closed set, say, B such that $f(A) < B < 1$. Now, $B < 1 \Rightarrow f^{-1}(B) \neq 1$. Then by $f(A) < B < 1$, there exists a proper fuzzy α-$\mathcal{I}$-closed set C in $(X, \tau, \mathcal{I})$ such that $C \geq f^{-1}(B)$. But by (i), $f^{-1}(B) > f^{-1}(f(A)) \geq A$, that is, $C > A$. This implies that there exists a proper fuzzy α-$\mathcal{I}$-closed set C such that $C > A$, which is a contradiction, since A is fuzzy α-$\mathcal{I}$-dense.
(iii)⇒(i): Let A be any fuzzy open set in (Y, σ) and suppose $f^{-1}(A) \neq 0$ and hence $A \neq 0$. Suppose $\alpha\mathcal{I}Int(f^{-1}(A)) = 0$. Then $\alpha\mathcal{I}Cl(1 - f^{-1}(A)) = 1 - \alpha\mathcal{I}Int(f^{-1}(A))$ $= 1 - 0 = 1$. This means that $1 - f^{-1}(A)$ is a fuzzy α-$\mathcal{I}$-dense set in X. By (iii), $f(1 - f^{-1}(A))$ is a fuzzy dense in Y. That is, $Cl(f(1 - f^{-1}(A))) = 1$, but $f(1 - f^{-1}(A)) = f(f^{-1}(1 - A)) \leq 1 - A = 1$, since $A \neq 0$. Since $1 - A$ is fuzzy closed and $f(1 - f^{-1}(A)) \leq 1 - A$, $Cl(f(f^{-1}(A))) \leq 1 - A$. That is, $1 \leq 1 - A \Rightarrow A \leq 0$ and hence $A = 0$, which is a contradiction to the fact that $A \neq 0$. Therefore, we must have $\alpha\mathcal{I}Int(f^{-1}(A)) \neq 0$. This means that, there exists a fuzzy α-$\mathcal{I}$-open set B in $(X, \tau, \mathcal{I})$ such that $0 \neq B \leq f^{-1}(A)$ and consequently f is somewhat fuzzy α-$\mathcal{I}$-continuous. □

Lemma 3.7. ([?]) *Let $g : X \to X \times Y$ be the graph of a function $f : X \to Y$. Then, if λ is a fuzzy set of X and μ is a fuzzy set of Y, $g^{-1}(\lambda \times \mu) = \lambda \wedge f^{-1}(\mu)$.*

Theorem 3.8. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a function, where X is product related to Y, and $g : X \to X \times Y$, the graph function of f. If g is somewhat fuzzy α-$\mathcal{I}$-continuous, then f is so.*

Proof. Let A be a non-zero fuzzy open set in Y. Then by Lemma 2.4 of [**?**], we have $f^{-1}(A) = 1 \wedge f^{-1}(A) = g^{-1}(1 \times A)$. Since g is somewhat fuzzy α-$\mathcal{I}$-continuous and $1 \times A$ is a non-zero fuzzy open set in $X \times Y$, there exists a non-zero fuzzy α-$\mathcal{I}$-open set B of $(X, \tau, \mathcal{I})$ such that $B \leq g^{-1}(1 \times A) = f^{-1}(A)$. This proves that f is a somewhat fuzzy α-$\mathcal{I}$-continuous function. □

Definition 3.9. A fuzzy ideal topological space $(X, \tau, \mathcal{I})$ is called a fuzzy $D_{\alpha\mathcal{I}}$-space (D-space) if for every nonzero fuzzy α-$\mathcal{I}$-open (fuzzy open) set in X is fuzzy α-$\mathcal{I}$-dense (fuzzy dense) in X.

Proposition 3.10. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{I})$ is a somewhat fuzzy α-$\mathcal{I}$-continuous surjection and $(X, \tau, \mathcal{I})$ is a fuzzy $D_{\alpha\mathcal{I}}$-space, then Y is a fuzzy D-space.*

Proof. Let A be a nonzero fuzzy open set in Y. We want to show that A is fuzzy dense in Y. Suppose not, then there exists a fuzzy closed set B in Y such that $A < B < 1$. Therefore, $f^{-1}(A) < f^{-1}(B) < f^{-1}(1) = 1$. Since $A \neq 0$, $f^{-1}(A) \neq 0$ and since f is somewhat fuzzy α-$\mathcal{I}$-continuous there exists a fuzzy α-$\mathcal{I}$-open set $C \neq 0$ in X such that $C < f^{-1}(A)$. Hence $C < f^{-1}(A) < f^{-1}(B) < \alpha\mathcal{I}Cl(f^{-1}(B)) < 1$. That is, $C < \alpha\mathcal{I}Cl(f^{-1}(B)) < 1$. This contradicts the fact that $(X, \tau, \mathcal{I})$ is a fuzzy $D_{\alpha\mathcal{I}}$-space; hence Y is a fuzzy D-space. □

Definition 3.11. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{I})$ is called somewhat fuzzy α-$\mathcal{I}$-open if and only if for any fuzzy α-$\mathcal{I}$-open set A, $A \neq 0$ in $(X, \tau, \mathcal{I})$ implies that there exists a fuzzy α-$\mathcal{I}$-open set B in $(Y, \sigma, \mathcal{I})$ such that $B \neq 0$ and $B \leq f(A)$.

Definition 3.12. A function $f : (X, \tau) \to (Y, \sigma)$ is called fuzzy open [**?**] if and only if for any fuzzy open subset A of X, $f(A) \in \sigma$.

It is clear that every fuzzy open function is somewhat fuzzy α-$\mathcal{I}$-open but the converse is not true as it can be seen from the following example.

Example 3.13. Let $X = \{a, b, c\}$, $\tau = \{0, A, 1\}$, $\sigma = \{0, B, 1\}$ and $\mathcal{I} = \{\varnothing\}$, where $A(a) = 1, A(b) = 1, A(c) = 0$ and $B(a) = 1, B(b) = 0, B(c) = 0$. Then the identity function $f : (X, \tau) \to (X, \sigma, \mathcal{I})$ is somewhat fuzzy α-$\mathcal{I}$-open but not fuzzy open.

Proposition 3.14. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{I})$ and $g : (Y, \sigma, \mathcal{I}) \to (Z, \gamma, \mathcal{I})$ are somewhat fuzzy α-$\mathcal{I}$-open functions, then $g \circ f : (X, \tau) \to (Z, \gamma, \mathcal{I})$ is somewhat fuzzy α-$\mathcal{I}$-open.*

Proof. Clear. □

Theorem 3.15. *For a surjective function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{I})$, the following statements are equivalent:*

(i) *f is somewhat fuzzy α-$\mathcal{I}$-open.*

(ii) *If A is a fuzzy α-$\mathcal{I}$-closed set in X such that $f(A) \neq 1$, then there exists a fuzzy α-$\mathcal{I}$-closed set B in Y such that $B \neq 1$ and $B > f(A)$.*

Proof. (i)⇒(ii): Let A be a fuzzy α-$\mathcal{I}$-closed set in X such that $f(A) \neq 1$. Then $1 - A$ is a fuzzy α-$\mathcal{I}$-open set such that $f(1 - A) = 1 - f(A) \neq 0$. Since f is somewhat fuzzy α-$\mathcal{I}$-open, there exists a fuzzy α-$\mathcal{I}$-open set D in $(Y, \sigma, \mathcal{I})$ such that $D \neq 0$ and $D \leq f(1 - A)$. Now $1 - D$ is fuzzy α-$\mathcal{I}$-closed set in Y such that $1 - D \neq 1$ and $D < f(1 - A)$. Put $1 - D = B$. Then $D > 1 - f(1 - A)= f(A)$.

(ii)⇒(i): Let A be a fuzzy α-$\mathcal{I}$-open of X such that $A \neq 0$. Then $1 - A$ is fuzzy α-$\mathcal{I}$-closed and $1 - A \neq 1$, $f(1 - A) = 1 - f(A) \neq 1$. Hence by hypothesis, there exists a fuzzy α-$\mathcal{I}$-closed set B in Y such that $B \neq 1$ and $B > f(1 - A) = 1 - f(A)$, that is, $f(A) > 1 - B$ and let $1 - B = C$. Clearly, C is a fuzzy α-$\mathcal{I}$-open set of Y such that $C < f(A)$ and $C \neq 0$. Hence f is somewhat fuzzy α-$\mathcal{I}$-open. □

Theorem 3.16. *For a surjective function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{I})$, the following statements are equivalent:*

(i) *f is somewhat fuzzy α-$\mathcal{I}$-open.*
(ii) *If A is a fuzzy α-$\mathcal{I}$-dense set of Y, then $f^{-1}(A)$ is fuzzy α-$\mathcal{I}$-dense set in X.*

Proof. (i)⇒(ii): Suppose A is fuzzy α-$\mathcal{I}$-dense and fuzzy α-$\mathcal{I}$-closed set of $(Y, \tau, \mathcal{I})$. We must to show that $f^{-1}(A)$ is fuzzy α-$\mathcal{I}$-dense in $(X, \tau, \mathcal{I})$. Suppose not, then there exists a fuzzy α-$\mathcal{I}$-closed set B in X such that $f^{-1}(B) < B < 1$. Since f is somewhat fuzzy α-$\mathcal{I}$-open and $1-B$ is fuzzy α-$\mathcal{I}$-open, there exists a fuzzy α-$\mathcal{I}$-open set D in Y such that $D < f(1 - B)$ and $D < 1 - f(B)$. From $f^{-1}(A) < B < 1$, we have $A < f(B) < 1$. Then $D < 1 - f(B) < 1 - A$. That is, $A < 1 - D < 1$. Since $1 - D$ is fuzzy α-$\mathcal{I}$-closed set in Y, this implies that A is not a fuzzy α-$\mathcal{I}$-dense, which is a contradicition. Therefore, $f^{-1}(A)$ must be a fuzzy α-$\mathcal{I}$-dense set in X.
(ii)⇒(i): Suppose $f^{-1}(A)$ is fuzzy α-$\mathcal{I}$-dense in $(X, \tau, \mathcal{I})$, where A is fuzzy α-$\mathcal{I}$-dense set in Y. We want to show that f is somewhat fuzzy α-$\mathcal{I}$-open. Assume that $A \neq 0$ and a fuzzy α-$\mathcal{I}$-open set in $(X, \tau, \mathcal{I})$. We have to show that $\alpha\mathcal{I}Int(f(A)) \neq 0$. Suppose not, then $\alpha\mathcal{I}Int(f(A)) = 0$ whenever A is fuzzy α-$\mathcal{I}$-open. Then $\alpha\mathcal{I}Cl(1 - f(A)) = 1 - \alpha\mathcal{I}Int(f(A)) = 1 - 0 = 1$. That is, $1 - f(A)$ is fuzzy α-$\mathcal{I}$-dense in Y. Therefore by assumption $f^{-1}(1 - f(A))$ is fuzzy α-$\mathcal{I}$-dense in X. Therefore, $1 = \alpha\mathcal{I}Cl(f^{-1}(1 - f(A))) = \alpha\mathcal{I}Cl(1 - A) = 1 - A$. This shows that $A = 0$, which is a contradiction and so $\alpha\mathcal{I}Int(f(A)) \neq 0$. □

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